Homework 6

MATH 301/601

Solutions to Graded Problems

Exercise 1 (Section 4.5, #34)**.** Let G be an abelian group of order pq, where $\gcd(p,q)=1$. If G contains elements a and b of order p and q, respectively, then show that G is cyclic.

Solution. Let $g = ab$, and let $n = |g|$. We will show that $n = pq$, and hence that g generates G. Firstly, using that G is abelian, we have that

$$g^{pq} = (ab)^{pq} = a^{pq}b^{pq} = (a^p)^q(b^q)^p = e,$$

so $n \leq pq$. To show the other inequality, we will argue that pq divides n. From a previous homework problem (Section 4.5 #30), we know that $\langle a\rangle \cap \langle b\rangle = \{e\}$ (this follows from the fact that the order of any element in $\langle a\rangle \cap \langle b\rangle$ has to divide both p and q, which are relatively prime). Again, using that G is abelian, we that $e = g^n = (ab)^n = a^nb^n$, so that $b^n = a^{-n}$. In particular, $a^n, b^n \in \langle a\rangle \cap \langle b\rangle$, and hence $a^n = b^n = e$. Since $a^n = e$, the order of a must divide n; in particular, $p \mid n$. Similarly, $q \mid n$. Now, as both p and q divide n, so does their least common multiple, which is pq. Therefore, $pq \leq n$, and as we have shown that $n \leq pq$, we have established that $n = pq$, as desired. □

Exercise 6. Prove that any two k-cycles in S_n are conjugate, that is, if $\sigma, \tau \in S_n$ are k-cycles, then there exists $\mu \in S_n$ such that $\mu\sigma\mu^{-1} = \tau$.

Solution. Let $\sigma, \tau \in S_n$ be k-cycles. There exist $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k \in \{1, 2, \ldots, k\}$ such that $\sigma(a_i) = a_{i+1}$ and $\tau(b_i) = b_{i+1}$, with indices read modulo k, and such that $\sigma(x) = x$ if $x \neq a_i$ for any i and $\tau(y) = y$ if $y \neq b_i$ for any i. Let μ be any permutation that satisfies $\mu(a_i) = b_i$ for all $i \in \{1, \ldots, k\}$. We claim that $\mu\sigma\mu^{-1} = \tau$. Indeed, we check this pointwise: for $i \in \{1, \ldots, k\}$,

$$\begin{aligned}
(\mu\sigma\mu^{-1})(b_i) &= \mu(\sigma(\mu^{-1}(b_i))) \\
&= \mu(\sigma(\mu^{-1}(\mu(a_i)))) \\
&= \mu(\sigma(a_i)) \\
&= \mu(a_{i+1}) \\
&= b_{i+1} \\
&= \tau(b_i),
\end{aligned}$$

where we again read the indices modulo k. Now, if $y \neq b_i$ for any i, then there exists x such that $\mu(x) = y$ and $x \neq a_i$ for any i. Hence, $\mu(\sigma(\mu^{-1}(y))) = \mu(\sigma(x)) = \mu(x) = y = \tau(y)$, establishing $\mu\sigma\mu^{-1} = \tau$. □