TURING DEGREE SPECTRA OF DIFFERENTIALLY CLOSED FIELDS

DAVID MARKER AND RUSSELL MILLER

Abstract. The degree spectrum of a countable structure is the set of all Turing degrees of presentations of that structure. We show that every nonlow Turing degree lies in the spectrum of some differentially closed field (of characteristic 0, with a single derivation) whose spectrum does not contain the computable degree $\mathbf{0}$. Indeed, this is an equivalence, for we also show that if this spectrum contained a low degree, then it would contain the degree $\mathbf{0}$. From these results we conclude that the spectra of differentially closed fields of characteristic 0 are exactly the jump-preimages of spectra of automorphically nontrivial graphs.

1. Introduction

Differential fields arose originally in work of Ritt examining algebraic differential equations on manifolds over the complex numbers. Subsequent work by Ritt, Kolchin and others brought this study into the realm of algebra, where numerous parallels appeared with algebraic geometry. The topic first intersected with model theory in the mid-twentieth century, in work of Abraham Robinson, and logicians soon discovered the theories of differential fields and of differentially closed fields to have properties which had been considered in the abstract, but had not previously been known to hold for any everyday theories in mathematics. It was the model theorists who provided the definitive resolution to the question of differential closure, several variations of which had previously been developed in differential algebra. In 1974, Harrington proved the existence of computable differentially closed fields, making the notion more concrete, although our grasp of this topic remains more tenuous than our understanding of algebraic closures in field theory.

In this article, we offer an analysis of the complexity of countable models of $\mathbf{DCF}_0$, the theory of differentially closed fields of characteristic 0. This work requires a solid

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background in differential algebra, in model theory, and in effective mathematics. Ultimately we will characterize the spectra of countable models of $\mathbf{DCF}_0$ as exactly the preimages, under the jump operation, of spectra of automorphically nontrivial countable graphs; or, equivalently, as exactly those spectra of such graphs which are closed under a simple equivalence relation on Turing degrees. To do so, we show that spectra of differentially closed fields have certain complexity properties, which are not known to hold of any other standard class of mathematical structures: every low differentially closed field of characteristic 0 is isomorphic to a computable one, whereas every nonlow degree computes a differentially closed field which has no computable copy. Indeed we will present a substantial class of fairly complex spectra that can all be realized by models of $\mathbf{DCF}_0$, including spectra with arbitrary proper α-th jump degrees, for every computable nonzero ordinal α. To explain what these results mean, we begin immediately with the necessary background. For supplemental information on computability theory, [29] is a standard source, while for more detail about model theory and differential fields, we suggest [15], [21], or the earlier [26].

1.1. **Background in Differential Algebra.** A differential ring is a ring with a *differential operator*, or *derivation*, on its elements. If the ring is a field, we call it a *differential field*. The differential operator δ is required to preserve addition and to satisfy the familiar *Leibniz Rule*: $\delta(x\cdot y) = (x\cdot\delta y)+(y\cdot\delta x)$. Examples include the field $\mathbb{Q}(x)$ of rational functions over $\mathbb{Q}$ in a single variable x, with the usual differentiation $\frac{d}{dx}$, or the field $\mathbb{Q}(t,\delta t,\delta^2 t,\ldots)$, with δ acting as suggested by the notation. In these examples, $\mathbb{Q}$ may be replaced by another differential field K, with the derivation δ on K likewise extended to all of $K(x)$ or $K(t,\delta t,\ldots)$. (The only possible derivation on $\mathbb{Q}$ maps all rationals to 0. In general, the *constants* of a differential field K are those $x\in K$ with $\delta x = 0$, and they form the *constant subfield* C_K of K.) We use angle brackets and write $K\langle y_i : i\in I\rangle$ for the smallest differential subfield (of a given extension of K) containing all the elements y_i; this is well-defined, and this subfield is said to be *generated* as a differential field by $\{y_i \mid i\in I\}$. Of course, the field generated by these same elements $\{y_i \mid i\in I\}$ may well be a proper subfield of this: in the examples above, $\mathbb{Q}\langle x\rangle = \mathbb{Q}(x)$, but $\mathbb{Q}(t)\subsetneq\mathbb{Q}\langle t\rangle = \mathbb{Q}(t,\delta t,\ldots)$. Differentiation of rational functions turns out to follow the usual quotient rule, noting that δ may map coefficients in a nonconstant ground field K to elements other than 0.

For the purposes of this article, we restrict ourselves to characteristic 0 and to *ordinary* differential rings and fields, i.e., those with only one derivation. *Partial* differential rings, with more differential operators, exist and have natural examples, as do differential rings of positive characteristic, but considering either would expand this article well beyond the scope we intend.

For a differential ring K with derivation δ, $K\{Y\}$ denotes the ring of all *differential polynomials* over K; it may be viewed as the ring of algebraic polynomials $K[Y,\delta Y,\delta^2 Y,\ldots]$, with Y and all its derivatives treated as separate variables. We then define $K\{Y_0,\ldots,Y_{n+1}\} = (K\{Y_0,\ldots,Y_n\})\{Y_{n+1}\}$. One sometimes differentiates a differential polynomial, treating each $\delta^{n+1}Y_i$ as the derivative of $\delta^n Y_i$. With only one derivation in the language, we often write Y' for δY, or $Y^{(r)}$ for $\delta^r Y$.

The *order* of a nonzero differential polynomial $q \in K\{Y\}$ is the greatest r such that the r-th derivative $Y^{(r)}$ appears nontrivially in q. Equivalently, it is the least r such that $q \in K[Y, Y', \ldots, Y^{(r)}]$. Having order 0 means that q is an algebraic polynomial in Y of degree > 0; nonzero elements of K within $K\{Y\}$ are said to have order -1, and in this article, the order of the zero polynomial is taken to be $+\infty$. Each nonzero polynomial in $K\{Y\}$ also has a *rank* in Y. For two such polynomials, the one with lesser order has lesser rank. If they have the same order r, then the one of lower degree in $Y^{(r)}$ has lesser rank. Having the same order r and the same degree in $Y^{(r)}$ is sufficient to allow us to reduce one of them, modulo the other, to a polynomial of lower degree in $Y^{(r)}$, and hence of lower rank: just take an appropriate K-linear combination of the two. So, for our purposes, the rank in Y is simply given by the order r and the degree of $Y^{(r)}$. Therefore, our ranks of nonzero differential polynomials will be ordinals in ω^2.

Our convention in this article is that the zero polynomial has order $+\infty$. Thus, for every element x in any differential field extension of K, the *minimal differential polynomial* of x over K is defined (up to a nonzero scalar from K) as the differential polynomial q in $K\{Y\}$ of least rank for which x is a zero (i.e., $q(x) = 0$). In particular, the zero polynomial is considered to be the minimal differential polynomial of an element differentially transcendental over K (such as t in $\mathbb{Q}\langle t\rangle$ above); this is simply for notational convenience.

The *differential closure* $\widehat{K}$ of a differential field K is the prime model of the theory $\mathbf{DCF}_0 \cup \Delta(K)$, the union of the atomic diagram $\Delta(K)$ of K with the (complete) theory $\mathbf{DCF}_0$. This theory was effectively axiomatized by Blum: her axioms for a differentially closed field F include the axioms for differential fields of characteristic 0 and state, for each pair (p, q) of differential polynomials with arbitrary coefficients from F and with $\mathrm{ord}(p) > \mathrm{ord}(q)$, that F must contain an element x with $p(x) = 0 \neq q(x)$. (By our convention on ranks, $\mathrm{ord}(p) > \mathrm{ord}(q)$ ensures that q is not the zero polynomial. However, q may equal 1, and so F must be algebraically closed. Notice here that, for all fields, model-theoretic algebraic closure implies field-theoretic algebraic closure.) Blum's proofs appear in [2, 3], and a summary of all these results can be found in [15].

Abraham Robinson showed that $\mathbf{DCF}_0$ has quantifier elimination. Blum's computable axiomatization makes $\mathbf{DCF}_0$ decidable, hence makes the quantifier elimination effective, both of which are particularly important for work involving computable-model-theoretic questions about $\mathbf{DCF}_0$. Every definable set in a computable model of $\mathbf{DCF}_0$ must now be decidable, and, given the original defining formula of the set, we can effectively find an equivalent quantifier-free formula, thereby passing uniformly to the decision procedure for the set. (Of course, this applies only to finitary defining formulas, not to computable infinitary formulas.)

Blum proved $\mathbf{DCF}_0$ to be ω-stable, and existing results of Morley then established that the theory $\mathbf{DCF}_0 \cup \Delta(K)$ always has a prime model, i.e., every differential field K has a differential closure. Subsequently, Shelah proved that, as the prime model extension of an ω-stable theory, the differential closure $\widehat{K}$ of K is unique and realizes exactly those types principal over K. Each principal 1-type has as generator a formula of the form $p(Y) = 0 \neq q(Y)$, where $(p,q) \in (K\{Y\})^2$ is a *constrained pair*. By definition, this means that $p(Y)$ is a monic, algebraically irreducible polynomial in $K\{Y\}$, that q has strictly lower rank in Y than p does, and that, in $\widehat{K}$ (and hence in every differential field extension of K), every y satisfying $p(y) = 0 \neq q(y)$ has minimal differential polynomial p over K. (A fuller definition appears in [17, Defn. 4.3].) Hence the elements satisfying the generating formula form an orbit under the action of those automorphisms of $\widehat{K}$ that fix K pointwise. For a pair (p,q) to be constrained is a Π_1^K property, and there exist computable differential fields K for which it is Π_1-complete. (This can happen even for a constant field K, such as the field $\mathbb{Q}[\sqrt{p_n} \ : \ n \in \varnothing']$; see [16].) If such a q exists, then p is said to be *constrainable*; clearly this property is Σ_2^K. Not all monic irreducible polynomials in $K\{Y\}$ are constrainable: for example, δY is not. More generally, no p in the image of $K\{Y\}$ under δ is constrainable, and certain polynomials p outside this image are also known to be unconstrainable. In fact, constrainability has been shown in [17] to be Σ_2^0-complete for certain computable differential fields K. The exact complexity of constrainability over the constant differential field $\mathbb{Q}$ is unknown: it might even be decidable. We note that p is constrainable over K if and only if some $y \in \widehat{K}$ has minimal differential polynomial p over K. (This equivalence will be extremely useful in the $\mathcal{S}_m$-substages of the construction for Theorem 4.1.) The equivalent condition proves again that constrainability is Σ_2^K, provided that there exists a K-computable presentation of $\widehat{K}$, which we get from a theorem of Harrington.

Theorem 1.1 (Harrington; [7], Corollary 3)**.** *For every computable differential field K, there exists a computable differential field L and a computable differential field homomorphism $g : K \to L$ such that L is a differential closure of the image $g(K)$. Moreover, indices for g and L may be found uniformly in an index for K.*

So this L is in fact a differential closure of K – or at least, of the image $g(K)$, which is computably isomorphic to K via g. In [23], Rabin proved the original analogue of this theorem for fields and their algebraic closures. We note that the exposition in [7] does not consider uniformity of the procedure it describes, but a close reading of the proof there indicates that the algorithm giving g and L is indeed uniform in an index for K. In particular, the following lemma is proven simply by uniformizing the proof of [7, 2(b), Lemma 2] and noting that the argument in the ensuing section 2(c) is uniform.

