Homework 8

MATH 590

Due Wednesday, March 16, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

The following may be useful:
Theorem 1. *Sine and cosine are continuous functions from $\mathbb{R} \to \mathbb{R}$.*

Exercise 1. Let $f\colon X \to Y$ be a continuous surjection. Prove that f is a quotient map if f is open or closed (or both).

Exercise 2. Let $I = [0, 1]$ and partition I as follows:

- $\{0, 1\}$
- $\{x\}$ for $x \in (0, 1)$.

Let $\sim$ be the equivalence relation induced by this partition. Prove that $I/\sim$ is homeomorphic to $\mathbb{S}^1$.

Exercise 3. (a) Let $p\colon X \to Y$ be a continuous map. Prove that if there is a continuous map $f\colon Y \to X$ such that $p \circ f$ equals the identity map on Y, then p is a quotient map.

(b) If $A \subset X$, a *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Exercise 4 (*). (a) Prove that the annulus retracts onto a circle.

(b) Let $I = [0, 1]$ and partition I^2 as follows:

- $\{(0, y), (1, 1 - y)\}$ for $y \in I$
- $\{(x, y)\}$ for $(x, y) \in (0, 1) \times I$

Let $\sim$ be the induced equivalence relation. Let $M = I^2/\sim$, then M is a called the *Möbius band*. Prove that M retracts onto a circle.

Exercise 5. Let $\pi_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points (x, y) for which either $x \geq 0$ or $y = 0$ (or both); let $q\colon A \to \mathbb{R}$ be obtained by restricting π_1. Prove that q is a quotient map that is neither open or closed.

Exercise 6. Let $f\colon X \to Y$ be a quotient map. Prove or disprove the following statements:

(a) If X is connected, then so is Y.

(b) If X is Hausdorff, then so is Y.

(c) If X is compact, then so is Y.

Exercise 7 (*)**.** Prove that $\mathbb{RP}^n$ is an n-manifold.

Exercise 8. Identify the closed 2-disk $\bar{\mathbb{D}}^2$ with the closed unit ball in $\mathbb{R}^2$. Partition $\bar{\mathbb{D}}^2$ into the following sets:

- $\partial\bar{\mathbb{D}}^2 = \mathbb{S}^1 = \{(x, y) \in \bar{\mathbb{D}}^2 : x^2 + y^2 = 1\}$
- $\{(x, y)\}$ for each $(x, y) \in \mathbb{D}^2$.

Let $\sim$ denote the equivalence relation defined by the above partition. Prove that $\bar{\mathbb{D}}^2/\sim$ is homeomorphic to $\mathbb{S}^2$. (Note: (1) $\bar{\mathbb{D}}^2/\sim$ is usually denoted by $\bar{\mathbb{D}}^2/\mathbb{S}^1$. (2) More generally, a similar proof yields $\bar{\mathbb{D}}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$.)

Exercise 9 (*)**.** Identify the closed 2-disk $\bar{\mathbb{D}}^2$ with the closed unit ball in $\mathbb{R}^2$. Partition $\bar{\mathbb{D}}^2$ into the following sets:

- $\{(x, y), (-x, -y)\}$ for $(x, y) \in \partial\bar{\mathbb{D}}^2 = \mathbb{S}^1 = \{(x, y) \in \bar{\mathbb{D}}^2 : x^2 + y^2 = 1\}$
- $\{(x, y)\}$ for each $(x, y) \in \mathbb{D}^2$.

Let $\sim$ denote the equivalence relation defined by the above partition. Prove that $\bar{\mathbb{D}}^2/\sim$ is homeomorphic to $\mathbb{RP}^2$. (Note: more generally, you can make a similar claim for $\mathbb{D}^n$ and $\mathbb{RP}^n$.)

Exercise 10 (*)**.** Let $I = [0, 1]$ and partition I^2 as follows:

- $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$
- $\{(x, 0), (x, 1)\}$ for $x \in (0, 1) \subset \mathbb{R}$,
- $\{(0, y), (1, y)\}$ for $y \in (0, 1) \subset \mathbb{R}$,
- $\{(x, y)\}$ for $(x, y) \in (0, 1) \times (0, 1) \subset I^2$.

Let $\sim$ be the equivalence relation on I^2 induced by this partition. Set $Y = I^2/\sim$. Prove that $Y \cong \mathbb{S}^1 \times \mathbb{S}^1$.