Isomorphisms of Non-Standard Fields and Ash's Conjecture

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Abstract. Cohesive sets play an important role in computability theory. Here we use cohesive sets to build nonstandard versions of the rationals. We use Koenigsmann's work on Hilbert's Tenth Problem to establish that these nonstandard fields are rigid. As a consequence we obtain results about automorphisms of the lattices of computably enumerable vector spaces arising in the context of Ash's conjecture.

1 Introduction

This paper is motivated by the 30-year open problem of finding automorphisms of the lattice $\mathcal{L}^*(V_\infty)$. As in Metakides and Nerode [16], the space V_∞ is the canonical computable $\aleph_0$-dimensional vector space over a computable field F. The lattice of computably enumerable (c.e.) subspaces of V_∞ is denoted by $\mathcal{L}(V_\infty)$. The lattice $\mathcal{L}(V_\infty)$ modulo finite dimension is denoted by $\mathcal{L}^*(V_\infty)$. Guichard [7] established that there are countably many automorphisms of $\mathcal{L}(V_\infty)$ because they are generated by computable semilinear transformations. Ash conjectured that the automorphisms of $\mathcal{L}^*(V_\infty)$ are generated by special computable semilinear transformations.

Definition 1. *An automorphism of $\mathcal{L}^*(V_\infty)$ is called an Ash automorphism if it is generated by a semilinear transformation with finite dimensional kernel and co-finite dimensional image in V_∞.*

Conjecture 1. (Ash) Every automorphism of $\mathcal{L}^*(V_\infty)$ is an Ash automorphism.

Definition 2. *(1) An infinite set $C \subset \omega$ is* cohesive *if for every c.e. set W either $W \cap C$ or $\overline{W} \cap C$ is finite.*

(2) A set M is maximal *if M is c.e. and $\overline{M}$ is cohesive.*

(3) A set B is quasimaximal *if it is the intersection of finitely many maximal sets.*

For sets A and B we use $A =^* B$ to denote that A and B differ on at most finitely many elements, and $A \subset_* B$ to denote that all but finitely many elements of A are also elements of B. For vector spaces we use the same notation where "finitely many elements" is replaced by "finite dimension." Let A be a computable basis of V_∞ and let B be a quasimaximal subset of A. Let $\mathcal{E}^*(B,\uparrow)$ denote the principal filter of B in the lattice $\mathcal{E}^*$ of c.e. sets modulo $=^*$. It is known that $\mathcal{E}^*(B,\uparrow)$ is isomorphic to a finite Boolean algebra $\mathbf{B_n}$. Let $V = cl(B)$ be the closure of B in V_∞. In contrast to $\mathcal{E}^*(B,\uparrow)$, the principal filter of V in $\mathcal{L}^*(V_\infty)$, $\mathcal{L}^*(V,\uparrow)$, is not always isomorphic to $\mathbf{B_n}$. Rather, as shown in [2] and [3], these filters are isomorphic to either:

(1) a finite Boolean algebra,

(2) a lattice of subspaces of an n-dimensional vector space W over a certain extension of F (denoted by $L(n, \widetilde{F})$), or

(3) a finite product of structures from the previous two cases.

The extension $\widetilde{F}$ of F mentioned in (2), which is denoted by $\prod\limits_C F$, is called the cohesive power of F, and is defined below. In the context of computable vector spaces the main interesting cases occur when F is finite or $F = \mathbb{Q}$. For finite F we have $\prod\limits_C F \cong F$. The first key result in this paper is that $\prod\limits_{\overline{M_1}} \mathbb{Q} \cong \prod\limits_{\overline{M_2}} \mathbb{Q}$ iff the maximal sets M_1 and M_2 are of the same 1-degree up to finitely many elements. This result implies the following theorem when $F = \mathbb{Q}$.

Theorem 1. *(i) The principal filters $\mathcal{L}^*(V,\uparrow)$ of type (2) fall into infinitely many non-isomorphic classes even when these filters are isomorphic to lattices of subspaces of finite dimensional vector spaces of the same dimension (≥ 3).*

(ii) Every automorphism of $\mathcal{L}^*(V_\infty)$ preserves m-degrees of the spaces in (2).

