

# The Hilbert's-Tenth-Problem Operator

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## Abstract

For a ring  $R$ , Hilbert's Tenth Problem  $HTP(R)$  is the set of polynomial equations over  $R$ , in several variables, with solutions in  $R$ . We view  $HTP$  as an operator, mapping each set  $W$  of prime numbers to  $HTP(\mathbb{Z}[W^{-1}])$ , which is naturally viewed as a set of polynomials in  $\mathbb{Z}[X_1, X_2, \dots]$ . For  $W = \emptyset$ , it is a famous result of Matiyasevich, Davis, Putnam, and Robinson that the jump  $\emptyset'$  is Turing-equivalent to  $HTP(\mathbb{Z})$ . More generally,  $HTP(\mathbb{Z}[W^{-1}])$  is always Turing-reducible to  $W'$ , but not necessarily equivalent. We show here that the situation with  $W = \emptyset$  is anomalous: for almost all  $W$ , the jump  $W'$  is not diophantine in  $HTP(\mathbb{Z}[W^{-1}])$ . We also show that the  $HTP$  operator does not preserve Turing equivalence: even for complementary sets  $U$  and  $\bar{U}$ ,  $HTP(\mathbb{Z}[U^{-1}])$  and  $HTP(\mathbb{Z}[\bar{U}^{-1}])$  can differ by a full jump. Strikingly, reversals are also possible, with  $V <_T W$  but  $HTP(\mathbb{Z}[W^{-1}]) <_T HTP(\mathbb{Z}[V^{-1}])$ .

## 1 Introduction

The original version of Hilbert's Tenth Problem demanded an algorithm deciding which polynomial equations from  $\mathbb{Z}[X_0, X_1, \dots]$  have solutions in integers. In 1970, Matiyasevic [8] completed work by Davis, Putnam and

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Robinson [3], showing that no such algorithm exists. In particular, these authors showed that there exists a 1-reduction from the Halting Problem  $\emptyset'$  to the set of such equations with solutions, by proving the existence of a single polynomial  $h \in \mathbb{Z}[X, \vec{Y}]$  such that, for each  $n$  from the set  $\mathbb{N}$  of nonnegative integers, the polynomial  $h(n, \vec{Y}) = 0$  has a solution in  $\mathbb{Z}$  if and only if  $n$  lies in  $\emptyset'$ . Since the membership in the Halting Problem was known to be undecidable, it followed that Hilbert's Tenth Problem was also undecidable.

One naturally generalizes this problem to all rings  $R$ , defining Hilbert's Tenth Problem for  $R$  to be the set

$$HTP(R) = \{f \in R[\vec{X}] : (\exists r_1, \dots, r_n \in R^{<\mathbb{N}}) f(r_1, \dots, r_n) = 0\}.$$

Here we will examine this problem for one particular class: the subrings  $R$  of the field  $\mathbb{Q}$  of rational numbers. Notice that in this situation, deciding membership in  $HTP(R)$  reduces to the question of deciding this membership just for polynomials from  $\mathbb{Z}[\vec{X}]$ , since one readily eliminates denominators from the coefficients of a polynomial in  $R[\vec{X}]$ . So, for us,  $HTP(R)$  will always be a subset of  $\mathbb{Z}[X_1, X_2, \dots]$ . In turn, sets of polynomials, such as  $HTP(R)$ , will be viewed as subsets of  $\mathbb{N}$ , using a fixed computable bijection from  $\mathbb{N}$  onto  $\mathbb{Z}[\vec{X}] = \mathbb{Z}[X_0, X_1, \dots]$ .

