Homework 5

MATH 590

Due Wednesday, February 17, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Let X be a compact topological space, $\{A_\alpha : \alpha \in I\}$ an arbitrary collection of closed sets, which is closed with respect to finite intersections (that is, given $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $A_\alpha \cap A_\beta = A_\gamma$). If for an open set $U \subseteq X$ one has $\cap_\alpha A_\alpha \subseteq U$, then there exists $\alpha \in I$ for which $A_\alpha \subseteq U$.

Exercise 2. Let X be a compact Hausdorff space. Prove that X is *normal*, that is, given disjoint closed subspaces A and B of X, there exist disjoint open sets U and V containing A and B, respectively. (Note: a normal space is also referred to as a T_4 space.)

Exercise 3. Prove that if $f\colon X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a *closed* map (that is, f carries closed sets to closed sets).

Exercise 4 (*). Let $f\colon X \to Y$; let Y be compact Hausdorff. Prove that f is continuous if and only if the *graph* of f,

$$G_f = \{(x, f(x)) \in X \times Y : x \in X\}$$

is closed in $X \times Y$.

Exercise 5. Let $f\colon X \to Y$ be a closed continuous function between topological spaces. Prove that for every $y \in Y$ and every neighborhood U of $f^{-1}(y)$ there exists a neighborhood W of y such that $f^{-1}(W) \subseteq U$.

Exercise 6 (*). (a) Prove that $\mathrm{GL}_n(\mathbb{R})$ is not compact.

(b) Prove that the orthogonal group $O(n)$ is compact.

Exercise 7. (a) If $f\colon X \to Y$ is a proper continuous function between locally compact Hausdorff spaces, prove that f extends to a continuous function $\hat{f}\colon \hat{X} \to \hat{Y}$ of their one-point compactifications. This means that $\hat{f}|_X = f$ and $\hat{f}(\infty_X) = \infty_Y$.

(b) If $f\colon X \to Y$ is a homeomorphism between locally compact Hausdorff spaces, prove that f extends to a homeomorphism $\hat{f}\colon \hat{X} \to \hat{Y}$ of their one-point compactifications.

Exercise 8 (*). Prove that the one-point compactification of $\mathbb{R}$ is homeomorphic to $\mathbb{S}^1$. (The same ideas used in a proof here will also prove that the one-point compactification of $\mathbb{R}^n$ is $\mathbb{S}^n$. You should convince yourself of this fact, but there is no need to write it up.)

Exercise 9 (*)**.** Let $\mathbb{C}$ be the complex plane. For $z = x + iy \in \mathbb{C}$, define $\bar{z} = x - iy$ and $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. Then the function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by $d(z, w) = |z - w|$ is a metric. We equip $\mathbb{C}$ with the topology induced by this metric. It is clear that $\mathbb{C} \cong \mathbb{R}^2$.

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of $\mathbb{C}$. $\hat{\mathbb{C}}$ is called the *Riemann sphere* (by Exercises 7 and 8, it is indeed a sphere). Let $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Prove that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$f(z) = \frac{az + b}{cz + d}$$

is a homeomorphism. A function of this form is called a *Möbius transformation*; they are fundamental to the study of complex analysis, dynamics, hyperbolic geometry, and low-dimensional topology.