A bijective semilinear map Φ on a vector space W over a field F is defined by

$$\Phi(av + bw) = f(a)\Phi(v) + f(b)\Phi(w), \tag{1}$$

where f is an automorphism of F. Such bijective semilinear map Φ on the space W in (2) above generates an automorphism τ_Φ of $L(n, \widetilde{F})$. Moreover, by the fundamental theorem of projective geometry, all automorphisms of $L(n, \widetilde{F})$ for $n \geq 3$ are generated by such semilinear maps. By (2) above we can regard τ_Φ as an automorphism of $\mathcal{L}^*(V,\uparrow)$. When Φ is merely semilinear, τ_Φ is not the restriction of any Ash automorphism to $\mathcal{L}^*(V,\uparrow)$. When Φ is linear, τ_Φ has a natural extension $\overline{\tau_\Phi}$ to an automorphism of $\mathcal{L}^*(V_\infty)$ as described in the construction in the proof of Theorem 2.1 in [5]. This $\overline{\tau_\Phi}$ is an Ash automorphism of $\mathcal{L}^*(V_\infty)$. In certain cases we may hope to generalize this construction in the case when Φ is merely semilinear and thereby generating a non-Ash automorphism. However, our second key result is that $\prod\limits_C F$ has only the trivial automorphism when $F = \mathbb{Q}$ and we can establish the following results.

Theorem 2. *(i) Any automorphism of $\mathcal{L}^*(V_\infty)$ of the form $\overline{\tau_\Phi}$ for every bijective semilinear map Φ is an Ash automorphism.*

(ii) Any automorphism of $\mathcal{L}^*(V, \uparrow)$, where $\mathcal{L}^*(V, \uparrow)$ is of type (2) and $n \geq 3$, can be extended to an automorphism of $\mathcal{L}^*(V_\infty)$.

The result that the cohesive power $\prod_{\overline{M}} \mathbb{Q}$ is rigid, and its proof are of independent interest. Our proof uses a recent number theoretic result by Koenigsmann on Hilbert's Tenth Problem [11], which allows us to apply work in nonstandard models of arithmetic to our problem.

In Section 2, we define a cohesive power of a computable structure $\mathcal{A}$ over a cohesive set C of natural numbers, $\prod_C \mathcal{A}$. We give a natural way of embedding $\prod_C \mathbb{N}$ into $\prod_C \mathbb{Q}$. In Section 3, we prove that $\prod_C \mathbb{N}$ is definable in both $\prod_C \mathbb{Z}$ and $\prod_C \mathbb{Q}$. The main result in this section implies that if M_1 and M_2 are maximal sets of natural numbers, then $\prod_{\overline{M_1}} \mathbb{Q} \cong \prod_{\overline{M_2}} \mathbb{Q}$ iff $M_1 \equiv_1^* M_2$. Finally, in Section 4, we prove that if C is a a co-maximal (hence co-c.e.) set, then $\prod_C \mathbb{Q}$ is rigid.

2 Effective Ultraproducts and Isomorphisms

Homomorphic images of the semiring of recursive functions have been studied as models of fragments of arithmetic in [6], [8], and [12]. Let C be an r-cohesive set. Fefferman, Scott, and Tannenbaum considered the quotient structure $\mathcal{R}/\sim_C$ where $\mathcal{R}$ is the set of all unary (totally) computable functions and $\sim_C$ is the equivalence relation on $\mathcal{R}$ defined by:

$$f \sim_C g \Leftrightarrow C \subseteq^* \{n \in \omega \mid f(n) = g(n)\}. \tag{2}$$

