

Homework 2

MATH 590

Due Wednesday, January 27, 2016

Instructions. Write up (in $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.

Exercise 1 (*). Let $(a, b) \subset \mathbb{R}$ and equip this set with the metric topology coming from restricting the Euclidean metric to (a, b) . Prove that (a, b) is homeomorphic to $(0, 1)$. Similarly, prove that $[a, b]$ is homeomorphic to $[0, 1]$.

Exercise 2. Let X be a set. Let $\mathcal{S} \subset 2^X$ be a collection of subsets satisfying

$$\bigcup_{U \in \mathcal{S}} U = X$$

and define τ to be the collection of arbitrary unions of finite intersections of elements of \mathcal{S} . Prove that τ is a topology of X . This topology is called the *topology generated by \mathcal{S}* .

Exercise 3. Let $f: X \rightarrow Y$ be a function between topological spaces and let \mathcal{S} be a subbasis for Y . Prove that f is continuous if and only if for every $U \in \mathcal{S}$ the set $f^{-1}(U)$ is open in X .

Exercise 4. (Extra credit) How many non-pairwise homeomorphic topologies are there on the set $X = \{a, b, c\}$?

Exercise 5. (a) Show that the function $s, p: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$\begin{aligned} s(x, y) &= x + y \\ p(x, y) &= xy \end{aligned}$$

are continuous. (If you are new to ϵ - δ proofs, see Exercise 12 in §21 for hints.)

(b) Prove that $\mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n$.

(c) Let X be a topological space and let $f, g: X \rightarrow \mathbb{R}$ be continuous functions. Show that $f \pm g$ and fg , are continuous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.

(d) Let $P: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial in n variables. Prove that P is continuous.

Exercise 6 (*). (a) Prove that $m: M_n(\mathbb{R}) \times M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ given by $m(A, B) = AB$ is continuous. (*Hint: You should think of $M_n(\mathbb{R})$ as \mathbb{R}^{n^2} .*)

(b) Prove that

$$\text{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is invertible}\}$$

is open in $M_n(\mathbb{R})$.

(c) Given $A \in M_n(\mathbb{R})$, the *transpose* of A , denoted A^T , is defined by

$$[A^T]_{i,j} = [A]_{j,i}$$

for all $1 \leq i, j, \leq n$. Prove that

$$O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$$

is closed in $M_n(\mathbb{R})$.

Exercise 7. Prove the following statements.

(a) If $f: X \rightarrow Y$ is a map, $A \subseteq X$ a subspace, then

$$f|_A: A \rightarrow Y$$

given by $f|_A(x) = f(x)$ whenever $x \in A$ is a continuous function.

(b) Let X, Y be topological spaces, $Y_1 \subseteq Y_2 \subseteq Y$ subspaces, $f: X \rightarrow Y_2$ a continuous function. Then f as a function $X \rightarrow Y$ is also continuous. If $f(X) \subseteq Y_1$, then f as a function from $X \rightarrow Y_1$ is continuous as well.

Exercise 8. Equip $GL_n(\mathbb{R})$ with the subspace topology coming from $M_n(\mathbb{R})$. Let $A \in GL_n(\mathbb{R})$ and let $f_A, g_A: GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ be given by

$$\begin{aligned} f_A(B) &= AB \\ g_A(B) &= BA \end{aligned}$$

for all $B \in GL_n(\mathbb{R})$. Prove that both f_A and g_A are homeomorphisms.

Exercise 9 (*). An n -sphere \mathbb{S}^n is a topological space homeomorphic to the subspace $\partial B_{\mathbb{R}^{n+1}}(0, 1)$ of \mathbb{R}^{n+1} given by

$$\partial B_{\mathbb{R}^{n+1}}(0, 1) = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i^2 = 1 \right\}.$$

For this exercise, identify \mathbb{S}^n with $\partial B_{\mathbb{R}^{n+1}}(0, 1)$.

(a) Prove that \mathbb{S}^n is a closed subset of \mathbb{R}^{n+1} .

(b) Prove that the the upper hemisphere

$$H = \mathbb{S}^n \cap \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_n \geq 0\}$$

is homeomorphic to the closed n -ball \bar{D}^n . Here, the closed n -ball \bar{D}^n refers to any topological space homeomorphic to the closed unit ball in \mathbb{R}^n :

$$\bar{B}_{\mathbb{R}^n}(0, 1) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1 \right\}.$$