Homework 11

MATH 301/601

Solutions to Graded Problems

Exercise 4. Let G be a cyclic group, let a be a generator of G, and let $\varphi, \psi \colon G \to H$ be homomorphisms. Prove that if $\varphi(a) = \psi(a)$, then $\varphi = \psi$. (This says that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.)

Solution. In order to show that the functions φ and ψ are equal, we need to show that $\varphi(g) = \psi(g)$ for each $g \in G$. So, let $g \in G$. As a is a generator of G, there exists $n \in \mathbb{Z}$ such that $g = a^n$. We have previously shown that isomorphisms respect exponentiation, so we have:

$$\begin{aligned}\varphi(g) &= \varphi(a^n)\\ &= \varphi(a)^n\\ &= \psi(a)^n\\ &= \psi(a^n)\\ &= \psi(g).\end{aligned}$$

As g was an arbitrary element of G, we have established that $\varphi = \psi$. □

****Exercise 6.** The subgroup of a group G generated by the set $\{xyx^{-1}y^{-1} : x, y \in G\}$ is called the *commutator subgroup of* G and is denoted G' (or $[G, G]$).

(a) Prove that G' is normal in G.

(b) Prove that G/G' is abelian.

(c) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if $G' \subset N$.

(The group G/G' is called the *abelianization* of G.)

Solution.

(a) The product $xyx^{-1}y^{-1}$ is called a *commutator* and is usually denoted by $[x, y]$. It is readily verified that $g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$. This says that the generating set of G' is invariant under conjugation, and hence G' is invariant under conjugation, i.e., $gG'g^{-1} = G$; in other words, G' is normal.

(b) There is a bit of redundancy in parts (b) and (c), so let us prove the following: If N is a normal subgroup of G containing G', then G/N is abelian. To see this, let $a, b \in G$. We need to show that $(ab)N = (ba)N$. From one of our lemmas, we know that $(ab)N = (ba)N$ if and only if $(ba)^{-1}(ab) \in N$. Here, $(ba)^{-1}(ab) = a^{-1}b^{-1}ab = [a^{-1}, b^{-1}] \in G'$, and as $G' < N$, we have that $(ab)N = (ba)N$. We have established that G/N is abelian.

(c) We have already established the reverse direction, so assume that G/N is abelian. We want to show that $G' < N$. By definition, G' is the smallest subgroup that contains every commutator, so to show that $G' < N$, we need to show that N contains every commutator. Let $x, y \in G$. As G/N is abelian, we have that $(x^{-1}y^{-1})N = (y^{-1}x^{-1})N$.

This implies that $(y^{-1}x^{-1})^{-1}(x^{-1}y^{-1}) \in N$, or equivalently, $[x, y] \in N$. As x and y were arbitrary elements of G, we have shown that N contains every commutator of elements in G, and hence it must contain G'.

$\square$