Subrings  $R$  of  $\mathbb{Q}$  correspond bijectively to subsets  $W$  of the set  $\mathbb{P}$  of all primes, via the map  $W \mapsto \mathbb{Z}[\frac{1}{p} : p \in W]$ . We write  $R_W$  for the subring  $\mathbb{Z}[\frac{1}{p} : p \in W]$ , which (as a subset of  $\mathbb{Q}$ ) is Turing-equivalent to  $W$  itself (as a subset of  $\mathbb{P}$ ). The *HTP operator* is the map sending each  $W \subseteq \mathbb{P}$  to  $HTP(R_W)$ . This operator, and its relation to Turing reducibility  $\leq_T$ , are the focus of our work here. Recall that  $A \leq_T B$  intuitively means that, if one knew which numbers lie in  $B$ , one could decide which numbers lie in  $A$ . ( $A <_T B$  just means that  $A \leq_T B$  but  $B \not\leq_T A$ .) One would expect that, for more complicated rings  $R$ ,  $HTP(R)$  would be more difficult to compute than for simpler rings. We will confound this expectation, employing properties of the polynomials  $X^2 + qY^2 - 1$  to produce subrings  $R$  and  $S$  of  $\mathbb{Q}$  such that  $R <_T S$  – so it is strictly easier to decide which rationals lie in  $R$  than which lie in  $S$  – yet  $HTP(S) <_T HTP(R)$  – i.e., it is strictly easier to decide which polynomials have solutions in  $S$  than to decide which have solutions in  $R$ .

Of course, we have a computable bijection between subsets of  $\mathbb{N}$  and subsets of  $\mathbb{P}$ , using the function mapping  $n \in \mathbb{N}$  to the  $n$ -th prime  $p_n$ , starting with  $p_0 = 2$ . Since this bijection preserves Turing degrees, Turing reductions can use sets  $W$  in  $2^{\mathbb{P}}$  as their oracles either by converting them to subsets of

$\mathbb{N}$  or by viewing them as subsets of  $\mathbb{N}$  already (which happen to be subsets of  $\mathbb{P}$ ). The *HTP* operator defined above specifically uses a subset of  $\mathbb{P}$  as its input, so we will generally stick to subsets of  $\mathbb{P}$  in our notation in this article.

Occasionally we will consider the jump  $HTP(R)'$  of a set  $HTP(R)$ , using the conversion just described. Recall that the *jump*  $W'$  of a set  $W \subseteq \mathbb{N}$  is essentially the Halting Problem, relativized to  $W$ :

$$W' = \{e \in \mathbb{N} : \Phi_e^W(e) \text{ halts}\},$$

where  $\Phi_e$  is the  $e$ -th *oracle Turing machine*, and  $\Phi_e^W$  denotes the partial function computed by this machine when it runs with  $W$  as its oracle. (Details about jumps, Turing reducibility, and oracle Turing computation may be found in many standard sources, such as [13, Chap. III].)

One normally views subsets of  $\mathbb{P}$  as paths through the tree  $2^{<\mathbb{P}}$ , a complete binary tree whose nodes are the functions from initial segments of the set  $\mathbb{P}$  into the set  $\{0, 1\}$ . This allows us to introduce a topology on the space  $2^{\mathbb{P}}$  of paths through  $2^{<\mathbb{P}}$ , and thus on the class  $\mathbf{Sub}(\mathbb{Q})$  of all subrings of  $\mathbb{Q}$ . Each basic open set  $\mathcal{U}_\sigma$  in this topology is described by a node  $\sigma$  on the tree:  $\mathcal{U}_\sigma = \{W \subseteq \mathbb{P} : \sigma \subset W\}$ , where  $\sigma \subset W$  denotes that when  $W$  is viewed as a function from  $\mathbb{P}$  into the set  $2 = \{0, 1\}$  (i.e., as an infinite binary sequence),  $\sigma$  is an initial segment of that sequence. Also, we put a natural measure  $\mu$  on  $\mathbf{Sub}(\mathbb{Q})$ : just transfer to  $\mathbf{Sub}(\mathbb{Q})$  the obvious Lebesgue measure on the power set  $2^{\mathbb{P}}$  of  $\mathbb{P}$ . Thus, if we imagine choosing a subring  $R$  by flipping a fair coin (independently for each prime  $p$ ) to decide whether  $\frac{1}{p} \in R$ , the *measure* of a subclass  $\mathcal{S}$  of  $\mathbf{Sub}(\mathbb{Q})$  is the probability that the resulting subring will lie in  $\mathcal{S}$ .

It is also natural, and in certain respects more productive, to consider Baire category theory on the space  $\mathbf{Sub}(\mathbb{Q})$ , as an alternative to measure theory. For background regarding Baire category theory on subrings of  $\mathbb{Q}$ , we refer the reader to [9], while parallel discussion of measure theory occurs in [10]. Due to the common subject matter of those articles and this one, there is a substantial overlap between the introductions and background sections of the three papers, which we trust the reader to forgive. Naturally, we have also made every effort to maintain the same notation across both papers.

