Homework 8

MATH 301/601

Solutions to Graded Problems

Exercise 1 (Section 6.5 #18)**.** If $[G : H] = 2$, prove that $gH = Hg$.

Proof. As the index of H in G is two, we know that H has two left cosets; moreover, we know that these two left cosets are disjoint and that their union is all of G (as they partition G). Therefore, the two left costs of H in G are H and $G \setminus H$. The same holds for right cosets, so the two right cosets of H in G are also H and $G \setminus H$.

Now, let $g \in G$. If $g \in H$, then $gH = H$ and $Hg = H$, so $gH = Hg$. Otherwise, $g \in G \setminus H$, implying $gH = G \setminus H$ and $Hg = G \setminus H$, so $gH = Hg$. Thus, $gH = Hg$. □

****Exercise 6.** Let G be a group acting on a set X. Let $x \in X$.

(a) Let $g, h \in G$. Prove that $gx = hx$ if and only if $h^{-1}g \in \mathrm{Stab}_G(x)$.

Solution. If $gx = hx$, then $h^{-1}(gx) = h^{-1}(hx) = (h^{-1}h)x$, and so $(h^{-1}g)x = x$, implying $h^{-1}g \in \mathrm{Stab}_G(x)$. For the converse, if $h^{-1}g \in \mathrm{Stab}_G(x)$, then

$$x = (h^{-1}g)x = h^{-1}(gx),$$

implying $hx = gx$. □

(b) Let $\mathcal{L}$ be the set of left cosets of $\mathrm{Stab}_G(x)$ in G. Let $\psi\colon \mathcal{L} \to \mathcal{O}_x$ be given by $\psi(g\mathrm{Stab}_G(x)) = gx$.

(i) Prove that ψ is a well-defined.

Solution. Suppose $g\mathrm{Stab}_G(x) = h\mathrm{Stab}_G(x)$. Then there exists $s \in \mathrm{Stab}_G(x)$ such that $gs = h$, and hence

$$\psi(g\mathrm{Stab}_G(x)) = gx = g(sx) = (gs)x = hx = \psi(h\mathrm{Stab}_G(x)),$$

implying ψ is well-defined. □

(ii) Prove that ψ is bijective.

Solution. The surjectivity of ψ follows immediately from the definition. Suppose that $\psi(g\mathrm{Stab}_G(x)) = \psi(h\mathrm{Stab}_G(x))$. Then $gx = hx$, which implies that $h^{-1}g \in \mathrm{Stab}_G(x)$. Therefore, $g\mathrm{Stab}_G(x) = h\mathrm{Stab}_G(x)$, and ψ is injective. □

(c) The previous part implies that $|\mathcal{O}_x| = [G : \mathrm{Stab}_G(x)]$. Apply Lagrange's theorem to obtain $|G| = |\mathcal{O}_x| \cdot |\mathrm{Stab}_G(x)|$.

Solution. By Lagrange's theorem, $|G| = |\mathrm{Stab}_G(x)| \cdot [G : \mathrm{Stab}_G(x)]$, and hence $|G| = |\mathrm{Stab}_G(x)| \cdot |\mathcal{O}_x|$ by part (b). □

(d) Now suppose G is a finite group, and let G act on itself by conjugation, that is, the action is given by $g \cdot a = gag^{-1}$. Apply the orbit–stabilizer theorem to show that, for $a \in G$, the cardinality of the set $\{gag^{-1} : g \in G\}$ (that is, the conjugacy class of a) divides $|G|$.

Solution. The orbit of a under the conjugation action is exactly the conjugacy class of a. By the Orbit-Stabilizer theorem, $|G|$ is a multiple of $|\mathcal{O}_a|$. $\square$