Homework 4

MATH 301/601

Solutions to Graded Problems

Exercise 1 (#32 in Section 3.5)**.** Show that if G is a finite group of even order, then there is an $a \in G$ such that a is not the identity and $a^2 = e$.

Solution. Let $E = \{g \in G : g^{-1} \neq g\}$. Pairing each element of E with its inverse, we see that E has an even number of elements. As $|G| = |E| + |G \setminus E|$ and both $|G|$ and $|E|$ are even, $G \setminus E$ has an even number of elements. Note if $g \in G \setminus E$, then $g = g^{-1}$, and hence $g^2 = e$. Therefore, to finish, we need to show that $G \setminus E$ contains a non-identity element. We know that $e \in G \setminus E$, so $G \setminus E$ has at least one element. But, $G \setminus E$ is even and hence has at least two elements. Therefore, there exists $a \in G \setminus E$ such that $a \neq e$. As noted above, this means that $a^2 = e$, and a is our desired element of G. □

****Exercise 3.** Let G be a finite group. Prove that there exists $N \in \mathbb{N}$ such that $g^N = e$ for each $g \in G$.

Solution. Fix $g \in G$. Consider the subset $\{g^k : k \in \mathbb{N}\} \subset G$. As G is finite, the above subset has only finitely many elements, and hence, there exists $i, j \in \mathbb{N}$ such that $g^j = g^i$ and $j > i$. It follows that $g^j g^{-i} = e$, and setting $N_g = j - i$, we have that $N_g \in \mathbb{N}$ and $g^{N_g} = e$.

Now, let $N = \prod_{g \in G} N_g$, and, for $g \in G$, let $N'_g = N/N_g = \prod_{h \in G \setminus \{g\}} N_h \in \mathbb{N}$. We then have that, for any $g \in G$,

$$g^N = g^{N_g N'_g} = \left(g^{N_g}\right)^{N'_g} = e^{N'_g} = e,$$

and hence N is as desired. □