Lemma 1.2. *There exists a single computable function γ, the* type function *for $\boldsymbol{DCF}_0$, such that, for every computable differential field F of characteristic* 0*, every* *index e for the atomic diagram $\Delta(F)$ of F, and every irreducible differential polynomial $f \in F\{X\}$, $\gamma(e,f)$ is an index of the characteristic function $\varphi_{\gamma(e,f)}$ of a 1-type $\Gamma(x)$ that is complete and principal over $\boldsymbol{DCF}_0 \cup \Delta(F)$ and contains the formula $f(X) = 0$.* □

This type function will enable us to extend individual formulas $f(X) = 0$ uniformly to principal 1-types over differential fields we have already built. However, while the type $\Gamma(x)$ given by the type function will always be principal, the lemma does not promise to identify any specific formula as a generator of the type. The characteristic function merely decides which formulas belong to the type and which do not: at some point it will come across a generating formula and include it, but having done so, it will simply continue including and excluding other formulas, although from then on the type is in fact completely determined.

1.2. **Background in Model Theory.** Proposition 3.1 will require some background beyond Subsection 1.1, which we provide here, referring the reader to [15] and [22] for details and further references regarding these results. Model theorists have made dramatic inroads in the study of differential fields and $\mathbf{DCF}_0$; here we restrict ourselves to describing the results necessary to prove Proposition 3.1, without giving complete definitions of all the relevant concepts.

Let K be a differentially closed field, with subfield C_K of constants. For $a \in K \smallsetminus C_K$, consider the elliptic curve E_a given by

$$y^2 = x(x-1)(x-a).$$

Let $E_a^\sharp$ be the Kolchin closure of the set of all torsion points in the usual group structure on E_a. (The Kolchin topology is the differential analogue of the Zariski topology.) The set $E_a^\sharp$ is known as the *Manin kernel* of this abelian variety, as it is the kernel of a certain homomorphism of differential algebraic groups. One construction of Manin kernels appears in [14]. In the proof of Proposition 3.1 we will use Manin kernels $E_{a_m a_n}^\sharp$, meaning $E_a^\sharp$ as above with $a = a_m + a_n$.

Theorem 1.3. *The family $\{E_a^\sharp : a' \neq 0\}$ is definable. Indeed, it can be defined uniformly in each a with $a' = a^3 - a^2 \neq 0$, by a quantifier-free formula.*

The definability is claimed in [9] but done more clearly in [20, Sec. 2.4]. Of course, quantifier elimination for $\mathbf{DCF}_0$ allows us to take the definition to be quantifier-free. The condition $a' = a^3 - a^2$ will be relevant below.

Theorem 1.4. *If $a' \neq 0$, then $E_a^\sharp$ is strongly minimal and locally modular. Moreover, $E_a^\sharp$ and $E_b^\sharp$ are non-orthogonal if and only if E_a and E_b are isogenous. In particular if a and b are algebraically independent over $\mathbb{Q}$, then $E_a^\sharp$ and $E_b^\sharp$ are orthogonal.*

These results are due to Hrushovski and Sokolović [10], whose manuscript was never published. A proof of the first fact is given in Section 5 of [14], and proofs of both results appear in Section 4 of [22].

Corollary 1.5. *For every element* (b_0, b_1) *of* $E_a^\sharp$ *in the differential closure of* $\mathbb{Q}\langle a\rangle$*, both* b_0 *and* b_1 *are algebraic over* $\mathbb{Q}\langle a\rangle$*.*

Proof. Let $\psi(b_0, b_1)$ be the formula over $\mathbb{Q}\langle a\rangle$ isolating the type of (b_0, b_1). If ψ defined an infinite subset of $E_a^\sharp$, then it would contain a torsion point. But if ψ contains an n-torsion point, every point in ψ would be an n-torsion point, yet there are only n^2 n-torsion points in E_a, a contradiction. Thus $\psi(b_0, b_1)$ defines a finite set, so this pair is model-theoretically algebraic over a, hence lies in the field-theoretic algebraic closure of $\mathbb{Q}\langle a\rangle$. □

Lemma 1.6. *Let* X *and* Y *be strongly minimal sets defined over a differentially closed field* K*. If* X *and* Y *are orthogonal, then for any new element* $x \in X$ *the differential closure of* $K\langle x\rangle$ *contains no new elements of* Y*.*

Lemma 1.6 appears as [15, 7.2], while Lemma 1.7 is found in [15, Sec. 6].

Lemma 1.7. *Let* K *be a differentially closed field and*

$$A = \{y \in K : y \neq 0 \ \&\ y \neq 1 \ \&\ y' = y^3 - y^2\}.$$

Then A *is a strongly minimal set of indiscernibles.*

It follows from indiscernibility that A must be a trivial strongly minimal set and hence A is orthogonal to each of the sets $E_a^\sharp$. (Also, the set A is computable in the Turing degree of the differential field K, as defined in the next subsection.)

Lemma 1.8. *If* $a, b, c, d, \in A$*,* $a \neq b$*,* $c \neq d$ *and* $\{a, b\} \neq \{c, d\}$*, then* $a + b$ *and* $c + d$ *are algebraically independent.*

Proof. Suppose $p(X, Y) \in \mathbb{Q}[X, Y]$ such that $p(a+b, c+d) = 0$. There are only finitely many y with $p(a + b, y) = 0$. Suppose without loss of generality that $d \notin \{a, b\}$. Then by indiscernibility $p(a + b, c + e) = 0$ for every $e \in A \setminus \{a, b, c\}$, a contradiction. □

1.3. **Background in Computable Model Theory.** Now we describe the necessary concepts from computable model theory. For Proposition 3.1 and Theorem4.1, only Definition 1.9 is essential, but the rest of the subsection will make clear why the broad results in Section 5 are of interest.

Let $\mathcal{S}$ be a first-order structure on the domain ω, in a computable language (e.g., any language with finitely many function and relation symbols). The *(Turing) degree* $\deg(\mathcal{S})$ is the Turing degree of the atomic diagram of $\mathcal{S}$; in a finite language, this is the join of the degrees of the functions and relations in $\mathcal{S}$. $\mathcal{S}$ is *computable* if this degree is the computable degree $\mathbf{0}$. A structure isomorphic to a computable structure is said to be *computably presentable*; many countable structures fail to be computably presentable. A more exact measure of the presentability of (the isomorphism type of) the structure is given by its *Turing degree spectrum*.

Definition 1.9. The *spectrum* of a countable structure $\mathcal{S}$ is the set of all Turing degrees of *copies* $\mathfrak{M}$ of $\mathcal{S}$:

$$\{\deg(\mathfrak{M}) : \mathfrak{M} \cong \mathcal{S} \ \&\ \mathrm{dom}(\mathfrak{M}) = \omega\}.$$

When dealing with fields, we often write $\{x_0, x_1, \ldots\}$ for the domain; otherwise the element 1 in ω might easily be confused with the multiplicative identity in the field, for instance. In [12], Knight proved that spectra are always closed upwards, except in a few "automorphically trivial" cases (such as the complete graph on countably many vertices, whose spectrum is $\{\mathbf{0}\}$).

A wide range of theorems is known about the possible spectra of specific classes of countable structures. Many classes, including directed and undirected graphs, partial orders, lattices, nilpotent groups (see [8] for these results), and fields (see [18]), are known to realize all possible spectra. We will use the following theorem of Hirschfeldt, Khoussainov, Shore, and Slinko.

Theorem 1.10 (see Theorem 1.22 in [8])**.** *For every countable, automorphically non-trivial structure $\mathfrak{M}$ in any computable language, there exists a (symmetric, irreflexive) graph with the same spectrum as $\mathfrak{M}$.*

Richter showed in [24] that linear orders, trees and Boolean algebras fail to realize any spectrum containing a least degree under Turing reducibility, except when that least degree is $\mathbf{0}$, whereas undirected graphs can realize all such spectra. Boolean algebras were then distinguished from these other two classes when Downey and Jockusch showed that every low Boolean algebra has the degree $\mathbf{0}$ in its spectrum; this has subsequently been extended as far as low_4 Boolean algebras, in [4, 13, 31]. In contrast, Jockusch and Soare showed in [11] that each low degree does lie in the spectrum of some linear order with no computable presentation, although it remains open whether there is a single linear order whose spectrum contains all nonzero degrees but not $\mathbf{0}$. (There does exist a graph whose spectrum contains all degrees except $\mathbf{0}$, by results in [28, 32]. A useful survey of related results appears in [6].)

Of relevance to our investigations are the algebraically closed fields, the models of the closely related theories $\mathbf{ACF}_0$ and $\mathbf{ACF}_p$. Here the spectrum question has long been settled: every countable algebraically closed field has every Turing degree in its spectrum. On the other hand, every field becomes a constant differential field when given the zero derivation, which adds no computational complexity, and so the result from [18] for fields, mentioned above, shows that every possible spectrum is the spectrum of a differential field. These bounds leave a wide range of possibilities for spectra of differentially closed fields, and this is the subject of the present paper. It should be noted that, although every differentially closed field K is also algebraically closed and therefore is isomorphic (as a field) to a computable field, it may be impossible to add a computable derivation to the computable field in such a way as to make it isomorphic (as a differential field) to K.

We will show in Proposition 3.1 that countable differentially closed fields do realize a substantial number of quite nontrivial spectra, derived in a straightforward way from the spectra of undirected graphs. In particular, differentially closed fields can have all possible proper α-th jump degrees (as defined in that section), for all computable ordinals $\alpha > 0$. Section 2 is devoted to general background material for the proof of Proposition 3.1. On the other hand, we then prove Theorem 4.1, paralleling the original Downey-Jocksuch result: it shows that if the spectrum of a countable model of $\mathbf{DCF}_0$ contains a low degree, then it must also contain the degree $\mathbf{0}$. $\mathbf{DCF}_0$ thus becomes the second theory known to have this property (apart from trivial examples such as $\mathbf{ACF}_0$). Our positive results in Section 3, however, show that this theorem does not extend to low$_2$ degrees, let alone to low$_4$ degrees, as holds for Boolean algebras. Thus $\mathbf{DCF}_0$ realizes a collection of spectra not currently known to be realized by the models of any other theory in everyday mathematics. Finally, in Section 5, we relativize Theorem 4.1 and combine it with the results from Section 3 to characterize the spectra of models of $\mathbf{DCF}_0$ precisely as the preimages under the jump operation of the spectra of automorphically nontrivial graphs, and also as those spectra of such graphs which are closed under first-jump equivalence.

2. Eventually Non-isolated Types

The model-theoretic basis of Proposition 3.1 is ENI-DOP, the *Eventually Non-Isolated Dimension Order Property*, developed by Shelah [27] in proving Vaught's Conjecture for ω-stable theories. In this section we give a simple example of how this property can be used to code graphs into models of theories satisfying ENI-DOP. The example may help demystify the coding in Section 3, which is a more complicated example of the same phenomenon

In our simple example, we have a language with two sorts A and F, and three unary function symbols $\pi_1, \pi_2 : F \to A$, and $S : F \to F$. Our theory T includes axioms saying that A is infinite, the map $(\pi_1, \pi_2) : F \to A^2$ is onto, $\pi_i \circ S = \pi_i$, and S is a permutation of F with no cycles. This T is complete and has quantifier elimination. Its prime model consists of a countable set A with one $\mathbb{Z}$-chain F_{ab} (under S) in F for each pair $(a, b) \in A^2$. (F_{ab} is the preimage $(\pi_1, \pi_2)^{-1}(a, b)$ and is called the *fiber above* (a, b).) Every permutation of A extends to an automorphism of the prime model, and so A is a set of indiscernibles, in this model and also in every other model of T.

