Homework 1 MATH 301/601

Solutions to graded problems

***Exercise 6.** Let $n \in \mathbb{N}$. Prove that the remainder obtained from dividing n^2 by 4 is either 0 or 1.

Proof. By the division algorithm, there exists $k \in \mathbb{Z}$ such that either $n = 2k$ or $n = 2k + 1$, that is, either n is even or odd, respectively. Also, by the division algorithm, there exists unique $q, r \in \mathbb{Z}$ such that $n^2 = 4q + r$ with $r \in \{0, 1, 2, 3\}$. If $n = 2k$, then $n^2 = (2k)^2 = 4k^2$, and hence, $q = k^2$ and $r = 0$. Otherwise, $n = 2k + 1$ and $n^2 = (2k + 1)^2 = 4(k^2 + k) + 1$, and hence, $q = k^2 + k$ and $r = 1$. In either case, $r \in \{0, 1\}$, as desired. $\square$

Exercise 7. Define the ordering $<$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) < (c, d)$ if $a < c$ or $a = c$ and $b < d$ (this is called the *lexicographical ordering*). Prove that $(\mathbb{N} \times \mathbb{N}, <)$ is well ordered, that is, show that given a nonempty subset S of $\mathbb{N} \times \mathbb{N}$ there exists $s \in S$ such that $s < s'$ for all $s' \in S \setminus \{s\}$.

Proof. Let S be a nonempty subset of $\mathbb{N} \times \mathbb{N}$. Let

$$T = \{a \in \mathbb{N} : \text{there exists } b \in \mathbb{N} \text{ such that } (a, b) \in S\}.$$

As S is nonempty, so is T. Therefore, we can apply the well-ordering principle to T to obtain its least element, call it x. Now, let $U = \{b \in \mathbb{N} : (x, b) \in S\}$. By the construction of x, we know that U is nonempty, so we can apply the well-ordering principle to U to obtain its least element, call it y. By construction, $s = (x, y) \in S$, and we claim that s is the least element of S. Indeed, let $s' = (x', y') \in S$. Then, $x' \in T$, and hence $x \leq x'$. If $x < x'$, then $s < s'$; otherwise, $x' = x$, and $y' \in U$. Therefore, $y' \leq y$, and hence $s = (x, y) \leq (x, y') = s'$. In either case, we have $s \leq s'$, as desired. $\square$