## 2 Subrings of the Rationals

Now we recall certain specific results about subrings of  $\mathbb{Q}$ . For all  $W \subseteq \mathbb{P}$ , writing  $R_W$  for  $\mathbb{Z}[W^{-1}]$  as before, we have the Turing reductions

$$W \oplus \text{HTP}(\mathbb{Q}) \leq_T \text{HTP}(R_W) \leq_T W'.$$

Indeed, each of these two Turing reductions is a 1-reduction; details appear in [10, §2.2]. Recall that the *semilocal* subrings of  $\mathbb{Q}$  are precisely those of the form  $R_W$  where the set  $W$  is cofinite in  $\mathbb{P}$ , containing all but finitely many primes. It will be important for us to know that whenever  $R$  is a semilocal subring of  $\mathbb{Q}$ , we have  $\text{HTP}(R) \leq_1 \text{HTP}(\mathbb{Q})$ . Indeed, both the Turing reduction and the 1-reduction are uniform in the complement. This result, which follows from Corollary 2.2 below, began with work of Julia Robinson in [11]. A proof by Eisenträger, Park, Shlapentokh, and the author appears in [4], based in turn on work by Koenigsmann in [6].

**Proposition 2.1** (see Proposition 5.4 in [4]) *For every prime  $p$ , there is a polynomial  $g_p(Z, X_1, X_2, X_3)$  such that for all rationals  $q$ , we have*

$$q \in R_{(\mathbb{P}-\{p\})} \iff g_p(q, \vec{X}) \in \text{HTP}(\mathbb{Q}).$$

Moreover,  $g_p$  may be computed uniformly in  $p$ .

**Corollary 2.2** *For each finite subset  $A_0 \subseteq \mathbb{P}$ , a polynomial  $f(Z_0, \dots, Z_n)$  has a solution in  $R_{(\mathbb{P}-A_0)}$  if and only if*

$$(f(\vec{Z}))^2 + \sum_{p \in A_0, j \leq n} (g_p(Z_j, X_{1,j,p}, X_{2,j,p}, X_{3,j,p}))^2$$

has a solution in  $\mathbb{Q}$ .

## 3 Diophantine Undefinability of the Jump

The full result of Matiyasevich, Davis, Putnam and Robinson says that not only is  $\text{HTP}(\mathbb{Z})$  undecidable, but in fact the Halting Problem  $\emptyset'$  is *diophantine in  $\mathbb{Z}$* , or expressible in  $\mathbb{Z}$  by a diophantine equation. That is, there exists a polynomial  $f \in \mathbb{Z}[X, Y_1, \dots, Y_n]$  (for some  $n$ ) such that

$$(\forall x \in \mathbb{Z}) [x \in \emptyset' \iff f(x, Y_1, \dots, Y_n) \in \text{HTP}(\mathbb{Z})].$$

Likewise, for any other ring  $R$ , the sets  $S$  diophantine in  $R$  are those subsets of  $R$  definable in the same way by a polynomial in  $R[X, Y_1, \dots, Y_n]$  for some  $n$ . When dealing with subrings  $R$  of  $\mathbb{Q}$ , we usually consider only subsets of  $\mathbb{Z}$ , often of  $\mathbb{N}$ , using a computable bijection between  $\mathbb{Q}$  and  $\mathbb{N}$  if needed.

In our collection of subrings of  $\mathbb{Q}$ ,  $\mathbb{Z}$  is the subring  $R_\emptyset$ , and one naturally asks whether the proof above carries over to all  $W \subseteq \mathbb{P}$ : is  $W'$  always diophantine in  $R_W$ ? Of course, a positive answer would immediately prove the undecidability of  $HTP(\mathbb{Q})$ , by taking  $W = \mathbb{P}$ . In fact, though, the answer is quickly seen to be negative. Indeed, with a little more work, we will show it to be negative in almost all cases.

The easy negative answers arise from taking the set  $W$  to be computably enumerable but not computable. For example,  $W$  might be the image in  $\mathbb{P}$  of  $\emptyset'$  itself, under the computable bijection from  $\mathbb{N}$  onto  $\mathbb{P}$ . We would then apply the following basic result. (The reader may wish to recall the notion of a 1-reduction from  $A$  to  $B$ , which is a computable total function  $h : \mathbb{N} \rightarrow \mathbb{N}$  such that all  $x \in \mathbb{N}$  satisfy  $(x \in A \iff h(x) \in B)$ .)