The type over a and b of a single element x of the fiber F_{ab} is isolated by the formula $(\pi_1(x) = a \,\&\, \pi_2(x) = b)$. However, over one realization c of this type, the type of a new element of F_{ab} (not in the $\mathbb{Z}$-chain of c) over a, b, and c is not isolated. This makes the type of x over a and b an example of an *eventually non-isolated type*: over sufficiently many realizations of itself, the generic realization of the type is non-isolated.

The important point here is that we can add a new point to F_{ab} without forcing any new points to appear in any other fiber or in A. (Indeed, we can continue adding points to various fibers without ever forcing any unintended points to appear in other fibers or in A.) This is roughly what is meant by saying that the types of generic elements of distinct fibers are *orthogonal*.

We use dimensions to code a graph G on A into a model of this theory T. (The *dimension* of F_{ab} is the number of $\mathbb{Z}$-chains in F_{ab}.) Starting with the prime model of T, we add one new $\mathbb{Z}$-chain to each fiber F_{ab} for which the graph has an edge between a and b. The orthogonality ensures the accuracy of this coding, by guaranteeing that this process does not accidentally give rise to new elements in any fiber F_{ab} for which the graph had no edge between a and b. This builds a new model $\mathfrak{M}$ of T, and the permutations of A which extend to automorphisms of $\mathfrak{M}$ are exactly the automorphisms of G.

It now follows that there exist continuum-many countable pairwise non-isomorphic models of T, since an isomorphism f between two such structures $\mathfrak{A}$ and $\mathfrak{B}$ would have to map the set of indiscernibles in $\mathfrak{A}$ onto that in $\mathfrak{B}$, hence likewise for the fibers, and therefore f on the indiscernibles would define an isomorphism between the graphs coded into $\mathfrak{A}$ and $\mathfrak{B}$. Moreover, the graph G coded into $\mathfrak{A}$ can be recovered from the computable infinitary Σ_2-theory of $\mathfrak{A}$ – that is, we can compute a copy of G if we know this theory – and in fact we can enumerate the edges in a copy of G just from the computable infinitary Π_1-theory of $\mathfrak{A}$, since this much information allows us to recognize any two elements of F_{ab} in $\mathfrak{A}$ that realize the nonisolated 2-type.

We will use this same strategy to code graphs into countable models K of $\mathbf{DCF}_0$, using the set A of indiscernibles given by Lemma 1.7. The fiber F_{mn} for $a_m, a_n \in A$ will be the Manin kernel $E^{\#}_{a_m a_n}$, defined in Theorem 1.3 and shown in Theorem 1.4 to have the appropriate properties, and the non-isolated computable infinitary Π_1-type in F_{mn} will be the type of an element of F_{mn} whose coordinates are both transcendental over $\mathbb{Q}\langle a_m + a_n\rangle$. With this background, the reader should be ready to proceed with Proposition 3.1.

Although we will not attempt to generalize here, it is reasonable to conjecture that the procedure in Section 3 should work for other classes of countable structures for which similar conditions hold. Analogues of its converse (Theorem 4.1, essentially) for such classes may be more challenging.

3. Noncomputable Differentially Closed Fields

In this section we consider countable models of the theory $\mathbf{DCF}_0$ which have no computable presentations. Using countable graphs with known spectra, we show how to construct differentially closed fields with spectra derived from those of the graphs. In particular, we create numerous countable differentially closed fields which are not computably presentable. We show that models of $\mathbf{DCF}_0$ can have proper α-th jump degree for every computable nonzero ordinal α. However, we will see in Section 5 that this is impossible when $\alpha = 0$: no countable model of $\mathbf{DCF}_0$ can have a least degree in its spectrum, unless that degree is $\mathbf{0}$. We encourage the reader to review Section 2 in order to understand the framework for the proof of the following theorem.

Proposition 3.1. *Let G be a countable symmetric irreflexive graph. Then there exists a countable differentially closed field $\widehat{K}$ of characteristic 0 such that*

$$Spec(\widehat{K}) = \{\boldsymbol{d} : \boldsymbol{d}' \text{ can enumerate a copy of } G\}.$$

(Saying that a degree $\boldsymbol{c}$ *can* enumerate a copy of G *means that there is a graph on* ω*, isomorphic to* G*, whose edge relation is* $\boldsymbol{c}$*-computably enumerable.)*

Proof. Taking G to have domain ω, we first describe one presentation of $\widehat{K}$, on the domain ω, without regard to effectiveness. We begin with $\widehat{\mathbb{Q}}$, the differential closure of the constant field $\mathbb{Q}$. Recall from Subsection 1.2 that the following is a computable infinite set of indiscernibles:

$$A = \{y \in \widehat{\mathbb{Q}} : y' = y^3 - y^2 \ \&\ y \neq 0 \ \&\ y \neq 1\}.$$

Writing $A = \{a_0 < a_1 < \cdots\}$, we use a_n to represent the node n from G.

For each a_m and a_n with $m < n$, let $E_{a_m a_n}$ be the elliptic curve defined by the equation $y^2 = x(x-1)(x - a_m - a_n)$. The type of a differential transcendental is orthogonal to each strongly minimal set defined over $\widehat{\mathbb{Q}}$. Thus, for each $m < n$, the Manin kernel $E^\sharp_{a_m a_n}$ contains only points differentially algebraic over $\mathbb{Q}\langle a_m, a_n\rangle$. These sets are also orthogonal to A. The points of $E_{a_m a_n}$ in $(\widehat{\mathbb{Q}})^2$ form an abelian group, with (for each $k > 0$) exactly k^2 points whose torsion divides k, and with no non-torsion points, since $\widehat{\mathbb{Q}}$ is the prime model of $\mathbf{DCF}_0$ over $\mathbb{Q}$. We will code our graph using these Manin kernels $E^\sharp_{a_m a_n}$, by adding a new point to $E^\sharp_{a_m a_n}$ (with coordinates transcendental over $\mathbb{Q}\langle a_m + a_n\rangle$) to our differential field just if the graph contains an edge from m to n. Any two of these Manin kernels are orthogonal, so adding a point to one (or even to infinitely many) of them will not add points to any other. Similarly, adding points to the Manin kernels will not add new points to A.

Now we build a differential field extension K of $\widehat{\mathbb{Q}}$, by adjoining to $\widehat{\mathbb{Q}}$ exactly one new point x_{mn} of $E^\sharp_{a_m a_n}$ for each $m < n$ such that G has an edge between its nodes m and n. (We note that, by orthogonality, the type of each generic point of $E^\sharp_{a_m a_n}$ over the differential field L generated by the preceding points $x_{m'n'}$ is computable: it is given by saying that x_{mn} is in $E^\sharp_{a_m a_n}$ but is not algebraic over $L\langle a_m + a_n\rangle$.) Adjoining all these x_{mn} yields a differential field K, and the differential field we want is the differential closure $\widehat{K}$ of this K. The principal relevant feature of $\widehat{K}$ is that, because of the mutual orthogonality of the Manin kernels, $\widehat{K}(E^\sharp_{a_m a_n})$ contains a point non-algebraic over $\mathbb{Q}\langle a_m + a_n\rangle$ if *and only if* there is an edge between m and n in G.

Now we claim that the spectrum of this $\widehat{K}$ contains exactly those Turing degrees whose jumps can enumerate a copy of G. To show that every degree in the spectrum has this property, suppose that $L \cong \widehat{K}$ has degree $\boldsymbol{d}$. Then with a $\boldsymbol{d}$-oracle, we can decide the set of all nontrivial solutions $b_0 < b_1 < \cdots$ in L to $y' = y^3 - y^2$. (The only trivial solutions are 0 and 1.) We build a graph H, with domain ω, using a $\boldsymbol{d}'$-oracle. The oracle tells us, for each $m < n$ and each solution $(x, y) \in L(E^{\#}_{b_m b_n})$, whether or not x is algebraic over $\mathbb{Q}\langle b_m + b_n\rangle$. If so, then we go on to the next point in $L(E^{\#}_{b_m b_n})$. If x is not algebraic, then we enumerate an edge between m and n into our graph H. The graph H thus enumerated is isomorphic to G: the isomorphism f from L onto $\widehat{K}$ must map the set $\{b_0, b_1, \ldots\}$ bijectively onto the set $\{a_0, a_1, \ldots\}$, and the map sending each $m \in H$ to the unique $n \in G$ with $f(b_m) = a_n$ will be an isomorphism of graphs. Thus $\boldsymbol{d}'$ has enumerated a copy H of G.

Conversely, suppose that the Turing degree $\boldsymbol{d}'$ enumerates a graph H isomorphic to G. Specifically, for a fixed set $D \in \boldsymbol{d}$, there is a Turing functional Φ for which the edge relation on H^2 is the domain of the partial function $\Phi^{D'}$. The description of $\widehat{K}$ above explains how to build a differentially closed field $\widehat{L}$ below a $\boldsymbol{d}$-oracle with $\widehat{L} \cong \widehat{K}$. Using Theorem 1.1, start building a computable copy of $\widehat{\mathbb{Q}}$, in which we enumerate all nontrivial solutions b_n to $y' = y^3 - y^2$, but build this solution slowly, with one new element at each stage, so that each step L_s in this construction is actually a finite fragment of the differential field L we wish to build. Then, with the $\boldsymbol{d}$-oracle, enumerate the jump D' of the set $D \in \boldsymbol{d}$: say $D' = \cup_{s\in\omega} D'_s$. Whenever we find a stage s such that some $\langle m, n\rangle$ lies in $\operatorname{dom}(\Phi_s^{D'_s})$ (and did not lie in this domain for $s-1$), we adjoin to L_s a new point $(x_{m,n,s}, y_{m,n,s})$ in $E^{\sharp}_{b_m b_n}$, such that $x_{m,n,s}$ does not yet satisfy any nonzero differential polynomial at all over L_s, and is specified not to be a zero of the first s polynomials of degree $\leq s$ over L_s. Of course, $y_{m,n,s}$ is a zero of the curve $E^{\sharp}_{b_m b_n}$ over $x_{m,n,s}$; this fully determines $y_{m,n,s}$ and its derivatives in terms of L_s and $x_{m,n,s}$ and its derivatives.

At the next stage, if we still have $\langle m, n\rangle \in \operatorname{dom}(\Phi_{s+1}^{D'_{s+1}})$, we declare that $x_{m,n,s+1} = x_{m,n,s}$ is not a zero of any of the first $s+1$ polynomials of degree $\leq s+1$ over L_{s+1}. If we ever reach a stage $t > s$ with $\langle m, n\rangle \notin \operatorname{dom}(\Phi_t^{D'_t})$ (which is possible, if the oracle has changed from the previous stage), then we turn $(x_{m,n,s}, y_{m,n,s})$ into a k-torsion point, with $k \geq t$ being the smallest value for which this is consistent with the finite fragment L_{t-1} built up till then. Since the types of torsion points are dense in the space of all types, the finitely many facts we have enumerated so far about L_{t-1} cannot possibly force this point to be a non-torsion point, so for some k this will be possible, and by searching we can identify such a k, using the decidability of $\mathbf{DCF}_0$. As we subsequently continue to build L (including the cofinite portion of $\widehat{\mathbb{Q}}$ which is yet to be constructed), we will take this k-torsion point into account, treating it as part of $\widehat{\mathbb{Q}}$. The decidability of $\mathbf{DCF}_0$ makes it easy to include the point into $\widehat{\mathbb{Q}}$ and still know what to build at each subsequent step.

