Homework 10

MATH 301/601

Solutions to Graded Problems

Exercise 1 (Section 9.4 #22)**.** Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5, respectively, such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

Solution. Since we are given that every element of H commutes with every element of K, we only need to check that $H \cap K = \{e\}$ and $G = HK$. Let us first show that the intersection is trivial. If $a \in H \cap K$, then by Lagrange's Theorem, the order of a must divide the order of H and also the order of K. As the orders of H and K are relatively prime, we must have that $|a| = 1$, and hence $a = e$; in particular, $H \cap K = \{e\}$.

Now, the set $HK = \{hk : h \in H, k \in K\}$ has at most $|H| \cdot |K|$ elements; we claim it has exactly that many elements. Suppose $h, h' \in H$ and $k, k' \in K$ such that $hk = h'k'$. Rearranging, this tells us that $(h')^{-1}h = k'k^{-1}$. The product on the left-hand side is in H and the product on the right-hand side is in K. But $H \cap K = \{e\}$, so we must have that $(h')^{-1}h = k'k^{-1} = e$, implying that $h' = h$ and $k' = k$. This establishes that $|HK| = |H| \cdot |K|$. Therefore, as $|H| \cdot |K| = |G|$, we have that $|G| = |HK|$, implying $G = HK$. We have now shown that G is the internal direct product of H and K. □

****Exercise 5.** Let N be a group, and let H be a subgroup of Aut(N), the automorphism group of N. The *(external) semidirect product* of N and H is the group $N \rtimes H$ whose underlying set is $N \times H$ and whose group operation is defined by $(a, \varphi)(b, \psi) = (a\varphi(b), \varphi \circ \psi)$.

(a) Prove that $N \rtimes H$ is a group (yes, I said it was a group in the definition, but that needs a proof).

(b) Let G be a group, and let N and H be subgroups of G such that

   (i) $N \cap H = \{e\}$,

   (ii) $G = NH = \{nh : n \in N, h \in H\}$, and

   (iii) $hnh^{-1} \in N$ for all $n \in N$ and all $h \in H$.

   (Note that condition (ii) and (iii) together imply that N is a normal subgroup of G, see the Week 12 notes.) Condition (iii) says that each element of H induces an automorphism of N via conjugation, that is, for $h \in H$ we can define $\varphi_h \in$ Aut(N) by $\varphi_h(n) = hnh^{-1}$ for all $n \in N$. Identifying h with φ_h, we can view H as a subgroup of Aut(N).

   Under these hypotheses and under this identification of H with a subgropup of Aut(N), prove that $G \cong N \rtimes H$. Here, we say G is the *(internal) semidirect product* of N and H.

Solution.

(a) Associativity follows from the associativity of the group operations of N and H; I'll leave the details to you. There is an identity element, namely (e_N, e_H). It is left to check inverses. If $(a, \varphi) \in N \rtimes H$, then observe that $(a, \varphi)(\varphi^{-1}(a^{-1}), \varphi^{-1}) = (e_N, e_H)$, so $(a, \varphi)^{-1} = (\varphi^{-1}(a^{-1}), \varphi^{-1})$.

(b) Define $\Phi\colon N \rtimes H \to G$ by $\Phi(n, h) = nh$. Condition (i) tells us that Φ is injective, and condition (ii) tells us that Φ is surjective. To finish, let $(n, h), (n', h') \in N \rtimes H$. Then

$$\begin{aligned} \Phi((n, h)(n', h')) &= \Phi(nhn'h^{-1}, hh') \\ &= nhn'h^{-1}hh' \\ &= nhn'h' \\ &= \Phi(n, h)\Phi(n', h'). \end{aligned}$$

All together, we have that Φ is an isomorphism.

□