# Homework 3

MATH 301/601

Due Thursday, February 22, 2024

**Instructions.** *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

**Exercise 1.** Complete Exercise #1 from Section 3.5 in the course textbook

**Definition 1.** An *equivalence relation* on a set $S$ is a binary relation $\sim$ that is:

(i) *reflexive*, that is, $a \sim a$ for all $a \in S$;

(ii) *symmetric*, that is, $a \sim b$ implies $b \sim a$ for all $a, b \in S$; and

(iii) *transitive*, that is, $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in S$.

**Exercise 2.** Let $n \in \mathbb{N}$. Prove that equivalence modulo $n$ is an equivalence relation on $\mathbb{Z}$.

***Exercise 3.** Let $n \in \mathbb{N}$. Prove that given any $m \in \mathbb{Z}$ there exists a unique element $a \in \{0, 1, 2, \ldots, n-1\}$ such that $m \equiv a \pmod{n}$. (Hint: Use the division algorithm.)

**Exercise 4.** Let $n \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Prove that if $a \equiv b \pmod{n}$, then

$$\gcd(a, n) = \gcd(b, n).$$

***Exercise 5.** Let $n \in \mathbb{N}$ with $n > 1$, and let $a \in \mathbb{Z}$.

(a) Prove that if $\gcd(a, n) = 1$ and $b, c \in \mathbb{Z}$ such that $ab \equiv ac \pmod{n}$, then $b \equiv c \pmod{n}$.

(b) Give an example of integers $n, a, b, c$ such that $a \not\equiv 0 \pmod{n}$, $b \not\equiv c \pmod{n}$, and $ab \equiv ac \pmod{n}$.

****Exercise 6.** Let $m, n \in \mathbb{N}$ be relatively prime, and let $a, b \in \mathbb{Z}$. Prove that there exists $x \in \mathbb{Z}$ such that

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n}. \end{aligned}$$

(Hint: Start by writing 1 as a linear combination of $m$ and $n$.)

(Turn page over.)

**Exercise 7.** Let $n \in \mathbb{N}$.

(a) Prove that $10^n \equiv 1 \pmod 9$. (There are numerous ways to see this. One way is to use induction.)

(b) (Divisibility by 9) Define $h\colon \mathbb{N} \to \mathbb{Z}$ by

$$h(n) = \sum_{j=0}^{k} a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $h(n)$ is the sum of the digits of $n$ when written in base 10. For example, if $n = 27301$, then $h(n) = 1 + 0 + 3 + 7 + 2 = 13$. Prove the following statement: Let $n \in \mathbb{N}$. Then, $9 \mid n$ if and only if $9 \mid h(n)$. (Hint: You will have to use part (a).)

***Exercise 8.** Let $n \in \mathbb{N}$.

(a) Prove that $10^n \equiv (-1)^n \pmod{11}$. (Hint: use induction.)

(b) (Divisibility by 11) Define $f\colon \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sum_{j=0}^{k} (-1)^j a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $f(n)$ is the alternating sum of the digits of $n$ when written in base 10. For example, if $n = 27301$, then $f(n) = 1 - 0 + 3 - 7 + 2 = -1$. Prove the following statement: Let $n \in \mathbb{N}$. Then, $11 \mid n$ if and only if $11 \mid f(n)$. (Hint: You will have to use part (a).)