Notes for the QC Pure Math
Summer Research Opporunity

1 Introduction

This is set of notes should be viewed as a "quick start guide" for the summer's project. Details are sparse, and the introduction of the content is unmotivated; however, everything here will be useful for the project.

We will spend much of our time thinking about ordinals, and in particular, thinking about them as topological spaces. We will first introduce the notion of an ordinal and then introduce some basics from point-set topology.

2 Ordinals

Ordinals are something you will probably not see in your math education. I did not see them, or really know much about them, until I needed to use them, so I am not expert. In fact, it has been helpful to me to write these notes. With that said, you will find poking around the Wikipedia entries around the various terms in this section surprisingly informative. Ordinals are defined in terms of ordered sets, so we begin with the definition of an ordered set.

Definition 2.1 (Ordered set) Let X be a set. A binary relation $\leq$ on X is:

(i) *reflexive* if $x \leq x$ for all $x \in X$.

(ii) *transitive* if $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$.

(iii) *antisymmetric* if $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

(iv) *strongly connected* if $x \leq y$ or $y \leq x$ for all $x, y \in X$.

(v) *well-founded* if every non-empty subset of X has a minimum, i.e., if $S \subset X$ is non-empty, then there exists $s \in S$ such that $s \leq t$ for all $t \in S$.

The pair binary relation $\leq$ is a:

- *preorder* if it satisfies (i) and (ii).
- *partial order* if it satisfies (i)–(iii), and the pair $(X, \leq)$ is called a *partially ordered set*, or *poset*.
- *total order* if it satisfies (i)–(iv), and the pair $(X, \leq)$ is called a *totally ordered set*.
- *well order* if it satisfies (i)–(v), and the pair $(X, \leq)$ is called a *well-ordered set*.

Everyone learns about orders starting in elementary school: everyone is familiar with the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ and their standard ordering, i.e., $1 < 2 < 3 < \cdots$. The *well-ordeing axiom* says that the natural numbers with this order is a well-ordered set. This axiom allows us to establish induction. In fact, it is possible to induct on any well-ordered set; this is called transfinite induction, and we will discuss it below.

Observe that the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, and the reals $\mathbb{R}$ with their standard orders are totally ordered, but not well-ordered. For an example of a partially ordered set, let S be any set, and let $\mathcal{P}(S)$ be the *power set* of S, i.e., the set of subsets of S. Set containment is a partial order on $\mathcal{P}(S)$.

Definition 2.2 Two partially ordered sets $(X, \leq)$ and $(Y, \preceq)$ are *order isomorphic* if there exists a bijection $f\colon X \to Y$ such that $x \leq y$ if and only if $f(x) \preceq f(y)$; the function f is called an *order isomorphism*.

Exercise 2.1 Prove that no two of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are order isomorphic.

The notion of order isomorphism gives an equivalence relation on the class of well-ordered sets (note: the collection of well-ordered sets is not itself a set, and so there are some issues with the notion of an equivalence relation, but it's safe for us to ignore these technicalities).

Definition 2.3 An *ordinal*, or *order type*, is an equivalence class of the order isomorphism equivalence relation on the class of well-ordered sets.

That is an easy definition, but it is not so useful. It is a fact that given two order isomorphic well-ordered sets there exists a unique order isomorphism between them (this is not so hard once you understand transfinite induction). It therefore makes sense to try to construct canonical representatives for a given ordinal.

Let $(X, \leq)$ be a well-ordered set. Given $x \in X$, let $I_x = \{y \in X : y < x\}$. Let $\Lambda(X) = \{I_x : x \in X\}$, and observe that $(\Lambda(X), \subseteq)$ is a well-ordered set; in particular, $I_x \subseteq I_y$ if and only if $x \leq y$. Therefore, the function $f\colon X \to \Lambda(X)$ given by $f(x) = I_x$ is an order isomorphism. This tells us that every ordinal can be represented by a well-ordered $(S, \leq)$ such that every element of s of S satisfies $s = \{x \in S : x < s\}$. This says that every well-ordered set is isomorphic to a transitive set:

Definition 2.4 (Transitive set) A well-ordered set X is *transitive* if every element of X is also a subset of X.

