Homework 7

MATH 301/601

Solutions to Graded Problems

Exercise 1 (Section 5.4 #25)**.** Prove that in A_n with $n \geq 3$, any permutation is a product of cycles of length 3.

Proof. Let us first show that the product two transpositions is either trivial, a 3-cycle, or a product of two 3-cycles. Let τ_1 and τ_2 be transpositions. Then we can write $\tau_1 = (a\ b)$ and $\tau_2 = (c\ d)$ for some $a, b, c, d \in \{1, \ldots, n\}$. Up to relabelling, there are three cases: (1) $a = c$ and $b = d$, (2) $a \neq c$ and $b = d$, and (3) both a and b are distinct from c and d. In the first case, $\tau_1\tau_2$ is the identity. In the second case, $\tau_1\tau_2 = (a\ b\ c)$. In the third case, we have

$$\begin{aligned} \tau_1\tau_2 &= (a\ b)(c\ d) \\ &= (a\ b)(b\ c)(b\ c)(c\ d) \\ &= (a\ b\ c)(b\ c\ d) \end{aligned}$$

Now, if $\sigma \in A_n$, then σ is a product of a non-zero even number of transpositions (this is true for the identity as well). So, there exists transpositions $\tau_1, \tau_2, \ldots, \tau_{2k}$ such that $\sigma = \tau_1\tau_2 \cdots \tau_{2k}$. For $j \in \{1, \ldots, k\}$, let $\sigma_j = \tau_{2j-1}\tau_{2j}$. Then, our above argument implies that σ_j is either the identity, a 3-cycle, or a product of two 3-cycles. Therefore, $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ expresses σ as a product of 3-cycles. □

****Exercise 5.** Let $\sigma = (1\ 2\ 3\ 4\ 5) \in A_9$ and $\tau = (5\ 6\ 7\ 8\ 9) \in A_9$. Prove that the permutation $(1\ 2\ 3)$ can be written as a word in $\{\sigma, \tau\}$, i.e., there exists $r \in \mathbb{N}$ and $n_1, \ldots, n_r, m_1, \ldots m_r \in \mathbb{Z}$ such that $(1\ 2\ 3) = \sigma^{n_1}\tau^{m_1}\sigma^{n_2}\tau^{m_2} \cdots \sigma^{n_r}\tau^{m_r}$.

Solution. Talk to Eric. I forgot to write down his solution, so I can't share it. I didn't take the time to figure it out on my own, so I gave everyone full credit. The goal was to get you to think visually about a group, so I hope you spent some time doing that. □