They proved that there is a specific Π_3^0 sentence σ such that $\mathbb{N} \models \sigma$ but $\mathcal{R}/\sim_C \not\models \sigma$ (see Theorem 2.1 in [12]). Lerman [12] further proved that if $R_1 \equiv_m R_2$ are r-maximal sets, then $\mathcal{R}/\sim_{\overline{R_1}} \cong \mathcal{R}/\sim_{\overline{R_2}}$. Moreover, Corollary 2.4 in [12] states that if M_1 and M_2 are maximal sets of different m-degrees, then $\mathcal{R}/\sim_{\overline{M_1}}$ and $\mathcal{R}/\sim_{\overline{M_2}}$ are not even elementary equivalent. These models of fragments of arithmetic have been further studied by Hirschfeld, Wheeler, and McLaughlin in [8], [9], [13], [14], and [15], and are special cases of what we call cohesive powers. The cohesive powers of fields, which were used in [3] to characterize the principal filters of quasimaximal spaces, motivated the following general definition in [4]. As usual, we will denote the equality of partial functions by $\simeq$.

Definition 3. *Let $\mathcal{A}$ be a computable structure for a computable language L and with domain A, and let $C \subset \omega$ be a cohesive set. The* cohesive power of $\mathcal{A}$ over C*, denoted by $\prod_C \mathcal{A}$, is a structure $\mathcal{B}$ for L with domain B such that the following holds.*

1. *The set $B = (D/=_C)$, where $D = \{\varphi \mid \varphi : \omega \to A$ is a partial computable function, and $C \subseteq^* dom(\varphi)\}$.*

For $\varphi_1, \varphi_2 \in D$, we have $\varphi_1 =_C \varphi_2$ iff $C \subseteq^* \{x : \varphi_1(x) \downarrow= \varphi_2(x) \downarrow\}$.
The equivalence class of φ with respect to $=_C$ will be denoted by $[\varphi]_C$, or simply by $[\varphi]$ (when the reference to C is clear from the context).

2. *If $f \in L$ is an n-ary function symbol, then $f^{\mathcal{B}}$ is an n-ary function on B such that for every $[\varphi_1], \ldots, [\varphi_n] \in B$, we have $f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi]$, where for every $x \in A$,*
$$\varphi(x) \simeq f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)). \tag{3}$$

3. *If $P \in L$ is an m-ary predicate symbol, then $P^{\mathcal{B}}$ is an m-ary relation on B such that for every $[\varphi_1], \ldots, [\varphi_m] \in B$,*
$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad iff \quad C \subseteq^* \{x \in A \mid P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}. \tag{4}$$

4. *If $c \in L$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the (total) computable function on A with constant value $c^{\mathcal{A}}$.*

Remark 1. Let C and B be as in Definition 3.
(i) The requirement that C is cohesive can be weakened to C being r-cohesive.
(ii) If C is co-c.e., then for every $[\varphi] \in B$ there is a computable function f such that $f =_C \varphi$. In this case the structures $\prod_C \mathbb{N}$ and $\mathcal{R}/\sim_C$ are isomorphic.

Versions of restricted Łoś's theorem were given in [13], [14] for models of fragments of arithmetic. The version of Łoś's theorem for cohesive powers of computable structures was given in [4] and was called the fundamental theorem of cohesive powers. Here is a part of the theorem that we will use in the proof of Proposition 1.

Theorem 3. *[4] If $\Phi(y_1, \ldots, y_n)$ is a formula in L that is a Boolean combination of Σ_1^0 and Π_1^0 formulas and $[\varphi_1], \ldots, [\varphi_n] \in B$, then*
$$\prod_C \mathcal{A} \models \Phi([\varphi_1], \ldots, [\varphi_n]) \quad iff \quad C \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}. \tag{5}$$

The structure $\prod_C \mathbb{N}$ can be embedded naturally into $\prod_C \mathbb{Q}$ by mapping the equivalence class of $[\varphi] \in \prod_C \mathbb{N}$ to the larger equivalence class of the same $[\varphi]$ in $\prod_C \mathbb{Q}$. With the following general approach we can also obtain $\prod_C \mathbb{Q}$ from $\prod_C \mathbb{N}$.