Thus the existence of a nonalgebraic point on $E^{\sharp}_{b_m b_n}$ in the field L built by this process is equivalent to $\langle m, n\rangle$ actually lying in $\operatorname{dom}(\Phi^{D'})$, and for all $\langle m, n\rangle$ not in this domain, every pair $(x_{m,n,s}, y_{m,n,s})$ ever defined (for any s) was eventually turned into a torsion point, so that it wound up in the subfield $\widehat{\mathbb{Q}}$ of L, since this subfield contains all k^2 of the k-torsion points for $E_{b_m b_n}$ in L. Therefore, the L that we finally built is the differential field extension of $\widehat{\mathbb{Q}}$ by one nontorsion point for each edge in H, hence is isomorphic to the differential field K built above. So the differential closure $\widehat{L}$ of L is isomorphic to $\widehat{K}$, and is also $\boldsymbol{d}$-computable, by Theorem 1.1. □

We note that in Proposition 3.1, it is reasonable to replace the graph G, which the $\boldsymbol{d}'$-oracle can enumerate, by another countable graph H which the same oracle can actually compute. The converse is accomplished by the technique known as a Marker $\exists$-extension. The forwards direction too is a simple question of coding.

Lemma 3.2. *Let H be a countable (symmetric irreflexive) graph. Then there exists a countable graph G such that*

$$\mathit{Spec}(H) = \{\boldsymbol{d} : \boldsymbol{d} \text{ can enumerate a copy of } G\}.$$

Conversely, for every countable graph G, there exists a countable graph H satisfying this same equation. □

Recall that, for a computable ordinal α, the α-th *jump degree* of a countable structure $\mathfrak{M}$ is the least degree in the set $\{\boldsymbol{d}^{(\alpha)} : \boldsymbol{d} \in \mathrm{Spec}(\mathfrak{M})\}$.

Theorem 3.3. *For every graph H, there exists a differentially closed field K such that*

$$\mathit{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \in \mathit{Spec}(H)\}.$$

In particular, for every computable ordinal $\alpha > 0$ and every degree $\boldsymbol{c} >_T \mathbf{0}^{(\alpha)}$, there is a differentially closed field which has α-th jump degree $\boldsymbol{c}$, but has no γ-th jump degree whenever $\gamma < \alpha$.

Using ordinal addition, one can re-express the second result by stating that, for every $\beta < \omega_1^{CK}$ and every $\boldsymbol{c}$ with $\boldsymbol{c} >_T \mathbf{0}^{(1+\beta)}$, there is a differentially closed field K with proper $(1+\beta)$-th jump degree $\boldsymbol{c}$.

Proof. Given H, use Lemma 3.2 to get a graph G whose copies are enumerable by precisely the Turing degrees in $\mathrm{Spec}(H)$. Then apply Proposition 3.1 to this G to get the differentially closed field K required, with

$$\mathrm{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \text{ can enumerate a copy of } G\} = \{\boldsymbol{d} : \boldsymbol{d}' \in \mathrm{Spec}(H)\}.$$

Now, for every computable ordinal β and every degree $\boldsymbol{c} \geq \mathbf{0}^{(\beta)}$, there exists a graph H with β-th jump degree $\boldsymbol{c}$, but with no γ-th jump degree for any $\gamma < \beta$. (This is shown for linear orders in [1] and [5] for all $\beta \geq 2$, and Theorem 1.10 then transfers the result to graphs. For $\beta < 2$ it is a standard fact; see e.g. [6].) If $\alpha > 0$ is finite, let β be its predecessor and apply the first part of the corollary to the H corresponding to $\boldsymbol{c}$ and to this β. Then

$$\{\boldsymbol{d}^{(\beta)} : \boldsymbol{d} \in \mathrm{Spec}(H)\} = \{(\boldsymbol{d}')^{(\beta)} : \boldsymbol{d} \in \mathrm{Spec}(K)\} = \{\boldsymbol{d}^{(\alpha)} : \boldsymbol{d} \in \mathrm{Spec}(K)\},$$

so $\boldsymbol{c}$ is the α-th jump degree of K. When $\beta \geq \omega$, the degree $(\boldsymbol{d}')^{(\beta)}$ is just $\boldsymbol{d}^{(\beta)}$ itself, and so, for every infinite computable ordinal α, the above analysis with $\beta = \alpha$ shows that again K has α-th jump degree $\boldsymbol{c}$. In both cases, this also proves that for each $\gamma < \alpha$, K has no γ-th jump degree. □

4. Low Differentially Closed Fields

Theorem 3.3 demonstrated that, for every nonlow Turing degree $\boldsymbol{d}$, there exists a $\boldsymbol{d}$-computable differentially closed field with no computable presentation: with $\boldsymbol{d}' > \boldsymbol{0}'$, just take the model of $\mathbf{DCF}_0$ given by the corollary with jump degree $\boldsymbol{d}'$. (The corollary showed specifically that every degree whose jump computes $\boldsymbol{d}'$ lies in the spectrum, so the structure has a $\boldsymbol{d}$-computable copy.) Of course, there exist noncomputable low Turing degrees $\boldsymbol{d}$, i.e., degrees with $\boldsymbol{d} > \boldsymbol{0}$ but $\boldsymbol{d}' = \boldsymbol{0}'$. Theorem 3.3 yields no proof of the same result for these degrees. Indeed, the surprising answer is that when $\boldsymbol{d}$ is low, every $\boldsymbol{d}$-computable differentially closed field has the degree $\boldsymbol{0}$ in its spectrum.

Theorem 4.1. *Every low differentially closed field K of characteristic 0 is isomorphic to a computable differential field.*

Before beginning the full proof, we give some idea how it will go. Our goal is to construct a computable differential field F, with elements $y_0, y_1, \ldots$, isomorphic to K, whose elements are $x_0, x_1, \ldots$. The isomorphism $x_n \mapsto y_{h(n)}$ will be Δ^0_2, and we construct finite approximations h_s to h. We must ensure that the limit of these h_s exists and is a bijection. The requirement $\mathcal{R}_n$ is that $\lim_s h_s(n)$ exists; the requirement $\mathcal{S}_m$ is that $\lim_s h_s^{-1}(m)$ exists. Since each h_s will define a finite partial isomorphism into F from the current approximation K_s to K, the limit h will then define an isomorphism from K onto F.

For a single element $x_n \in K$, the basic module for satisfying $\mathcal{R}_n$ is not difficult. Since K is low, we can guess effectively at the minimal differential polynomial p_n of x_n over the finitely many higher-priority elements x_i of K. Assuming at stage s that our current guess is correct, we simply check through the finitely many elements currently in F to see whether this p_n is currently the minimal polynomial of any of them over the corresponding $y_{h_s(i)}$ in F. If so (and if that element is not already claimed by a higher-priority requirement), then we choose it as the image of x_n. If not, then we add a new element to F, making it a solution of p_n, and define it to be the image of x_n at this stage. Once our guesses at p_n have stabilized, this element will be $y_{h(n)}$, the image of x_n under our Δ^0_2-isomorphism. In the meantime, if our guess at p_n changes, we simply start the process over, leaving a leftover element in F.

Our construction will define the atomic diagram of F only in ways consistent with the complete decidable theory $\mathbf{DCF}_0$. (If $\mathcal{R}_n$ wants p_n to be given a zero in F, but $\mathbf{DCF}_0$ refuses to allow it, then the construction waits for a change in the guesses $p_0, \ldots, p_n$, which must happen, since K satisfies $\mathbf{DCF}_0$.) Therefore, a leftover element still can rely on $\mathbf{DCF}_0$'s assurance that there exists some zero of that p_n: K must contain some such zero, although x_n turned out not to be such a zero. Our next task, in building F, is to find a preimage for each leftover element y_m, as required by $\mathcal{S}_m$. Of course, once y_m is made into a root of a certain differential polynomial, it must remain a root of that polynomial; however, it might later be made into a root of another differential polynomial of lesser rank, so that the first one might not be its *minimal* differential polynomial. Since ranks are ordinals, this can only happen finitely often.

While y_m is believed to have minimal differential polynomial f over the higher-priority elements of F, and while $h_s^{-1}(m)$ is undefined, we search for an element of K which appears to have the same minimal differential polynomial over the corresponding higher-priority elements of K. If we find one, we make it the preimage of y_m. However, the existence of such an element in K is guaranteed only if f is constrainable (over the differential subfield generated by the higher-priority elements), which may not be decidable. Moreover, even if we find an $x \in K$ which appears to have the correct minimal differential polynomial, we could turn out to be mistaken, since we have only a computable approximation to minimal differential polynomials in K. There is a danger that no x with the correct minimal differential polynomial actually exists in K, but that K keeps offering us different possible elements x forever, each appearing to have the minimal differential polynomial we want. (In this sense, K "cannot be trusted" ever to give us a correct preimage, nor to cease supplying possibilities which turn out to be incorrect.) Therefore, while searching for a zero of f in K, we use the type function γ from Lemma 1.2 to determine a principal type containing the formula $f = 0$, and make this the type of y_m. The ground field (providing the e in Lemma 1.2) is the differential subfield of F generated by the higher-priority elements, under the assumption that no higher-priority requirement ever acts again. Obeying the type function ensures that eventually y_m will settle as a zero of a polynomial which is constrainable (over the higher-priority elements of F), and this in turn ensures that K will contain an element with that same minimal differential polynomial, which we will eventually find and define to be the preimage of y_m. The construction is therefore a finite-injury procedure, using these basic modules for the two types of requirements.

Proof. Our goal is to build a computable differential field F, with domain $\{y_0, y_1, \ldots\}$, and a sequence of uniformly computable finite partial functions $h_s : \omega \to \omega$ such that, for all n, $\lim_s h_s(n)$ converges to an element $h(n)$ so as to define an isomorphism $x_n \mapsto y_{h(n)}$ from K onto F. When $n \leq h(n)$, we will arrange that x_n and $y_{h(n)}$ have the same minimal differential polynomials over the differential subfields generated by the higher-priority elements in K and F:

$$\mathbb{Q}\langle x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \ldots, x_{n-1}, x_{h^{-1}(n-1)}\rangle \subseteq K$$
$$\mathbb{Q}\langle y_{h(0)},\ y_0,\ y_{h(1)},\ y_1,\ \ldots, y_{h(n-1)},\ y_{n-1}\rangle \subseteq F.$$

More precisely, there will be a differential polynomial $p_n \in \mathbb{Q}\{X_0, Y_0, X_1, \ldots, Y_{n-1}, X_n\}$ such that $p_n(x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n-1)}, X_n)$ is the minimal differential polynomial of x_n over the first subfield and $p_n(y_{h(0)}, y_0, y_{h(1)}, y_1, \ldots, y_{n-1}, Y_n)$ is the minimal differential polynomial of $y_{h(n)}$ over the second subfield.

Likewise, when $n > h(n)$, we will arrange that x_n and $y_{h(n)}$ have the same minimal differential polynomials over the differential subfields generated by higher-priority elements:

$$\mathbb{Q}\langle x_0, x_{h^{-1}(0)}, x_1, x_{h^{-1}(1)}, \ldots, x_{h^{-1}(h(n)-1)}, x_{h(n)}\rangle \subseteq K$$
$$\mathbb{Q}\langle y_{h(0)},\ y_0,\ y_{h(1)},\ y_1,\ \ldots, y_{h(n)-1},\ y_{h(h(n))}\rangle \subseteq F.$$

(With $n > h(n)$, the lower index $h(n)$ gives the priority of the pair $(x_n, y_{h(n)})$. Those pairs containing any of the elements $x_0, \ldots, x_{h(n)}$ and $y_0, \ldots, y_{h(n)-1}$ will have higher priority and so will be considered first.)

