Homework 10 MATH 590

Due Wednesday, March 30, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Let $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{R})$ and let $f_A\colon T^2 \to T^2$ be the corresponding homeomorphism of the torus. Using the model of the torus $I^2/{\sim}$, where $\sim$ is the appropriate equivalence relation (see previous homework), describe the behavior of the homeomorphism f_A on the torus.

Exercise 2. Recall that the torus T^2 is a topological group. Let $e \in T^2$ denote the identity. For an element $g \in T^2$ define the homeomorphism $f_g\colon T^2 \to T^2$ by $f_g(h) = gh$ for all $h \in T^2$. For $A \in \mathrm{SL}(2,\mathbb{Z})$, let $F_A\colon T^2 \to T^2$ be the corresponding homeomorphism of the torus. Prove that $F_A = f_g$ if and only if A is the identity matrix and $g = e$.

Exercise 3 (*). Define the upper half plane $\mathbb{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$. Prove that $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{U}$ via $\alpha\colon \mathrm{SL}(2,\mathbb{R}) \times \mathbb{U} \to \mathbb{U}$ where

$$\alpha\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], z\right) = \frac{az+b}{cz+d}.$$

(Note: (1) Let $I_2 \in \mathrm{SL}(2,\mathbb{R})$ be the identity matrix. Then $\alpha(-I_2, z) = z$ for every $z \in \mathbb{C}$. (2) If $\Gamma < \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$ is a discrete subgroup – with respect to the subspace topology – acting freely on $\mathbb{U}$, then Γ is a called a *Fuchsian group* and $\mathbb{U}/\Gamma$ is called a *hyperbolic surface*.)

Exercise 4. Let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ via $\alpha\colon \mathbb{Z}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\alpha((n,m),(x,y)) = (x+n, y+m)$$

(i.e. the standard action of $\mathbb{Z}^2 < (\mathbb{R}^2,+)$ on $\mathbb{R}^2$). Prove that this action is both proper and free.

Exercise 5 (*). As a set let $\mathbb{Z}_n = \{0,\ldots,n-1\}$ and define $a +_n b = (a+b) \mod n$, then $(\mathbb{Z}_n, +_n)$ is a group. Give a proper and free action of $\mathbb{Z}_n$ on $\mathbb{S}^1$. Compute the space $\mathbb{S}^1/\mathbb{Z}_n$. (Provide the necessary details.)

Exercise 6. Let G be a discrete group acting on a compact Hausdorff topological space X. Prove that G is finite (i.e. G has a finite number of elements) if and only if the action of G on X is proper.

Exercise 7 (*). Let G be a topological group acting on topological space X. Prove that the quotient map $\pi\colon X \to X/G$ is open.

Exercise 8. Let X be a first-countable topological space and let $A \subset X$.

(a) Prove that $x \in \bar{A}$ if and only if there exists a sequence $\{x_i\} \subset A$ such that $x_i \to x$ (i.e. x_i converges to x).

(b) In addition, suppose that X is compact. Prove that X is sequentially compact (i.e. every sequence in X has a convergent subsequence).

Definition 1. Let G be a topological group acting on a topological space X. The action is called *wandering* if for every $x \in X$ there exists a neighborhood $U_x \subset X$ of x such that the set

$$G_{U_x} = \{g \in G : (g \cdot U_x) \cap U_x \neq \emptyset\}$$

is finite.

Exercise 9 (*)**.** Let G be a discrete group acting properly on a manifold M. Prove that the action is wandering.

Exercise 10 (Extra credit)**.** Let $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Let $M = \mathbb{R}^2 \smallsetminus \{0\}$ and define the action $\alpha \colon \mathbb{Z} \times M \to M$ by $\alpha(n, v) = A^n \cdot v$.

(a) Prove that this action is wandering.

(b) Prove that this action is *not* proper.

(c) Prove that $M/\mathbb{Z}$ satisfies all the properties of being a 2-manifold except for the Hausdorff condition.