**Lemma 3.1** *If  $W$  is a computably enumerable subset of  $\mathbb{P}$ , then  $HTP(R_W)$  is also c.e., and therefore  $HTP(R_W) \leq_T \emptyset'$ . Indeed there exists a 1-reduction, i.e.,  $HTP(R_W) \leq_1 \emptyset'$ .*

*More generally, when  $W$  is c.e. relative to an oracle  $V$ ,  $HTP(R_W) \leq_1 V'$ .*

*Proof.* In the more general setting, let  $\langle W_s \rangle_{s \in \mathbb{N}}$  be a  $V$ -computable enumeration of  $W$ . Then  $HTP(R_W)$  is just the set of those  $f \in \mathbb{Z}[\vec{X}]$  for which

$$(\exists s)(\exists \vec{x}, y \in \mathbb{Z}) \left[ f \left( \frac{x_1}{y}, \dots, \frac{x_n}{y} \right) = 0 \text{ \& all primes dividing } y \text{ lie in } W_s \right].$$

Thus  $HTP(R_W)$  is defined by a condition existential relative to  $V$  (since  $V$  can compute every  $W_s$  uniformly in  $s$ ), and so  $HTP(R_W)$  is  $V$ -c.e. It is then standard (see [13, Thm III.2.3(iii)]) that  $HTP(R_W) \leq_1 V'$ . ■

So in the case where  $W \subseteq \mathbb{P}$  is the image of  $\emptyset'$  (under our computable bijection from  $\mathbb{N}$  onto  $\mathbb{P}$ ), we have  $HTP(R_W) \leq_T \emptyset'$  but  $W' = \emptyset'' \not\leq_T \emptyset'$ , and therefore  $W' \not\leq_T HTP(R_W)$ . Indeed, this holds whenever the set  $W$  is c.e. but *non-low*. Recall that a set  $W$  is *low* if  $W' \leq_T \emptyset'$ : this holds of all computable sets  $W$ , of course, but also of certain noncomputable sets  $W$ , both c.e. and otherwise. However, many non-low c.e. sets exist, including  $\emptyset'$  itself, and all of these satisfy  $W' \not\leq_T HTP(R_W)$ .

Generalizing this result requires theorems proven by Jockusch in [5] and by his student Kurtz in his Ph.D. thesis [7].

**Theorem 3.2 (Jockusch & Kurtz)** *The set*

$$\{S \subseteq \mathbb{N} : (\exists V <_T S) S \text{ is c.e. relative to } V\}$$

*has measure 1 in  $2^{\mathbb{N}}$  (Kurtz) and is comeager there, in the sense of Baire category (Jockusch).*

A set  $S$  in this collection is said to be *relatively c.e.*, as it is c.e. in  $V$  but not computable in  $V$ . Every set  $S$  is c.e. relative to itself, of course, but the condition that  $S \not\leq_T V$  implies that the jumps of these sets satisfy  $S' \not\leq_1 V'$ . (This follows from the Jump Theorem; see, e.g., [13, Theorem III.2.3].)

We can now apply Lemma 3.1 to each relatively c.e. set  $S$ , with image  $W \subseteq \mathbb{P}$ , to see that

$$W' \not\leq_1 \text{HTP}(R_W).$$

Indeed, with  $V$  as in the definition, the lemma proves  $\text{HTP}(R_W) \leq_1 V'$ , whereas  $W' \not\leq_1 V'$  as noted above. In particular, this yields our next corollary.

**Corollary 3.3** *The set of those  $W \subseteq \mathbb{P}$  such that the jump  $W'$  is not diophantine in  $R_W$  has full Lebesgue measure within  $2^{\mathbb{P}} = \mathbf{Sub}(\mathbb{Q})$ , the space of all subrings of  $\mathbb{Q}$ , and is comeager there in the sense of Baire category.*

*Proof.* If  $W'$  is diophantine in  $R_W$ , via an  $f \in \mathbb{Z}[X, \vec{Y}]$  such that

$$(\forall x \in \mathbb{N}) [x \in W' \iff f(x, Y_1, \dots, Y_n) \in \text{HTP}(R_W)],$$

then we have a 1-reduction from  $W'$  to  $\text{HTP}(R_W)$ , by mapping each  $x \in \mathbb{N}$  to the polynomial  $f(x, Y_1, \dots, Y_n)$ . But the discussion above shows that such 1-reductions exist only for a measure-0, meager class of sets  $W \subseteq \mathbb{P}$ . ■