This will establish that h defines an embedding of differential fields. We will also ensure that $h : \omega \to \omega$ is a bijection, hence defines an isomorphism. Since F is computable, this will prove the theorem.

Our key asset in this construction is a computable approximation not only of the atomic diagram of the differential field K, but also of the minimal differential polynomial of each element x_n (in the domain $\{x_0, x_1, \ldots\}$ of K) over the differential subfield $\mathbb{Q}\langle x_0, \ldots, x_{n-1}\rangle$. Indeed, we use slightly more: we can effectively approximate the minimal differential polynomial $p_{n,\rho}$ of any x_n over $\mathbb{Q}\langle x_{n_1}, \ldots, x_{n_k}\rangle$, where $\rho = \langle n_1, \ldots, n_k\rangle \in \omega^{<\omega}$. This holds because the computable infinitary Σ_1-diagram of K is computable in the jump $(\deg(K))'$, i.e., in $\mathbf{0}'$. Recall that by our convention in this article, the minimal differential polynomial of a differential transcendental is the zero polynomial, and the comments above apply to differential transcendentals as well, since one jump over $\deg(K)$ is enough to decide whether x_n satisfies any nonzero differential polynomial at all over $\mathbb{Q}\langle x_{n_1}, \ldots, x_{n_k}\rangle$.

So, for each n, we will guess at some $p_n \in \mathbb{Q}\{X_0, Y_0, \ldots, Y_{n-1}, X_n\}$ giving the minimal differential polynomials of x_n and $y_{h(n)}$ over the relevant differential subfields, as described earlier. Our requirements to satisfy are:

$$\mathcal{R}_n : h(n) = \lim_s h_s(n) \text{ exists}$$
$$\mathcal{S}_m : h^{-1}(m) = \lim_s h_s^{-1}(m) \text{ exists},$$

with priority $\mathcal{R}_0 \prec \mathcal{S}_0 \prec \mathcal{R}_1 \prec \cdots$. If we can satisfy them, and maintain our rule that each p_n gives the minimal differential polynomial of both x_n and $y_{h(n)}$, then we will have built our isomorphism, which will in fact then be $\mathbf{0}'$-computable itself.

The strategy for satisfying a single requirement $\mathcal{R}_n$ is relatively simple. There exists a stage s by which our approximation to K will have settled on the true minimal differential polynomial $p_n(x_0, x_{h_s^{-1}(0)}, \ldots, x_{h_s^{-1}(n-1)}, X_n)$ of x_n over the higher-priority elements. If there already exists an element y_m in F for which $p_n(y_{h_s(0)}, y_0, \ldots, y_{n-1}, Y_n)$ is the minimal differential polynomial (over these higher-priority elements, according to the structure of F at this stage), we define $h_{s+1}(n) = m$. (This includes the situation where $m < n$ and $h_s(n) = m$ was already defined for the sake of the higher-priority $\mathcal{S}_m$.) Alternatively, if for some lower-priority y_m already in F it is consistent with $\mathbf{DCF}_0$ (given the current types of higher-priority elements of F), for y_m to become a zero of this polynomial, then again we define $h_{s+1}(n) = m$; otherwise, we add a new element y_m to F_{s+1}, making it a zero of this polynomial (provided this is consistent, the same as above) and set $h_{s+1}(n)$ equal to this new m. (If neither of these options is consistent, then we simply wait for our approximations to K to change.) Assuming that no higher-priority requirement ever again injures $\mathcal{R}_n$, and that the guess p_n never again changes, this y_m will continue to have this minimal differential polynomial throughout the rest of the construction: neither $\mathcal{R}_n$ nor any higher-priority requirement will ever need to change it, and no lower-priority requirement will ever be allowed to do so. (K itself witnesses that it is consistent with $\mathbf{DCF}_0$ for $x_0, x_{h^{-1}(0)}, \ldots, x_n$ to have the minimal differential polynomials that we have found, so $\mathbf{DCF}_0$ will not require any further changes to y_m.) Therefore $\mathcal{R}_n$ will never again injure any lower-priority requirement. Also, any similar actions taken by $\mathcal{R}_n$ before we reached this stage s will not impede us from satisfying $\mathcal{R}_n$ or any higher-priority requirement. The strategy for satisfying a single requirement $\mathcal{S}_m$ is more complicated; we will describe it in the construction, before the instructions for the $\mathcal{S}$-substages.

Notation 4.2. *To avoid cumbersome subscripts, we adopt the convention of writing "$[s]$" at the end of an expression to indicate that all items in the expression have the values assigned to them as of stage s. For example, $p_{n_i,\rho_i}(y_{h(n_0)}, \ldots, y_{h(n_i)})[s]$ will denote $p_{n_{i,s},\rho_{i,s},s}(y_{h_s(n_{0,s})}, \ldots, y_{h_s(n_{i,s})})$.*

Having F_s be a finite fragment of a differential field will allow us to lean heavily on the theory $\mathbf{DCF}_0$ for guidance in constructing F_{s+1}. This theory is complete and decidable, and so, given the finite fragment F_s containing (say) $y_0, \ldots, y_r$, we can write out the entire relational atomic diagram $\psi(y_0, \ldots, y_r)$ of these elements. When considering how to build F_{s+1}, we can then ask whether $\mathbf{DCF}_0$ contains the sentence

$$\exists Y_0 \cdots \exists Y_m [\psi(Y_0, \ldots, Y_r) \ \& \ g(Y_0, \ldots, Y_m) = 0].$$

(Here g is some polynomial over $\mathbb{Q}$ for which we might wish to declare $\vec{y}$ to be a zero.) If this is inconsistent, then the decision procedure for $\mathbf{DCF}_0$ will tell us so, and we will not set $g(\vec{y}) = 0$ in F_{s+1}. If it is consistent, it belongs to the complete theory $\mathbf{DCF}_0$, so some tuple of elements of K must realize $[\psi(\vec{X}) \ \& \ g(\vec{X}) = 0]$, and it is safe to set $g(\vec{y}) = 0$ in F_{s+1}, as K must contain preimages of these elements which are consistent with the minimal differential polynomials $p_0, p_{h^{-1}(0)}, p_1, \ldots$, up to the first p_n for which our approximations have not yet converged. (Notice that all formulas here are finitary. $\mathbf{DCF}_0$ cannot decide the consistency of computable infinitary formulas, so cannot be used to decide, for instance, whether a pair of differential polynomials is a constrained pair.) Of course, we must also verify that doing so will not change the minimal differential polynomial of any higher-priority element. Part of the purpose of Lemma 4.3 is to show how to do this verification effectively.

At stage 0, we set F_0 to contain $y_0 = 0$ and $y_1 = 1$ as the identity elements of F. The actual step is that we add Y_0 and $(Y_1 - 1)$ to the set U_0, i.e., to the computable enumeration of the set U of those differential polynomials $f \in \mathbb{Q}\{Y_0, Y_1, \ldots\}$ for which

$f(y_0, y_1, \ldots) = 0$ in F. This is equally strong and will simplify the construction, since it parallels our process for approximating K, which uses minimal differential polynomials rather than using the relations directly. In order to use the differential polynomials this way, we will need to be able to consider the finite set U_s at each stage and decide, for each m, just what minimal polynomial (over the higher-priority elements of F) we have committed y_m to satisfy. This requires the following lemma.

Lemma 4.3. *There is an algorithm which, when given as input (strong indices for) finite sets $V, W \subseteq \mathbb{Q}\{T_0, \ldots, T_r\}$ of differential polynomials and an $m \le r$ such that $\exists T_0 \cdots \exists T_r \psi$ lies in $\boldsymbol{DCF}_0$, where ψ is the formula*

$$\bigwedge_{g \in V} g(T_0, \ldots, T_r) = 0 \ \& \ \bigwedge_{g \in W} g(T_0, \ldots, T_r) \neq 0 \ \& \ \bigwedge_{i < j \le r} T_i \neq T_j,$$

outputs a differential polynomial $f = \sum_\theta f_\theta T_m^\theta$ in $\mathbb{Q}\{T_0, \ldots, T_m\}$ of least possible rank in T_m (written here using finitely many $f_\theta \in \mathbb{Q}\{T_0, \ldots, T_{m-1}\}$) such that $\boldsymbol{DCF}_0$ contains the sentence

$$(\forall T_0, \ldots \forall T_r [\psi \to \ f = 0]) \ \& \ \left(\exists T_0, \ldots \exists T_r \bigvee_\theta [\psi \ \& \ f_\theta \neq 0]\right).$$

(The point here is that committing ourselves to the finite set ψ of conditions will force T_m to be a zero of f, but will not force it to be a zero of any differential polynomial of lesser rank. So the algorithm is producing the apparent minimal differential polynomial f of T_m over $T_0, \ldots, T_{m-1}$, under the condition ψ, although of course ψ does not necessarily rule out the possibility of T_m satisfying some differential polynomial of smaller rank as well. The f produced is unique up to a scalar from $\mathbb{Q}^\times$.)

Proof. This is simply the algorithm originally developed by Ritt for reducing one differential polynomial modulo others of lower rank. It is given in full in [25], in a version which allows for several derivations, and is analogous to the reduction procedure for finding a principal generator of an algebraic ideal in the (non-differential) polynomial ring $L[T]$. Here we first convert the negative statements given by W to positive ones by adjoining variables S_g satisfying $1 = S_g \cdot g(T_0, \ldots, T_r)$ for each $g \in W$. Then we do Ritt's procedure, using all polynomials in V and these new equations from W, to get a minimal differential polynomial for T_0. If this polynomial lies in $\mathbb{Q}\{T_0\}$, then it is our output f_0 for the $m = 0$ case; if not, then f_0 is the zero polynomial. In either case, we then treat the quotient field of $\mathbb{Q}\{T_0\}/\{f_0\}$ as our ground field and repeat the process for T_1 over this ground field (still using all the equations from V and W) to produce f_1, then continue recursively up to f_m which is the desired f. □

Now we give the algorithm to be followed at stage $s+1$, using the function h_s and the set U_s from stage s. The domain of h_s contains finitely many elements of ω, which we view as indices of the elements x_n of K, while its range is a set of certain indices $m \le r$ of elements y_m of the finite set $F_s = \{y_0, \ldots, y_r\}$. We order the indices of elements of F_s according to priority:

$$h_s(0) \prec 0 \prec h_s(1) \prec 1 \prec \cdots \prec r,$$

and, after removing all repetitions from this list, we name these indices $m_{0,s} \prec m_{1,s} \prec \cdots$. If $h_s(n)$ is undefined for some n, we simply skip that spot in our list of indices $m_{i,s}$. The list ends once it contains all indices of elements of F_s, namely $\{0, 1, \ldots, r\}$. For each i, we define $n_{i,s} = h_s^{-1}(m_{i,s})$, if this inverse image exists. For the least j such that $n_{j,s}$ is not defined by this process, we set $n_{j,s}$ to be the least element not in $\mathrm{dom}(h_s)$, since we might be able to extend $\mathrm{dom}(h_{s+1})$ to include this element. Then, for each $i \leq j$, we set $\rho_{i,s}$ to be the finite tuple $(n_{0,s}, n_{1,s}, \ldots, n_{i-1,s})$ containing those elements of higher priority than $n_{i,s}$ in F_s.

