Homework 2

MATH 301/601

Due Wednesday, February 14, 2024[1]

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Let S be a finite subset of $\mathbb{N}$. Prove that S has a maximum, that is, prove that there exists $s \in S$ such that $s' \leq s$ for all $s' \in S$.

***Exercise 2.** Let $a, b \in \mathbb{Z} \setminus \{0\}$, and let $c \in \mathbb{Z}$. Prove that c is a linear combination of a and b if and only if $\gcd(a, b)$ divides c.

Exercise 3. Let $x \in \mathbb{Z} \setminus \{1\}$. Use induction to prove that

$$\frac{x^n - 1}{x - 1} = \sum_{i=0}^{n-1} x^i$$

for all $n \in \mathbb{N}$. (Note: you are inducting on n, not x.)

Exercise 4. Complete the following exercises from Section 2.4 in the course textbook:

5, 15, 16, 27, ***28** (you will need to use Exercise 3), 31

Note that 28 above is starred.

Definition. Given two nonzero integers a and b, an integer c is a *common multiple* of a and b if $a \mid c$ and $b \mid c$. The *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive common multiple of a and b.

***Exercise 5.** Let a and b be nonzero integers.

(1) Prove that the least common multiple of a and b exists.

(2) Prove that if $k \in \mathbb{Z}$ is a common multiple of a and b, then $\operatorname{lcm}(a, b)$ divides k. (Hint: divide k by $\operatorname{lcm}(a, b)$ using the division algorithm.)

****Exercise 6.** Let $a, b \in \mathbb{N}$.

(1) Prove that the product of $\operatorname{lcm}(a, b)$ and $\gcd(a, b)$ is equal to ab. (Hint: the product ab is divisible by $d = \gcd(a, b)$. Let $m = ab/d$. Now, let ℓ be the least common multiple of a and b. Write d as a linear combination in a and b, and use this to express the fraction ℓ/m as an integer.)

(2) Prove that $\operatorname{lcm}(a, b) = ab$ if and only if $\gcd(a, b) = 1$.

---

[1] As CUNY is closed on Monday, I will not have office hours. So, if you would like, you can turn in your assignment by Friday, February 16 by dropping it off in my mailbox in the math office, Kiely 243, or sliding it under my office door, Kiely 507.