Homework 5

MATH 301/601

Due Wednesday, March 13, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from #1 from Section 3.5 in the course textbook: # 39, 41, 43, 45, 46, 47, 48

***Exercise 2.** Let H be a subgroup of a group G. Define the relation $\sim$ on G by $a \sim b$ if $b^{-1}a \in H$. Prove that $\sim$ is an equivalence relation on G.

****Exercise 3.** Suppose H is a nonempty finite subset of a group G and that H is closed under multiplication (that is, $ab \in H$ for all $a, b \in H$). Prove that H is a subgroup of G.

Exercise 4. Complete the following exercises from Section 4.5 in the course textbook:

#1(a,b,c,d), 2(a,e,f), 3(b,c,e), 4(a,b,c), 9, 11, ***23**, 30, 31, 39

Definition 1. The *center* of a group G, denoted $Z(G)$, is the subgroup

$$Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\},$$

i.e., $Z(G)$ is the subgroup consisting of group elements that commute with every element of G. (You proved that $Z(G)$ is a subgroup in #48 in §3.5.)

Exercise 5. (a) Compute the center of $\mathrm{GL}(2, \mathbb{R})$. (Hint: use the following test matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.)

(b) Compute the center of $\mathrm{SL}(2, \mathbb{R})$.

***Exercise 6.** Suppose G is a nontrivial group in which the only two subgroups of G are itself and the trivial subgroup.

(a) Prove that G is cyclic.

(b) Using part (a), prove that G is a finite group of prime order.