The atomic diagram of $F_s = \{y_0, \ldots, y_r\}$ so far determined is denoted

$$\psi_s(Y_0, \ldots, Y_r) : \bigwedge_{i<j\leq r} Y_i \neq Y_j \ \& \ \bigwedge_{f \in U_s} f(Y_0, \ldots, Y_k) = 0 \ \& \bigwedge_{i<s \& g_i \notin U_s} g_i(Y_0, \ldots, Y_k) \neq 0.$$

(At the Final Step of each stage $s+1$, it is determined whether the s-th polynomial g_s lies in U or not.) Similarly, for each i with $n_{i,s}$ defined, $\sigma_{i,s}$ is the current approximation to K up to $x_{n_{i,s}}$, using the priority ordering:

$$\sigma_i(X_{n_0}, \ldots, X_{n_i})[s] : \bigwedge_{j \leq i} \left[p_{n_j,\rho_j}(X_{n_0}, \ldots, X_{n_j}) = 0 \ \& \ \bigwedge_{k<j} X_{n_k} \neq X_{n_j} \right] [s]$$

where, as defined earlier, $p_{n_i,\rho_i}(X_{n_0}, \ldots, X_{n_i})[s]$ is the current approximation to the minimal differential polynomial of $x_{n_{i,s}}$ over $\mathbb{Q}\langle \rho_{i,s} \rangle$. (Having $p_{n,\rho,s}$ be the zero polynomial when x_n appears to be differentially transcendental over $\mathbb{Q}\langle \rho \rangle$ suits this definition of $\sigma_{i,s}$ nicely.)

$\mathcal{R}_n$-substages. At stage $s+1$, we go through each $\mathcal{R}_n$ and $\mathcal{S}_n$ with $n \leq s$ in turn, with one substage for each, starting with $\mathcal{R}_0$. At the substage for a requirement $\mathcal{R}_n$, fix i such that $n = n_{i,s}$. (Such an i must exist, since we included the least index $\notin \mathrm{dom}(h_s)$ on our list of indices $n_{i,s}$. After this least element has been reached, no further substages will be executed at this stage.) Now we know that, for all $n_{k,s}$ with $k < i$, $h_{s+1}(n_{k,s}) = h_s(n_{k,s})$, since otherwise the stage would have ended already. First we check whether the sentence

$$\exists X_{n_0} \cdots \exists X_{n_i} \ \sigma_i(X_{n_0}, \ldots, X_{n_i})[s]$$

belongs to $\mathbf{DCF}_0$. If not, then we do nothing at this substage, and do not go on to the next substage, but instead go straight to the Final Step of stage $s+1$ (described below). In particular, $h_{s+1}(n_{k,s})$ is undefined for all $k \geq i$. As a simple example, if $p_{n_i,\rho_i} = X_{n_i} - a[s]$ and $p_{n_j,\rho_j} = X_{n_j} - a[s]$ for the same rational a and for some $j < i$, then the sentence would be rejected as inconsistent. If it is consistent, then we follow these instructions.

(1) If $h_{s+1}(n_{i,s})$ has been defined at an earlier substage, then we keep that value and go on to the next substage. (This happens if $h_{s+1}(n_{i,s}) < n_{i,s}$.)

(2) If $h_s(n_{i,s})\downarrow$ and Lemma 4.3 shows the minimal differential polynomial of $y_{h_s(n)}$ over $\mathbb{Q}\langle y_{h(n_0)},\dots,y_{h(n_{i-1})}\rangle[s]$ to be $p_{n,\rho_i}(y_{h(n_0)},\dots,y_{h(n_{i-1})},X)[s]$, then we preserve the map, setting $h_{s+1}(n)=h_s(n)$, and go on to the next substage. For instance, we do this if $\rho_{i,s-1}=\rho_{i,s}$ and $p_{n,\rho_i}[s-1]=p_{n,\rho_i}[s]$.

(3) Otherwise, either $h_s(n)$ is undefined, or else $h_s(n)=m'$ is defined with $m'\geq n$ but $p_{n,\rho_i}(y_{h(n_0)},\dots,y_{h(n_{i-1})},X)[s]$ is not the minimal differential polynomial of $y_{m'}$ over $\mathbb{Q}\langle y_{h(n_0)},\dots,y_{h(n_{i-1})}\rangle[s]$ in F_s. (This latter case happens if $p_{n,\rho_i}[s]\neq p_{n,\rho_i}[s-1]$.) In this case, x_n *abandons* this $y_{m'}$, if it existed at all, and we will need to choose a new value m for $h_{s+1}(n)$. The element $y_{m'}$ becomes *unattached*, and all lower-priority requirements will be injured at this stage.

If $p_{n,\rho_i}[s]$ is the zero polynomial, then x_n currently appears to be differentially transcendental, so we set $h_{s+1}(n)$ equal to the least number m such that $y_m\notin F_s$. Elements already in F_s already satisfy a polynomial, so we cannot define $h_{s+1}(n)$ to be an existing m. The new y_m is adjoined to F_{s+1}, with no change to U_{s+1} (so that y_m appears differentially transcendental in F_s).

If $p_{n,\rho_i}[s+1]$ was nonzero, then we wish to find some y_m for which we can make $p_{n,\rho_i}(y_{h(n_0)},\dots,y_{h(n_{i-1})},Y)[s]$ the minimal differential polynomial over $\mathbb{Q}\{y_{h(n_0)},\dots,y_{h(n_{i-1})}\}[s]$. For p_{n,ρ_i} of positive order, this can be done by taking $m=r+1$ if needed, since no facts about y_{r+1} have yet been stated in F. (Algebraic polynomials p_{n,ρ_i} will have no more roots in K than they are allowed to have in F, so either y_{r+1} or an existing y_m must suffice.) However, for the sake of $\mathcal{R}_n$, we need to choose m as small as possible without injuring higher-priority requirements. It is now necessary to define the process by which $\mathcal{R}_n$ *asks permission* from those requirements to add a polynomial to U_{s+1}; this appears directly below. For the least $m\leq r+1$ such that $\mathcal{S}_{n-1}$ (and hence all higher-priority requirements) grant permission, and such that y_m is not yet a root of any lower-order polynomial than p_{n,ρ_i}, we adjoin

$$p_{n,\rho_i}(Y_{h(n_0)},\dots,Y_{h(n_{i-1})},Y_m)[s]$$

to U_{s+1}; this means we are setting $p_{n,\rho_i}(y_{h(n_0)},\dots,y_{h(n_{i-1})},y_m)=0[s]$ in F, just as $p_{n,\rho_i}(x_{n_0},\dots,x_{n_{i-1}},x_n)=0[s]$ in K_s. With $h_{s+1}(n)=m$, our h_{s+1} still defines a partial isomorphism, based on the approximation K_s. If $m=r+1$, we also add x_{r+1} to F_{s+1}.

No matter which case held in item (3), we do not go on to the next substage, but continue instead with the Final Step of stage $s+1$ (described below).

This covers all the possibilities at substages dedicated to $\mathcal{R}$-requirements. Notice that, even if m lay in range(h_s) but not in range(h_{s+1}), y_m is still in F_{s+1}, and $U_s\subseteq U_{s+1}$. This is necessary in order for F to be computable. Eventually, $\mathcal{S}_m$ will choose an h-preimage for m respecting these conditions.

Asking permission to adjoin to U. Suppose $g\in\mathbb{Q}\{Y_{m_0},\dots,Y_{m_i}\}$ is a polynomial which we wish to add to U_{s+1}. To *ask permission* from a requirement $\mathcal{R}_n$ or $\mathcal{S}_m$ to do this, we choose the unique i with $m_{i,s} = m$ (for $\mathcal{S}_m$) or with $h_s(n) = m_{i,s}$ (for $\mathcal{R}_n$), and run the following process. If $\mathbf{DCF}_0 \vdash \psi_s \to g \neq 0$, then permission is immediately denied. Otherwise, let $E_{0,s} = \mathbb{Q}$, and define $E_{i,s}$ by recursion on $j < i$.

- If $n_j = h^{-1}(m_j) \leq m_j[s]$, then $\mathcal{R}_{n_j}$ controls y_{m_j}, and we set $E_{j+1,s}$ to be the computable differential field $E_{j,s}\{y_{m_j}\}/\langle p_{n_j,\rho_j}\rangle$, whose atomic diagram $\Delta(E_{j+1,s})$ is generated over $\mathbf{DCF}_0 \cup \Delta(E_{j,s})$ by the formula $p_{n_j,\rho_j}(y_{m_0},\ldots,y_{m_j}) = 0$ along with the statements that y_{m_j} is not a zero of any polynomial over $E_{j,s}$ of lower order than this p_{n_j,ρ_j}.
- If $n_j = h^{-1}(m_j) > m_j[s]$, then $\mathcal{S}_{m_j}$ controls y_{m_j}, and we set $E_{j+1,s}$ to be the computable differential field extending $E_{j,s}$ with one new generator y_{m_j} satisfying the type given by $\gamma(e_{j,s}, f)$, where γ is the type function from Lemma 1.2, $e_{j,s}$ is an index for $\Delta(E_{j,s})$, and f is the current minimal differential polynomial of y_{m_j} in F_s, as given by Lemma 4.3.

So $E_{i,s}$ is the differential field which the higher-priority requirements currently believe us to be building. (If the approximations given by K subsequently change, though, then $E_{i,s}$ could turn out not to be a subfield of the F we finally build.) Hence the theory $\mathbf{DCF}_0 \cup \Delta(E_{i,s})$ is complete and consistent, is decidable uniformly in i and s using quantifier elimination in $\mathbf{DCF}_0$, and contains constant symbols $y_{m_0},\ldots,y_{m_i}$. Now $g(y_0,\ldots,y_r)$ may have more variables than just these constants, so we check whether the formula

$$(\exists y_{k_0}\exists y_{k_1}\cdots\exists y_{k_l})[\psi_s \ \& \ g = 0]$$

lies in this theory, where $\{k_0,\ldots,k_l\} = \{k \leq r : k \notin \{m_0,\ldots,m_i\}\}$. If so, then the requirement *allows* g to be adjoined to U_{s+1}; if not, then it *denies permission* for this adjoinment. This completes the process of asking permission. (Notice that in fact we have received permission not just from the given requirement $\mathcal{R}_n$ or $\mathcal{S}_m$, but from all higher-priority requirements as well, via their subfields $E_{j,s}$ of $E_{i,s}$.)

$\mathcal{S}_m$-substages. Next we explain the instructions for a substage for the requirement $\mathcal{S}_m$. We fix the i (which must exist) such that $m_{i,s} = m$, and the current minimal differential polynomial f of y_m over $y_{m_0},\ldots,y_{m_{i-1}}[s]$. Now either $h^{-1}_{s+1}(m)$ has already been determined by some higher-priority $\mathcal{R}_n$ (so $\mathcal{S}_m$ has nothing to do), or $h_s(n) = m$ for some $n > m$, or y_m is currently unattached (i.e., $m \notin \text{range}(h_s)$). In these latter two cases, it is not clear that we will ever be able to find any $x \in K$ with minimal differential polynomial f over $x_{n_0},\ldots,x_{n_{i-1}}[s]$, since f might not be constrainable over these elements. (If $h^{-1}_s(m) = n$ is defined, then x_n currently appears to fill this role, but in the noncomputable differential field K, this could change at any time.) So the requirement $\mathcal{S}_m$ will use the type function γ from Lemma 1.2, knowing that γ must give us an index for a complete principal 1-type over $E_{i,s}$ which is consistent with ψ_s (and in particular with $f = 0$).