This seems very abstract, so let us bring things down to earth a bit, but first we state a theorem. We can order the ordinals as follows: if α and β are ordinals, we write $\alpha \leq \beta$ if there exist $A \in \alpha$ and $B \in \beta$ such that A is order isomorphic to an element of $\Lambda(B)$, that is, there exists $b \in B$ such that A is order isomorphic to $\{x \in B : x < b\}$.

Theorem 2.5 *Every set of ordinals is well-ordered.* □

Intuitively, the above theorem says that the "set of ordinals" is well-ordered. The issue is that the "set of ordinals" is not a set! It suffers from the same type of paradox that plagued Bertrand Russell: *is the set of all sets a set?*

Given the discussion so far and the theorem above, we give a canonical representative of an ordinal, which is due to von Neumann (from when he was 19!).

Definition 2.6 (von Neuman ordinal) Given an ordinal α, consider the set

$$\{\beta : \beta \text{ is an ordinal and } \beta < \alpha\},$$

which is well-ordered and has order type α. From here on out, we identify the ordinal α with the set of all ordinals less than α, which is sometimes called a *von Neumann ordinal*.

Well, maybe this all still seems hopelessly abstract. But, let us look at some of the first ordinals and get a sense of what is going on. In particular, we mentioned that $\mathbb{N}$ is well ordered, so let us figure out its order type, or rather, let us define the natural numbers! There is a unique well-ordered set with no elements: the empty set $\varnothing$. We define 0 to be the empty set, that is, $0 = \varnothing$. Then $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc. It is not hard to see that every finite well-ordered set must be order isomorphic to one of the well-ordered sets in this list, so with this notation, we can define the natural numbers $\mathbb{N}$ to be the non-empty finite ordinals (I tend not to count 0 as a natural number).

With this definition, the natural numbers are well-ordered set, and we let ω denote their order type, in which case, using the notation above, $\omega = \{0\} \cup \mathbb{N}$. The ordinal ω is first infinite ordinal. Also, other than being infinite, it is fundamentally different than the natural numbers, as it does not have an immediate predecessor in the order. To make this statement more precise and to build a better understanding of ordinals, we need to introduce ordinal arithmetic.

Definition 2.7 (Ordinal addition) Let $(X, \leq)$ and $(Y, \preceq)$ be well-ordered sets. We write $X + Y$ denote the well-ordered set $(X \sqcup Y, \leq_+)$ where $\leq_+$ is the binary relation defined as follows: $z_1 \leq_+ z_2$ if and only if either

- $z_1, z_2 \in X$ and $z_1 \leq z_2$,
- $z_1, z_2 \in Y$ and $z_1 \preceq z_2$, or
- $z_1 \in X$ and $z_2 \in Y$.

I picture this operation as follows: imagine the elements of X in a line (according to its order), then on the line add a copy of Y (according to its order) to the right of X. Then $X + Y$ is the union of this copy of X and this copy of Y ordered as you see them on the line. That is, $X + Y$ is the concatenation of X and Y (with Y following after X). I should note that not every well-ordered set can be embedded in the real line, but the I think the visual is useful.

There is a very important example of ordinal addition: adding 1.

Definition 2.8 (Successor ordinal) The *successor* of an ordinal α is the ordinal $\alpha + 1$. An ordinal α is a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$.

If we let $\alpha = [0, \alpha) = \{\beta : \beta < \alpha\}$, then $\alpha + 1 = [0, \alpha] = \{\beta : \beta \leq \alpha\}$. This notion of successor defines an operation on ordinals: $\text{succ}(\alpha) = \alpha + 1$. We can use this operation to define the standard addition on the natural numbers. We will not discuss this in depth, but this leads one to the Peano axioms.

Exercise 2.2 Prove that ω is not a successor ordinal.

Exercise 2.3 (Addition is not commutative) Prove that $1 + \omega$ is order isomorphic to ω. Use this to conclude that $1 + \omega$ is not order isomorphic to $\omega + 1$, and hence, addition of ordinals is not commutative.

Exercise 2.4 Give an explicit subset of $\mathbb{R}$ order isomorphic to $\omega + 1$. (Hint: think about convergent sequences.)

Let us now consider multiplication.