Definition 4. *Let $\mathcal{M}_1$ be a structure for the language $L = \{+, \cdot, 0, 1\}$, which satisfies the commutative semiring axioms.*

Let $\mathcal{M}_2$ be a ring with domain $(M_1 \times M_1)_{\equiv^+}$ where $(a_1, b_1) \equiv^+ (a_2, b_2)$ iff $a_1 + b_2 = b_1 + a_2$. Suppose that the natural definition of the ring operations of $\mathcal{M}_2$ is such that $\mathcal{M}_2$ is an integral domain.

Let $\mathcal{M}_3$ be a field with domain $(M_2 \times M_2)_{\equiv^\cdot}$ where $(a_1, b_1) \equiv^\cdot (a_2, b_2)$ iff $a_1 \cdot b_2 = b_1 \cdot a_2$ and the field (of quotients) operations of $\mathcal{M}_3$ are naturally defined.

Remark 2. Let $\mathcal{M}_1$ be as in Definition 4. If $\mathcal{M}_1 = \prod_C \mathbb{N}$, then $\mathcal{M}_2 \cong \prod_C \mathbb{Z}$ and $\mathcal{M}_3 \cong \prod_C \mathbb{Q}$. The natural embedding of $\mathcal{M}_1$ into $\mathcal{M}_3$ yields the natural embedding of $\prod_C \mathbb{N}$ into $\prod_C \mathbb{Q}$ mentioned earlier.

Lemma 1. *(i) Any automorphism of $\mathcal{M}_i$ induces an automorphism of $\mathcal{M}_j$ for $i < j \leq 3$.*

(ii) If $\mathcal{M}_i$ is definable in $\mathcal{M}_j$ for $i < j \leq 3$, then any automorphism of $\mathcal{M}_j$ induces an automorphism of $\mathcal{M}_i$.

(iii) If $\mathcal{M}_j$ is rigid, then so is $\mathcal{M}_i$ for $i < j \leq 3$.

(iv) If $\mathcal{M}_i$ is definable in $\mathcal{M}_j$ for $i < j \leq 3$ and $\mathcal{M}_i$ is rigid, then so is $\mathcal{M}_j$.

Proof. We will prove only (ii) and (iv) and leave the rest of the theorem to the reader.

(ii) Let Γ be an automorphism of $\mathcal{M}_3$. To define an automorphism $\Gamma_1 : \mathcal{M}_2 \to \mathcal{M}_2$, let $a \in M_2$. The natural embedding of $\mathcal{M}_2$ into $\mathcal{M}_3$ maps a to $[(a,1)]_{\equiv^{\cdot}}$. Let ϕ be a first-order formula in L that defines the set $\{ [(x,1)]_{\equiv^{\cdot}} \mid x \in M_2\}$ in $\mathcal{M}_3$. Then $\mathcal{M}_3 \models \phi([(a,1)]_{\equiv^{\cdot}})$, so $\mathcal{M}_3 \models \phi(\Gamma([(a,1)]_{\equiv^{\cdot}}))$. Then $\Gamma([(a,1)]_{\equiv^{\cdot}}) = [(c,1)]_{\equiv^{\cdot}}$ for a unique $c \in M_2$. Let $\Gamma_1(a) = c$. The proofs for the other cases are similar.

(iv) We will only prove that $\mathcal{M}_3$ is rigid provided that $\mathcal{M}_2$ is rigid. Suppose that Γ is an automorphism of $\mathcal{M}_3$. Since M_2 is first-order definable in $\mathcal{M}_3$, Γ_1 defined in (ii) is an automorphism of $\mathcal{M}_2$. Let $a \in M_3$, and let $b_1, b_2 \in M_2$ be such that $a = [(b_1, b_2)]_{\equiv^{\cdot}}$. Then

$$\Gamma(a) = \Gamma((b_1,1)) \cdot \Gamma((1,b_2)) = [(\Gamma_1(b_1), \Gamma_1(b_2))]_{\equiv^{\cdot}} = [(b_1,b_2)]_{\equiv^{\cdot}} = a \quad (6)$$

because $\mathcal{M}_2$ is rigid.