At a substage for a requirement $\mathcal{S}_m$ within stage $s+1$, we follow these instructions. Fix the unique i such that $m = m_{i,s}$. If there exists an $n \leq m$ such that $h_{s+1}(n)$

has already been defined to equal m, then we go on to the next substage. Also, if $h_s^{-1}(m)$ was defined and equal to some $n = n_{i,s} > m$, and $p_{n,\rho_i}[s] \neq p_{n,\rho_i}[s-1]$, then y_m becomes *unattached*. We make $h_{s+1}^{-1}(m)$ undefined and end this substage, and, instead of continuing to the next substage, we execute the Final Step of stage $s+1$.

Otherwise we create the computable differential field $E_{i,s}$ currently envisioned by the higher-priority requirements, exactly as defined above in the process for asking permission from the next-higher-priority requirement $\mathcal{R}_m$. Let $e_{i,s}$ be an index for the atomic diagram $\Delta(E_{i,s})$. For each of the first s irreducible differential polynomials $q_0, \ldots, q_s \in \mathbb{Q}\{Y_{m_0}, \ldots, Y_{m_i}\}$ of strictly lower order than f in Y_{m_i}, we compute $\varphi_{\gamma(e_{i,s},f)}(\ulcorner \psi_s \ \& \ q_j = 0 \urcorner)$; that is, we ask whether the formula $(\psi_s \ \& \ q_j = 0)$ belongs to the 1-type determined by γ for y_{m_i} over $E_{i,s}$, given that $f(y_{m_0}, \ldots, y_{m_{i-1}}, Y)$ is currently the minimal differential polynomial of y_{m_i} over $E_{i,s}$. If so, then for the least such j, we adjoin q_j to U_{s+1}, having already seen from $E_{i,s}$ that this will not injure any higher-priority requirements; we then end this substage and go directly to the Final Step of the stage. (This constitutes an injury to all lower-priority requirements, but since the order of the minimal polynomial of y_{m_i} can only decrease finitely often, there will be only finitely many such injuries.)

If there is no $j \le s$ for which $(\psi_s \ \& \ q_j = 0)$ belongs to the 1-type in question, then we keep $U_{s+1} = U_s$, and act according to the following three cases, which together complete the instructions for the $\mathcal{S}_m$-substage.

(1) If $h_s^{-1}(m)$ was defined and equal to some $n = n_{i,s} > m$, and no $n' < n$ with $n' \notin \{n_0, \ldots, n_{i-1}\}[s]$ has $p_{n',\rho_i}[s]$ equal to the apparent minimal differential polynomial f of y_m over $\{y_0, \ldots, y_{m_{i-1}}\}$ in F_s, then we keep $h_{s+1}(n) = m$ and go on to the next substage.

(2) If $h_s^{-1}(m)$ was defined and equal to some $n = n_{i,s} > m$, and some $n' < n$ with $n' \notin \{n_0, \ldots, n_{i-1}\}[s]$ has p_{n',ρ_i} equal to the apparent minimal differential polynomial f of y_m over $\{y_0, \ldots, y_{m_{i-1}}\}$ in F_s, then y_m becomes *unattached*. We make $h_{s+1}^{-1}(m)$ undefined and end this substage, and, instead of continuing to the next substage, we execute the Final Step of stage $s+1$. (At the $\mathcal{S}_m$-substage of the next stage, we will search for a new h-preimage for m, most likely the n' found above.)

(3) Otherwise, $h_s^{-1}(m)$ was undefined and $h_{s+1}^{-1}(m)$ has not been defined at an earlier substage of this stage. We check to see whether any $n \le s$ with $n \notin \{n_0, \ldots, n_{i-1}\}[s]$ has $p_{n,\rho_i}[s] = f$. If so, then we define $h_{s+1}(n) = m$ (for the least such n); if not, then $h_{s+1}^{-1}(m)$ remains undefined. In either case we proceed to the Final Step. (Eventually some such n will have to reveal itself, since, once our choice of f has stabilized, this f will be constrainable over the higher-priority elements of F, hence must have a zero in K over the corresponding elements there.)

Final Step. To finish stage $s+1$, after completing the last substage, consider the next differential polynomial $g_s(Y_0, \ldots, Y_k)$ in a fixed computable enumeration

$g_0, g_1, \ldots$ of $\mathbb{Q}\{Y_0, Y_1, \ldots\}$. Consider the lowest-priority element $y_{r'}$ currently in F_s. We ask permission either from the requirement $\mathcal{R}_n$ (where $h_{s+1}(n) = r' \geq n$, if such an n exists), or else from the requirement $\mathcal{S}_{r'}$, to adjoin g_s to U. If this permission is granted, then $g_s \in U_{s+1}$. If not, then U_{s+1} stays unchanged and we know $g_s \notin U$. (Thus U will be decidable.) This completes the Final Step, and ends stage $s+1$.

We set $F = \{y_m : m \in \omega\}$, but the important objects constructed were the decidable set $U = \bigcup_s U_s$ and the finite functions h_s, whose limit will be the isomorphism from K onto F. Notice that, every time any differential polynomial $g(Y_0, \ldots, Y_{k_f})$ was enumerated into U_s, the permission process confirmed that the formula

$$\exists Y_0 \cdots \exists Y_r (\psi_s \wedge g = 0)$$

belonged to the theory $\mathbf{DCF}_0$. It follows that the entire set of formulas ψ_s, for all s, is consistent with $\mathbf{DCF}_0$.

The bijection between F and K will follow once we prove these claims for all i:

- $n_i = \lim_s n_{i,s}$ exists, and the map $i \mapsto n_i$ is a permutation of ω;
- $m_i = \lim_s m_{i,s}$ exists, and the map $i \mapsto m_i$ is a permutation of ω;
- the function $h = \lim_s h_s$ is a bijection from ω onto ω, and hence defines a bijection $x_n \mapsto y_{h(n)}$ from K onto F; and
- the limit $p_i = \lim_s p_{n_{i,s}, \rho_{i,s}, s} \in \mathbb{Q}\{X_{n_0}, X_{n_1}, \ldots, X_{n_i}\}$ exists, and U contains $p_i(Y_{h(n_0)}, \ldots, Y_{h(n_i)})$, and no $q(Y_{h(n_0)}, \ldots, Y_{h(n_i)})$ in U has lower $Y_{h(n_i)}$-rank than p_i. (Here $\rho_i = (n_0, \ldots, n_{i-1}) = \lim_s \rho_{i,s}$, from the first claim.)

The first three claims here can be proven together by a single induction.

Lemma 4.4. *For every m, there exists a unique i with $\lim_s m_{i,s} = m$; likewise, for every n, there exists a unique i with $\lim_s n_{i,s} = n$. Thus every requirement $\mathcal{R}_n$ and $\mathcal{S}_m$ is satisfied by the foregoing construction.*

Proof. The uniqueness of i, for any single m or n, is immediate from our definitions of $m_{i,s}$ and $n_{i,s}$. We specifically excluded all repetitions from the first sequence, making $m_{i,s} \neq m_{j,s}$ for every $i < j$, and we made every h_s injective. Recall that by our definition, at stage s, every $n_{i,s}$ except the very last one lies in $\text{dom}(h_s)$. The injectivity of each h_s follows from its construction: we always included in $\psi_s(Y_0, \ldots, Y_r)$ the conditions that $Y_i \neq Y_j$ for all $i < j \leq r$, and similarly in $\sigma_{i,s+1}$ that $X_{n_{i,s}} \neq X_{n_{j,s}}$, and then we required the choice of each new $h_{s+1}(n)$ to have $\sigma_{i,s+1}(Y_{h_{s+1}(n_{0,s})}, \ldots, Y_{h_{s+1}(n_{i-1,s})}, Y_{h_{s+1}(n)})$ consistent with $\psi_s(Y_0, \ldots, Y_r)$.

We proceed by induction on these requirements, according to their priority order, starting with $\mathcal{R}_0$. The inductive hypothesis is that there exists a stage s_0 such that, for every $s \geq s_0$ and each higher-priority requirement $\mathcal{R}_{n'}$ or $\mathcal{S}_{m'}$, there are unique numbers j and k with $n_{j,s} = n'$ and $m_{k,s} = m'$ and $h_s(n') = h_{s_0}(n')$ and $h_s^{-1}(m') = h_{s_0}^{-1}(m')$. Turning to the minimal polynomials in K, we may also assume that s_0 is so large that, for every $n' = n_{j,s} < n$, $p_{n', \rho_{j,s}, s} = p_{n', \rho_{j,s}, s_0}$ (noting that $\rho_{j,s} = \rho_{j,s_0}$ by the previous part of the hypothesis). That is, all approximations to minimal polynomials of higher-priority elements of K have converged by stage s_0. It follows that, from stage s_0+1 on, every substage for a higher-priority requirement will do nothing. Moreover, at all subsequent stages s, the field $E_{i,s}$ will have stabilized as one particular differential subfield E_i of F (where i is chosen so that either $m = m_{i,s}$ or $h_s(n) = m_{i,s}$).

Suppose this inductive hypothesis holds of every requirement of higher priority than $\mathcal{R}_n$. If there exists an $m < n$ with $h_{s_0}(n) = m$, then the satisfaction of $\mathcal{S}_m$ shows that $\mathcal{R}_n$ is satisfied as well. So assume that there is no such m. Let $\rho = \rho_{i,s_0+1}$ be the sequence of indices of elements in K of higher priority than n. This too never changes at stages $> s_0$. But now the approximations $p_{n,\rho,s}$ to the minimal differential polynomial of x_n over $\mathbb{Q}\langle x_0,\ldots,x_{i-1}\rangle$ (with $x_j = \lim_s x_{j,s}$) must converge, to some limit $p_n(X_0,\ldots,X_i)$. Let $s_1 > s_0$ be a stage by which this convergence has occurred. If $h_{s_1}(n)$ is undefined, then at stage s_1+1 the construction will reach the substage for $\mathcal{R}_n$ and will act according to item (3) at that substage, and will choose a value $h_{s_1+1}(n) \le r+1$. This $y_{h_{s_1+1}(n)}$ therefore lies in F_s at all $s \ge s_1+1$. At the next stage s_1+2, n will lie in the domain of h_{s_1+1}, and therefore will have $n = n_{i,s_1+1}$ for some i, i.e., n will have been assigned a priority, corresponding to the requirement $\mathcal{R}_n$. From then on, item (2) in the substage for $\mathcal{R}_n$ will always apply, leaving the value of $h_s(n)$ unchanged. Moreover, in the process of asking permission, $E_{i,s}$ ensures that the minimal polynomial of $y_{h_s(n)}$ in F would only change if the rank of a higher-priority element changed, or if the approximation to p_n changed. By assumption neither of these ever changes again, so the minimal polynomial of $y_{h_s(n)}$ in F stays fixed forever. Therefore, $h_s(n)$ will never again change its value, and the requirement $\mathcal{R}_n$ is indeed satisfied. The existence of the (unique) i with $n = n_i = \lim_s n_{i,s}$ follows.