Definition 2.9 (Multiplication of ordinals) Let $(X, \leq)$ and $(Y, \preceq)$ be well-ordered sets. Then the product of X and Y, denoted $X \cdot Y$, is the well-ordered set $(X \times Y, \leq_\times)$, where $\leq_\times$ is the binary relation given by $(x_1, y_1) \leq_\times (x_2, y_2)$ if and only if either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \preceq y_2$. The order $\leq_\times$ is known as the *lexicographical order*.

It can be checked that if $n \in \omega$ and X is a well-ordered set, then $X \cdot n$ is order isomorphic to to X added to itself n times, e.g., $X \cdot 2$ is order isomorphic to $X + X$. Like addition, multiplication is not commutative: for example, $2 \cdot \omega$ is order isomorphic to ω, but $\omega \cdot 2$ is not (since not every element of $\omega \cdot 2$ is a successor).

Exercise 2.5 Come up with a way to visualize ω^2 and $\omega^2 + 1$, where $\omega^2 = \omega \cdot \omega$.

Next, we describe a recursive definition for multiplication, which we can natural build on to define exponentiation. As always, we need some more setup.

Definition 2.10 (Limit ordinal) A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

From the exercise above, ω is the first limit ordinal.

Exercise 2.6 Let S be a set of ordinals. Define the *supremum* of S, denoted $\sup(S)$, by $\sup(S) = \bigcup_{\alpha \in S} \alpha$. Prove that $\sup(S)$ is an ordinal, i.e., a transitive well-ordered set, and that it is the least ordinal containing S.

We can now describe ordinal multiplication via a transfinite recursion. Let α and β be ordinals, we define $\alpha \cdot \beta$ via a transfinite induction on β, as follows:

- $\alpha \cdot 0 = 0$.

- if $\beta = \delta + 1$ is a successor ordinal, then $\alpha \cdot \beta = \alpha \cdot \delta + \alpha$.
- if β is a limit ordinal, then $\alpha \cdot \beta = \bigcup_{\delta \in \beta} \alpha \cdot \delta$.

For example, $\omega^2 = \omega \cdot \omega = \bigcup_{n \in \omega} \omega \cdot n$. Does this lead to a different visualization of ω^2 and $\omega^2 + 1$ than you had before? Can you visualize $\omega^3 + 1$? What about $\omega^4 + 1$?

Definition 2.11 (Ordinal exponentiation) Let α and β be ordinals. The exponentiation of α by β, denoted α^β, is defined via transfinite induction as follows:

- $\alpha^0 = 1$.
- if $\beta = \delta + 1$ is a successor ordinal, then $\alpha^\beta = \alpha^\delta \cdot \alpha$.
- if β is a limit ordinal, then $\alpha^\beta = \bigcup_{\delta \in \beta} \alpha^\delta$.

You were asked to visualize $\omega^n + 1$ for $n \in \omega$ above, so to understand $\omega^\omega + 1$, you have to take the union of all these images in your mind.

Exercise 2.7 Try to build a mental model for $\omega^\omega + 1$.

Understanding exponentiation of ω is enough to understand all ordinals.

Theorem 2.12 (Cantor normal form) *If β is an ordinal, then there exists $k \in \omega$ and ordinals $\alpha_1, \ldots, \alpha_k$ and finite ordinals $n_1, \ldots, n_k \in \omega$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and*

$$\beta = \sum_{i=1}^{k} \omega^{\alpha_i} \cdot n_i.$$

Moreover, this representation is unique. □

Above we used transfinite induction/recursion. Let me give a formal statement and proof.

Theorem 2.13 (Transfinite induction) *Let $P(\alpha)$ be a property defined for all ordinals α. If $P(\alpha)$ is true whenever $P(\beta)$ is true for all $\beta < \alpha$, then P is true for all ordinals.*

Proof Suppose there P is not true for all ordinals. Then there exists a least ordinal α for which $P(\alpha)$ is false. It follows that $P(\beta)$ is true for all $\beta < \alpha$, and hence, by the hypothesis, $P(\alpha)$ is true, a contradiction. □

To satisfy a property P satisfies the hypothesis in the above theorem, we usually break the argument into three cases:

(i) $P(0)$ is true,

(ii) if $\beta < \alpha$, then $P(\beta)$ true implies $P(\beta + 1)$ true, and

(iii) if $\lambda \leq \alpha$ is a limit ordinal, then if $P(\beta)$ is true for all $\beta < \lambda$ then $P(\lambda)$ is true.