Homework 9

MATH 301/601

Due Wednesday, April 17, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Let $\varphi\colon G \to H$ be an isomorphism.

(a) Prove that $\varphi(e_G) = e_H$. (Hint: use the fact that $e_G e_G = e_G$.)

(b) Prove that $\varphi(g)^{-1} = \varphi(g^{-1})$ for all $g \in G$.

(c) Prove that $\varphi(g^n) = \varphi(g)^n$ for all $g \in G$ and for all $n \in \mathbb{Z}$.

Definition 1. An *automorphism* of a group G is an isomorphism $G \to G$.

***Exercise 2.** Let G be a finite abelian group of order n. Suppose $m \in \mathbb{N}$ is relatively prime to n. Prove that $\varphi\colon G \to G$ given by $\varphi(g) = g^m$ is an automorphism of G. (This says that every element of G has an m^{th}-root.)

Exercise 3. Let G be a group. Prove that the set of automorphisms of G, denoted $\mathrm{Aut}(G)$, is a group with respect to function composition (this group is called *the automorphism group of* G).

***Exercise 4.** Let G be a cyclic group, and let $\varphi, \psi \in \mathrm{Aut}(G)$. Prove that if $a \in G$ is a generator of G and $\varphi(a) = \psi(a)$, then $\varphi = \psi$.

Exercise 5. Complete the following exercises from Section 9.4 in the course textbook:

2, 8, 11, 12, 14, ***31**, 38, 39, 41, 46

(Hint for #38 and #39: use Exercise 4.)

****Exercise 6.** Let $\mathbb{Q}$ denote the group $(\mathbb{Q}, +)$, and let $\mathbb{Q}^\times$ denote the group $(\mathbb{Q} \setminus \{0\}, \cdot)$.

(a) Let $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ be an isomorphism. Prove that $\varphi(x) = x \cdot \varphi(1)$ for all $x \in \mathbb{Q}$. (This is saying that every automorphism of $\mathbb{Q}$ is $\mathbb{Q}$-linear.)

(b) Use part (a) to prove that if $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ is an isomorphism, then there exists $q \in \mathbb{Q} \setminus \{0\}$ such that $\varphi(x) = qx$ for all $x \in \mathbb{Q}$.

(c) Use part (b) to prove that $\mathrm{Aut}(\mathbb{Q}) \cong \mathbb{Q}^\times$.