Now we turn to the inductive step for a requirement $\mathcal{S}_m$, using the stage s_0 defined above by the inductive hypothesis on all higher-priority requirements. Once again, it follows that every higher-priority requirement will do nothing at its substage during each stage $> s_0$, and so the $\mathcal{S}_m$-substage will be reached at every such stage. If $h_{s_0}(n) = m$ for some $n \le m$, then the satisfaction of the higher-priority requirement $\mathcal{R}_n$ shows that $m = \lim_s h_s(n)$; so assume that this is not the case. Now F_s increases at infinitely many stages s, so eventually some F_{s_1} will include y_m. At this point, an i will be chosen for which $m_{i,s} = m$, since this happens for all indices of elements of F_s. Moreover, taking $s_1 > s_0$ and knowing that the higher-priority requirements never act again, we will have $m_{i,s} = m$ at all stages $> s_1$ as well; this proves existence of the i with $m = m_i = \lim_s m_{i,s}$, and its uniqueness was already seen.

At stage s_1, y_m has an apparent minimal differential polynomial f over the higher-priority requirements. Since E_i never again changes, and every subsequent adjoinment to U will require the permission of $\mathcal{S}_m$, we know that y_m must realize the type Γ over E_i given by the type function: $\varphi_{\gamma(e_i,f)}$ computes this type. Since Γ is principal, there must exist an s such that Γ contains a formula of the form $(q = 0 \ \&\ \psi_s)$ which generates Γ. This q is therefore constrainable (with ψ_s providing the constraint, if a nontrivial one is needed), and when q appears in an $\mathcal{S}_m$-substage, y_m will be defined to be a zero of this q.

(Lemma 1.2 did not actually claim that, whenever $q = 0$ lies in the type $\gamma(e, f)$ with q of smaller rank than f, the index $\gamma(e, q)$ must then define the same type as $\gamma(e, f)$. It can readily be arranged for this to be so, however; and even if it were not so, it would only contribute finitely many more injuries to the lower-priority requirements.)

So eventually y_m is found to be a zero of a constrainable q, in particular, of the smallest-rank q such that $q = 0$ lies in this type. Once this has happened, the differentially closed field K must reveal an x_n realizing this same type over the h-preimages of the higher-priority elements. For the least such n, once the K-approximation settles on q as the minimal differential polynomial of this x_n (and once all $x_{n'}$ with $n' < n$ have settled on their own minimal differential polynomials distinct from q), we will define $h(n) = m$, and will preserve $h(n) = m$ forever after. This completes the proof of the lemma. □

Finally we consider the last claim, for a fixed i. The first part of the claim has already been noted: we have seen above that the limit $n_j = \lim_s n_{j,s}$ exists for every j, and so, with $\rho_i = \lim_s \rho_{i,s}$, the computable approximations in K all converge to the actual minimal differential polynomials $p_i = \lim_s p_{n_i,\rho_{i,s},s}$. We have also seen above that $m_i = h(n_i) = \lim_s h_s(n_i)$ exists. But each map h_s defines a partial isomorphism from the approximation K_s into F, and so, once all the approximations for a given fragment of K have converged, the limit h on this fragment will define a partial isomorphism. Since h is also a bijection, it does in fact define an isomorphism $x_n \mapsto y_{h(n)}$. This completes the proof of the final claim.

It follows that the operations in F are computable. For instance, given any $y_i, y_j \in F$, the elements $x_{h^{-1}(i)}$ and $x_{h^{-1}(j)}$ of K have a sum x_k. Since h defines an isomorphism, the polynomial $Y_i + Y_j - Y_{h(k)}$ must lie in the decidable set U, and when we find it, we will know that $y_{h(k)} = y_i + y_j$. Multiplication and differentiation are similarly computable, so F is a computable structure, and the isomorphism h from K onto F establishes Theorem 4.1. □

Theorem 4.1 will remind many readers of the well-known theorem of Downey and Jockusch from [4], that every low Boolean algebra has a computable copy. However, the parallels between these results are few. The latter theorem has been extended to included low$_4$ Boolean algebras, in work by Thurber [31] and Knight and Stob [13], whereas by Theorem 3.3, the result for $\mathbf{DCF}_0$ does not even extend to the low$_2$ case. Moreover, the proof of Theorem 4.1 constructed a Δ^0_2-isomorphism from the low model of $\mathbf{DCF}_0$ to its computable copy, whereas for Boolean algebras, there is always a Δ^0_3-isomorphism but not always a Δ^0_2 one. The construction here relied heavily on the completeness and decidability of the theory $\mathbf{DCF}_0$, whereas the theory of Boolean algebras is certainly not complete. Conversely, the construction in [4] uses theorems of Vaught and Remmel which are specific to Boolean algebras, with no obvious analogue for $\mathbf{DCF}_0$.

The closer analogy is to the theory $\mathbf{ACF}_0$, for which Theorem 4.1 is trivially true, since *every* countable algebraically closed field has a computable presentation. All those of finite transcendence degree over $\mathbb{Q}$ are relatively computably categorical, meaning that every presentation of degree $\boldsymbol{d}$ has a $\boldsymbol{d}$-computable isomorphism onto a computable copy. The unique countable model of $\mathbf{ACF}_0$ of infinite transcendence degree over $\mathbb{Q}$ is not, but it is relatively Δ^0_2-categorical, since in one jump over the atomic diagram of the structure, one can compute a transcendence basis for the field over $\mathbb{Q}$. For low models of $\mathbf{ACF}_0$, one can give a much simpler version of the priority construction used in Theorem 4.1. For readers who find the construction in the proof of Theorem 4.1 daunting, carrying out this construction for $\mathbf{ACF}_0$ might be a useful prelude.

5. Spectra of Differentially Closed Fields

Proposition 5.1. *For every countable model K of $\boldsymbol{DCF}_0$ of Turing degree $\boldsymbol{c}$, every degree $\boldsymbol{d}$ with $\boldsymbol{d}' \geq \boldsymbol{c}'$ lies in the spectrum of K.*

Proof. One simply runs the same construction as in Theorem 4.1, relative to an oracle from $\boldsymbol{d}$. Since $\boldsymbol{d}' \geq \boldsymbol{c}'$, this oracle can compute all the necessary approximations to facts about K and about minimal differential polynomials in K, so this produces a $\boldsymbol{d}$-computable differential field isomorphic to K. As mentioned in Subsection 1.3, Knight's theorem from [12] then shows that $\boldsymbol{d} \in \text{Spec}(K)$, since no differentially closed field is automorphically trivial. □

Definition 5.2. *First-jump equivalence* is the relation $\sim_1$ on Turing degrees:

$$\boldsymbol{c} \sim_1 \boldsymbol{d} \iff \boldsymbol{c}' = \boldsymbol{d}'.$$

Proposition 5.1 shows that every spectrum of a model K of $\mathbf{DCF}_0$ *respects* $\sim_1$, in the sense that, whenever $\boldsymbol{c} \sim_1 \boldsymbol{d}$, we have $(\boldsymbol{c} \in \mathcal{S} \iff \boldsymbol{d} \in \mathcal{S})$. It follows that $\text{Spec}(K)$ is actually determined by its *jump spectrum* $\{\boldsymbol{d}' : \boldsymbol{d} \in \text{Spec}(K)\}$. Moreover, this proposition, along with Lemma 5.4 (which is easily proven using the methods of [29, Chapter VI]), yields a quick proof of a property for $\mathbf{DCF}_0$ which was already known to hold for linear orders, Boolean algebras, and trees (viewed as partial orders), by results of Richter in [24]. When the question of spectra of differentially closed fields first arose, this corollary was quickly observed by Andrews and Montalbán, who pointed out that it follows from [24].

Corollary 5.3 (cf. Andrews & Montalbán)**.** *No countable differentially closed field K of characteristic 0 intrinsically computes any noncomputable set $B \subseteq \omega$. That is, the spectrum of K cannot be contained within the upper cone $\{\boldsymbol{d} : \boldsymbol{b} \leq \boldsymbol{d}\}$ above a nonzero degree $\boldsymbol{b}$. In particular, if such a spectrum has a least degree under $\leq_T$ among its elements, then that degree is $\mathbf{0}$.*

Proof. Let K have degree $\boldsymbol{c}$. Lemma 5.4 below yields a degree $\boldsymbol{d}$ with $\boldsymbol{b} \nleq \boldsymbol{d}$ and $\boldsymbol{c}' \leq \boldsymbol{d}'$. But then $\boldsymbol{d} \in \text{Spec}(K)$ by Proposition 5.1. □

Lemma 5.4 (Folklore)**.** *For every noncomputable set B and every set C, there exists some set D with $B \not\leq_T D$ and $C' \leq_T D'$. Indeed $C' \leq_T \varnothing' \oplus D$.* $\square$

The main consequence of Proposition 5.1 is a very precise description of the spectra of models of $\mathbf{DCF}_0$ in terms of arbitrary spectra. Theorem 1.10 shows that items (2) and (3) of Theorem 5.5 could equally well allow G and J to vary over structures in all computable languages.

Theorem 5.5. *For a set $\mathcal{S}$ of Turing degrees, the following are equivalent.*

(1) $\mathcal{S}$ is the spectrum of some countable model K of $\boldsymbol{DCF}_0$.
(2) There exists a countable, automorphically nontrivial graph G for which $\mathcal{S} = \{\boldsymbol{d} : \boldsymbol{d}' \in Spec(G)\}$.
(3) $\mathcal{S}$ respects $\sim_1$ and there exists a countable, automorphically nontrivial graph J with $\mathcal{S} = Spec(J)$.

Proof. The implication $(2) \Longrightarrow (1)$ is precisely Theorem 3.3 above. Also, $(1) \Longrightarrow (3)$ follows from Proposition 5.1 and Theorem 1.10. To establish $(3) \Longrightarrow (2)$, given J, we appeal to the following theorem, proven by Soskova and Soskov in [30] and independently by Montalbán in [19] and first presented by Soskov in a talk in 2002.

Theorem 5.6 (see [19, 30])**.** *For every countable structure $\mathcal{A}$, there exists a countable structure $\mathcal{A}'$, the* jump of the structure $\mathcal{A}$*, such that $Spec(\mathcal{A}') = \{\boldsymbol{c}' : \boldsymbol{c} \in Spec(\mathcal{A})\}$.*

Using Theorem 1.10, we convert the jump J' of our J into a graph G, with $\text{Spec}(G) = \{\boldsymbol{c}' : \boldsymbol{c} \in \text{Spec}(J)\}$. Since J is automorphically nontrivial, so is G. Now each $\boldsymbol{d} \in \mathcal{S} = \text{Spec}(J)$ has $\boldsymbol{d}' \in \text{Spec}(G)$. Conversely, for every $\boldsymbol{d}$ with $\boldsymbol{d}' \in \text{Spec}(G)$, we have some $\boldsymbol{c} \in \text{Spec}(J) = \mathcal{S}$ with $\boldsymbol{c}' = \boldsymbol{d}'$, making $\boldsymbol{d} \in \mathcal{S}$ since $\mathcal{S}$ respects $\sim_1$. $\square$

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Dept. of Mathematics, Statistics, & Computer Science, University of Illinois at Chicago, Chicago, IL, U.S.A.
E-mail address: marker@math.uic.edu

Dept. of Mathematics, Queens College, & Ph.D. Programs in Mathematics & Computer Science, Graduate Center, City University of New York, USA
E-mail address: Russell.Miller@qc.cuny.edu
URL: http://qcpages.qc.cuny.edu/~rmiller/