So the situation for  $\mathbb{Z}$  is an anomaly among the subrings of  $\mathbb{Q}$ . This is not too surprising:  $\mathbb{Z}$  is very far from being a generic subring of  $\mathbb{Q}$ , in any sense of the word “generic.” Nevertheless, it is good to understand that most subrings of  $\mathbb{Q}$  do not behave the same way as  $\mathbb{Z}$ .

It is natural to ask whether one could extend the above result. In the original example, with  $W$  as the image of  $\emptyset'$  in  $\mathbb{P}$ , we had not only  $W' \not\leq_1 \text{HTP}(R_W)$ , but actually  $W' \not\leq_T \text{HTP}(R_W)$ : there was no Turing reduction,

let alone a 1-reduction. This would follow more generally for those sets  $W$  which are c.e. relative to some oracle  $V$  such that  $W' \not\leq_T V'$ . (One might call such a  $W$  *relatively non-low-c.e.*) However, these sets are far less common: the class of subsets of  $\mathbb{N}$  which are relatively non-low-c.e. in this sense has measure 0 in Cantor space, and is meager there. Of course, this does not automatically mean that  $HTP(R_W)$  must compute  $W'$  for the remaining sets  $W$  either. (If it did, then Corollary 1 from [9] would yield the dramatic conclusion that  $\emptyset' \leq_T HTP(\mathbb{Q})$ .)

## 4 Number Theory

Our principal tool for proving Theorem 5.1 and its corollaries, the chief remaining results in this article, will be the equations  $X^2 + qY^2 = 1$ . In this section we prove the relevant number-theoretic results. First we show that for each odd prime  $q$ , there is an infinite decidable set  $V$  of primes such that  $R_V$  contains no nontrivial solutions to  $X^2 + qY^2 = 1$ . (Here the trivial solutions are  $(\pm 1, 0)$ , which in Section 5 will be ruled out as solutions, at the cost of turning  $(X^2 + qY^2 - 1)$  into a messier polynomial.)

**Definition 4.1** For a fixed odd prime  $q$ , a prime  $p$  is *q-appropriate* if  $p$  is odd and  $p \neq q$  and  $\left(\frac{-q}{p}\right) = 1$  (that is,  $-q$  is a square modulo  $p$ ).

The crux of Lemmas 4.2 and 4.4 is that the  $q$ -appropriate primes are precisely the possible factors of the denominator in a nontrivial solution to  $x^2 + qy^2 = 1$ , thus justifying the term *q-appropriate*.

**Lemma 4.2** *Fix an odd prime  $q$ , and let  $x$  and  $y$  be positive rational numbers with  $x^2 + qy^2 = 1$ . Then every odd prime factor  $p$  of the least common denominator  $c$  of  $x$  and  $y$  must be  $q$ -appropriate.*

*For  $q \equiv 3 \pmod{4}$ ,  $p$  is  $q$ -appropriate if and only if  $p$  is a square modulo  $q$ .*

*For  $q \equiv 1 \pmod{4}$ , the situation is a little more complicated. Now a prime  $p$  is  $q$ -appropriate if and only if one of the following holds:*

- $p \equiv 1 \pmod{4}$  and  $p$  is a square modulo  $q$ .
- $p \equiv 3 \pmod{4}$  and  $p$  is not a square modulo  $q$ .

*Proof.* Suppose that  $a, b, c$  are positive integers, with no common factor, satisfying  $a^2 + qb^2 = c^2$ . If  $p$  divides  $c$ , then it cannot divide  $b$  (lest it also divide  $a$ ), and so  $\left(\frac{a}{b}\right)^2 \equiv -q \pmod{p}$ . Thus every such  $p$  is  $q$ -appropriate. Suppose in addition that  $q \equiv 3 \pmod{4}$ . If  $p \equiv 1 \pmod{4}$ , then  $-1$  is also a square mod  $p$ , so  $q$  is a square mod  $p$ , and by quadratic reciprocity  $p$  must be a square mod  $q$ . On the other hand, if  $p \equiv 3 \pmod{4}$ , then  $-1$  is not a square mod  $p$ , so  $q$  is not either; but with both  $p$  and  $q$  congruent to  $3 \pmod{4}$ , quadratic reciprocity now shows that  $p$  is again a square mod  $q$ . (The number-theoretic results here may be found in any standard text on the subject, e.g., [12].)