Remark 3. Note that if $\mathcal{M}_1$ is rigid in language L and a relation R is definable in $\mathcal{M}_1$, then $\mathcal{M}_1$ is rigid in $L \cup \{R\}$. We will later use this fact when the relation R is $<$.

3 Definability and Isomorphisms

We will now prove that $\prod_C \mathbb{N}$ is definable both in $\prod_C \mathbb{Z}$ and $\prod_C \mathbb{Q}$. The definability of $\mathbb{Z}$ in $\mathbb{Q}$ (by a Π^0_3 formula) has been established by J. Robinson in [10]. More recently, Koenigsmann [11] gave a Π^0_1 definition of $\mathbb{Z}$ in $\mathbb{Q}$. He proved that there is a positive integer n and a polynomial $p \in \mathbb{Z}[y, z_1, \ldots, z_n]$ such that

$$y \in \mathbb{Z} \iff \forall z_1 \ldots \forall z_n [p(y, z_1, \ldots, z_n) \neq 0]. \quad (7)$$

We note that the intended range of all quantified variables in formulas (7) through (10) is $\mathbb{Q}$. The proof of Proposition 1 below essentially uses the Koenigsmann's definition and cannot work with a definition of higher complexity.

The definability of $\mathbb{N}$ in $\mathbb{Z}$ (by various Σ_1^0 formulas) has been established by R. Robinson in [17]. We will use the formula that defines the natural numbers as sums of the squares of four integers. Using these results we obtain that $\mathbb{N}$ can be defined in $\mathbb{Q}$ as follows:

$$x \in \mathbb{N} \Leftrightarrow \exists y_1 \dots \exists y_4 [\bigwedge_{i \leq 4} y_i \in \mathbb{Z} \wedge x = y_1^2 + y_2^2 + y_3^2 + y_4^2] \tag{8}$$

$$x \in \mathbb{N} \Leftrightarrow \exists y_1 \dots \exists y_4 \forall z_1 \dots \forall z_n [\bigwedge_{i \leq 4} p(y_i, z_1, \dots, z_n) \neq 0 \wedge x = y_1^2 + y_2^2 + y_3^2 + y_4^2], \tag{9}$$

which we will abbreviate as

$$x \in \mathbb{N} \Leftrightarrow \exists \overline{y} \forall \overline{z} \theta(x, \overline{y}, \overline{z}) \tag{10}$$

where $\phi(x, \overline{y}, \overline{z})$ is a quantifier-free formula in the language of rings $L = \{+, \cdot, 0, 1\}$. Note that there is a natural embedding of $\prod_C \mathbb{N}$ into $\prod_C \mathbb{Z}$, and of $\prod_C \mathbb{Z}$ into $\prod_C \mathbb{Q}$.

Proposition 1. *The natural embedding of* $\prod_C \mathbb{N}$ *is definable in* $\prod_C \mathbb{Q}$ *by the same formula* $\exists \overline{y} \forall \overline{z} \theta(x, \overline{y}, \overline{z})$ *that defines* $\mathbb{N}$ *in* $\mathbb{Q}$.

Proof. First, assume that for some $[\varphi] \in \prod_C \mathbb{Q}$ we have

$$\prod_C \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta([\varphi], \overline{y}, \overline{z}) \tag{11}$$

and that $y_i = [\psi_i]$ are such that

$$\prod_C \mathbb{Q} \models \forall \overline{z} \theta([\varphi], \overline{[\psi_i]}, \overline{z}). \tag{12}$$

By Theorem 3, we have that:

$$C \subseteq^* \{n : \mathbb{Q} \models \forall \overline{z} \theta(\varphi(n), \overline{\psi_i(n)}, \overline{z}\}. \tag{13}$$

Using the definition of $\theta(x, \overline{y}, \overline{z})$ we immediately obtain that $C \subseteq^* \{n : \varphi(n) \in \omega\}$, which means that $[\varphi] \in \prod_C \mathbb{N}$.

