Homework 12

MATH 301/601

This is the last homework assignment of the semester. It will not be collected.

Instructions. *This assignment will not be collected, but the questions are fair game for Exam 3.*

Exercise 1. Complete #16 from Section 11.4 in the course textbook.

Exercise 2. Use Exercise 4 in Homework 11 to answer the following questions.

(a) Find all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_6$.

(b) Show that $\varphi\colon \mathbb{Z}_6 \to \mathbb{Z}_4$ defined by $\varphi(\bar{1}) = \bar{1}$ is not a homomorphism (there are multiple ways to see this, but one is to realize that it is not even well defined).

(c) Find all homomorphisms $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

****Exercise 3.** Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Prove that if A is a cyclic subgroup of $\mathbb{Z}^2$, then $\mathbb{Z}^2/A$ is isomorphic to $\mathbb{Z}$.

Exercise 4. Complete the following exercises from Section 16.7:

1(a)–(g), 7, 13(a)–(c)

Definition 1. Let F_1 and F_2 be fields. A function $\varphi\colon F_1 \to F_2$ is a *homomorphism* of fields if, for all $a, b \in F_1$,

- $\varphi(a + b) = \varphi(a) + \varphi(b)$,
- $\varphi(ab) = \varphi(a)\varphi(b)$, and
- $\varphi(1) = 1$.

An *isomorphism* of fields is a bijective homomorphism of fields.

Exercise 5. Let $\varphi\colon F_1 \to F_2$ be a homomorphism between fields. Prove that φ is injective.

Exercise 6. Let F be a field of characteristic $p \neq 0$. Define $\varphi\colon F \to F$ by $\varphi(x) = x^p$.

(a) Prove that φ is a homomorphism. (Hint: Use the Freshman's dream.)

(b) If F is finite, prove that φ is an isomorphism.

Exercise 7. The goal here is to explore the field of order 9.

(a) Find an irreducible quadratic polynomial p in $\mathbb{Z}_3[x]$.

(b) Then $\mathbb{F}_9 = \{a + b\beta : a, b \in \mathbb{Z}_3 \text{ and } p(\beta) = 0\}$ is a field of order 9. Find the multiplicative inverses of $1 + \beta, 2 + \beta$, and $1 + 2\beta$.