Now, assume that $[\varphi] \in \prod_C \mathbb{N}$. We will prove that

$$\prod_C \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta([\varphi], \overline{y}, \overline{z}). \tag{14}$$

Define the partial computable functions $\xi_i : \omega \to \mathbb{Q}$ $(i \leq 4)$ as follows.

If at stage s we have $\varphi^s(n) = m$ and $m \in \omega$, then find the least $(b_1, \dots, b_4) \in \omega^4$ such that $m = \sum_{i=1}^{4} b_i^2$ and let $\xi_i(n) = b_i$.

By the definition of the functions y_i, we have that

$$C \subseteq^* \{n : \mathbb{Q} \models [\bigwedge_{i\leq 4}(\xi_i(n) \in \mathbb{Z}) \wedge \varphi(n) = \sum_{i\leq 4}\xi_i(n)^2]\}. \tag{15}$$

Again, by Theorem 3, we obtain that

$$\prod_C \mathbb{Q} \models \forall\overline{z}\theta([\varphi], [y_1], \ldots, [y_4], \overline{z}), \tag{16}$$

which implies that

$$\prod_C \mathbb{Q} \models \exists\overline{y}\forall\overline{z}\theta([\varphi], \overline{y}, \overline{z}). \tag{17}$$

For convenience we will introduce additional notation. Let
(1) $\varphi_1(x) =_{def} \exists y_1 \ldots \exists y_4[x = y_1^2 + y_2^2 + y_3^2 + y_4^2]$, and
(2) $\varphi_2(x) =_{def} \forall z_1 \ldots \forall z_n[p(x, z_1, \ldots, z_n) \neq 0]$.

Definition 5. *Let $\varphi(\overline{x})$ be a formula in a prenex normal form. Define $\varphi^*(\overline{x})$ inductively as follows:*
(1) $\varphi^*(\overline{x}) =_{def} \varphi(\overline{x})$ if φ is a quantifier-free formula,
(2) $\varphi^*(\overline{x}) =_{def} \exists y[\varphi_2(y) \wedge \psi^*(\overline{x}, y)]$ if $\varphi(\overline{x}) = \exists y\psi(\overline{x}, y)$,
(3) $\varphi^*(\overline{x}) =_{def} \forall y[\varphi_2(y) \Rightarrow \psi^*(\overline{x}, y)]$ if $\varphi(\overline{x}) = \forall y\psi(\overline{x}, y)$.

Note that in this case: $x \in N \Leftrightarrow \varphi_1^*(x) \Leftrightarrow \exists\overline{y}\forall\overline{z}\theta(x, \overline{y}, \overline{z})$.

Definition 6. *Let $\varphi(\overline{x})$ be a formula in a prenex normal form. Define $\varphi^{\dagger}(\overline{x})$ inductively as follows:*
(1) $\varphi^{\dagger}(\overline{x}) =_{def} \varphi(\overline{x})$ if φ is a quantifier-free formula
(2) $\varphi^{\dagger}(\overline{x}) =_{def} \exists y[\varphi_1^*(y) \wedge \psi^{\dagger}(\overline{x}, y)]$ if $\varphi(\overline{x}) = \exists y\psi(\overline{x}, y)$,
(3) $\varphi^{\dagger}(\overline{x}) =_{def} \forall y[\varphi_1^*(y) \Rightarrow \psi^{\dagger}(\overline{x}, y)]$ if $\varphi(\overline{x}) = \forall y\psi(\overline{x}, y)$.

The idea for this definition is that $\phi^{\dagger}(\overline{x})$ essentially expresses the formula $\phi(\overline{x})$ with the scope of its quantifiers limited from $\mathbb{Q}$ to $\mathbb{N}$ (and from $\prod_C \mathbb{Q}$ to $\prod_C \mathbb{N}$ because of Proposition 1).

Proposition 2. *$\prod_C \mathbb{Q}$ and $\mathbb{Q}$ are not elementary equivalent.*

Proof. Let T be Kleene's predicate. By a result by Fefferman, Scott, and Tannenbaum (see Theorem 2.1 in [12]), we know that for

$$\phi = \forall x\exists t\forall e\forall z[(e < x \wedge T(e, x, z)) \Rightarrow z < t], \tag{18}$$

$$\mathbb{N} \models \phi \quad \textit{but} \quad \prod_C \mathbb{N} \not\models \phi. \tag{19}$$

We can assume that ϕ is a sentence in $L = (+, \cdot, 0, 1)$ since both Kleene's T predicate and $<$ are definable in $\mathbb{N}$. Then $\mathbb{Q} \models \phi^{\dagger}$ iff $\mathbb{N} \models \phi$, and $\prod_C \mathbb{Q} \not\models \phi^{\dagger}$ iff $\prod_C \mathbb{N} \not\models \phi$. This finally gives us that

$$\mathbb{Q} \not\equiv \prod_C \mathbb{Q}. \tag{20}$$

Definition 7. *([4]) The sets $A \subseteq \omega$ and $B \subseteq \omega$ have the same 1-degree up to $=^*$ (denoted by $A \equiv_1^* B$) if there are $C =^* A$ and $D =^* B$ such that $C \equiv_1 D$.*

Remark 4. Using Myhill's Isomorphism Theorem (see [18] p. 24),we conclude that $A \equiv_1^* B$ iff there is a computable permutation σ of ω such that $\sigma(A) =^* B$.

Proposition 3. *Let $M_1 \subseteq \omega$ and $M_2 \subseteq \omega$ be maximal sets.*

1. *If $M_1 \equiv_1^* M_2$, then $\prod_{\overline{M_1}} \mathbb{Q} \cong \prod_{\overline{M_2}} \mathbb{Q}$.*
2. *If $M_1 \not\equiv_1^* M_2$, then $\prod_{\overline{M_1}} \mathbb{Q} \not\equiv \prod_{\overline{M_2}} \mathbb{Q}$.*

Proof. (1) This fact has been proven in [4] for an arbitrary computable structure $\mathcal{A}$. If σ is a computable permutation of ω such that $\sigma(M_1) =^* M_2$, then the map $\Phi : \prod_{\overline{M_1}} \mathcal{A} \to \prod_{\overline{M_2}} \mathcal{A}$ such that $\Phi([\psi]) = [\psi \circ \sigma]$ is an isomorphism.

(2) Note that for maximal sets we have $M_1 \equiv_1^* M_2$ iff $M_1 =_m M_2$. For the proof in the nontrivial direction, assume that $M_1 \leq_m M_2$ via f and $M_2 \leq_m M_1$ via g. Since $\overline{M_1}$ is cohesive, $g \circ f(\overline{M_1}) \cap \overline{M_1}$ is infinite and, by Proposition 2.1 in [12], $g \circ f|_{\overline{M_1}}$ and $I|_{\overline{M_1}}$ differ only on finitely many elements. Then to define the computable permutation σ we enumerate M_1 and let

$$\sigma(n) = \begin{cases} n, & \text{if n is enumerated into } M_1 \text{ first;} \\ f(n), & \text{if} \quad g(f(n)) = n. \end{cases}$$

Note that $\sigma(n)$ will be defined for almost every $n \in \omega$ and let $\sigma(n) = n$ in the finitely many remaining cases.

If $M_1 \neq_m M_2$, then we apply Theorem 2.3 in [12]. In fact, Lerman provided a specific sentence θ (originally in the language $L_< = \{+, \cdot, 0, 1, <\}$) for which $\prod_{\overline{M_1}} \mathbb{N} \models \theta$ while $\prod_{\overline{M_2}} \mathbb{N} \models \neg\theta$. As before, we can assume that the sentence θ is equivalent to a sentence in the language L. Thus, for the relativisation $\theta^{\dagger}$, we have $\prod_{\overline{M_1}} \mathbb{Q} \models \theta^{\dagger}$, while $\prod_{\overline{M_2}} \mathbb{Q} \models \neg\theta^{\dagger}$.

4 Automorphisms

We now assume that C is a co-maximal (co-c.e. and cohesive) set and will prove that the field $\prod_C \mathbb{Q}$ is rigid (i.e., it has only the trivial automorphism). To do this we will show that $\prod_C \mathbb{N}$ is a special case of arithmetic (exactly Δ^0_1) ultrapowers studied by Hirschfeld, Wheeler, and McLaughlin. Specifically, they studied the structures $\mathcal{F}_n/U$, where U is a non-principal ultrafilter in the Boolean algebra of Δ^0_n sets and $\mathcal{F}_n$ is the set of all total functions with Σ^0_n graphs. In Theorem 2.11 of [14], McLaughlin proved that $\mathcal{F}_n/U$ is rigid for the language $L_< = \{+, \cdot, 0, 1, <\}$. To apply McLaughlin's result we need to make a few observations. First, clearly, the theorem also holds for the language $L = \{+, \cdot, 0, 1\}$ because of the definability of the relation "$<$". Second, we will see how the equivalence relation induced by the co-maximal set C is equivalent to the one induced by a Δ^0_1 ultrafilter. Finally, the domain of $\prod_C \mathbb{N}$ consists of partial computable functions, while the functions in $\mathcal{F}_1$ are total. The last two points are addressed in the following proposition.

Proposition 4. *(1)* $U_C = \{R \mid R \in \Delta^0_1$ *and* $C \subseteq^* R\}$ *is an ultrafilter in the Boolean algebra* Δ^0_1.
(2) $\prod_C \mathbb{N} \cong \mathcal{F}_1/U_C$

Proof. (1) It is straightforward to show that U_C is a filter. Since C is cohesive, we have $(\forall R \in \Delta^0_1)[C \subseteq^* R \vee C \subseteq^* \overline{R}]$ and therefore, U_C is maximal.

To prove (2) we will show that for every partial computable function φ for which $C \subseteq^* dom(\varphi)$, there is a computable function f_φ such that $[\varphi]_C = [f_\varphi]_C$.

We simply define

$$f_\varphi(n) = \begin{cases} \varphi(n), & \text{if } \varphi(n)\downarrow \text{ first;} \\ 0, & \text{if n is enumerated into } \overline{C} \text{ first.} \end{cases}$$

Obviously, $f_\varphi(n)$ is defined for all but finitely many n. For the finitely many n for which $f_\varphi(n)$ is not defined, we let $f_\varphi(n) = 0$. It is immediate that f_φ is computable and $[\varphi]_C = [f_\varphi]_C$. It is also immediate that if $[\varphi]_C = [\psi]_C$, then $A = \{n : f_\varphi(n) = f_\psi(n)\}$ is a computable set such that $C \subseteq^* A$ and so $A \in \mathcal{U}$ and $[f_\varphi] =_{\mathcal{U}} [f_\psi]$.

Then the map $\Phi : \prod_C \mathbb{N} \to \mathcal{F}_1/U_C$ given by $\Phi([\varphi]_C) = [f_\varphi]_{U_C}$ is an isomorphism.

Corollary 1. *The structure* $\prod_C \mathbb{N}$ *is rigid.*

Theorem 4. *The structure* $\prod_C \mathbb{Q}$ *is rigid.*

Proof. If $\mathcal{M}_1 = \prod_C \mathbb{N}$ and $\mathcal{M}_3 \cong \prod_C \mathbb{Q}$, then $\mathcal{M}_1$ is definable in $\mathcal{M}_3$ by Proposition 1, and $\mathcal{M}_1$ is rigid by Proposition 4. Then the rigidity of $\mathcal{M}_3$ follows from Lemma 1, part (4).

Corollary 2. *If co-maximal powers* $\prod_{\overline{M_1}} \mathbb{Q}$ *and* $\prod_{\overline{M_2}} \mathbb{Q}$ *are isomorphic, then there is a unique isomorphism between them.*

Proof. If f_1 and f_2 are isomorphisms that map $\prod_{M_1} \mathbb{Q}$ to $\prod_{M_2} \mathbb{Q}$, then $f_2^{-1} \circ f_1$ must be the identity automorphism of $\prod_{M_1} \mathbb{Q}$.

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