When  $q \equiv 1 \pmod{4}$ , a similar analysis, with careful use of quadratic reciprocity, gives the result stated in the lemma.  $\blacksquare$

**Corollary 4.3** *Let  $3 = q_0 < q_1 < \dots$  be the odd prime numbers. Then, for every  $e \in \mathbb{N}$ , there are infinitely many primes  $p$  that are  $q_e$ -appropriate but (for all  $i < e$ ) are not  $q_i$ -appropriate.*

*Proof.* A famous theorem of Dirichlet (see [12, Chap. 6, §4]) states that every arithmetic progression  $\{m + kn : k \in \mathbb{N}\}$  with  $m$  and  $n$  relatively prime contains infinitely many primes. Therefore, the corollary holds for  $e = 0$ , as Lemma 4.2 shows that all primes congruent to  $1 \pmod{3}$  are 3-appropriate. Lemma 4.2 also noted that the situation is more complicated for  $q_1 = 5$  than for  $q_0 = 3$ , because  $q_1 \equiv 1 \pmod{4}$ . Therefore the full property required for our inductive hypothesis (below) is that all primes  $p \equiv 5 \pmod{12}$  are 3-inappropriate.

Now assume inductively that there is some residue  $n$  modulo the product  $(4q_0 \cdots q_{e-1})$  for which  $n \equiv 1 \pmod{4}$  and no  $q_i$  with  $i < e$  divides  $n$ , and such that every prime  $p$  with residue  $n \pmod{(4q_0 \cdots q_{e-1})}$  is  $q_i$ -inappropriate for all  $i < e$ . (For  $e = 1$ , we saw above that  $n = 5$  works.) There are  $q_e$  distinct elements  $n + k(4q_0 \cdots q_{e-1})$  in  $\mathbb{Z}/(4q_0 \cdots q_e)$ , and their residues modulo  $q_e$  are all distinct, hence include all of the elements of  $\mathbb{Z}/(q_e)$ . Lemma 4.2 shows that there are  $\frac{q_e-1}{2}$  residues  $m$  in  $\mathbb{Z}/(4q_0 \cdots q_e)$  such that all primes with residue  $m$  there will be  $q_e$ -appropriate but  $q_i$ -inappropriate for all  $i < e$ . By Dirichlet's theorem, for each such  $m$ , the arithmetic progression  $\{m + k(4q_0 \cdots q_e) : k \in \mathbb{N}\}$  will contain infinitely many primes, proving the Corollary for  $e$ . On the other hand, another  $\frac{q_e-1}{2}$  distinct residues  $m$  in  $\mathbb{Z}/(4q_0 \cdots q_e)$  have the property that all primes with that residue are  $q_i$ -inappropriate for all  $i < e + 1$ , and that none of  $q_0, \dots, q_e$  divides  $m$ . Therefore the inductive hypothesis still holds for  $e$ , allowing the induction to proceed.



It should be noted that the factor of 4 in  $(4q_0 \cdots q_e)$  allowed us to avoid the bifurcation in Lemma 4.2. With the relevant residue  $n$  equivalent to  $1 \pmod{4}$ , we know that, for all primes  $p$  with that residue,  $q_e$ -appropriateness simply means being a square modulo  $q_e$ . ■

In fact, the solutions of  $X^2 + qY^2 = 1$  are precisely the pairs of the form

$$\left( \pm \frac{m^2 - qn^2}{m^2 + qn^2}, \pm \frac{2mn}{m^2 + qn^2} \right)$$

for relatively prime integers  $m, n \in \mathbb{N}$ , not both zero, with the trivial solutions  $(\pm 1, 0)$  corresponding to  $m = 0$  and to  $n = 0$ . Up to sign, the rational nontrivial solution  $(\frac{a}{c}, \frac{b}{c})$  arises from the integers  $m = a - c$  and  $n = b$ . For each prime  $q$ , we call a solution  $(a, b, c)$  to  $X^2 + qY^2 = Z^2$  *primitive* if  $a, b, c$  are pairwise relatively prime positive integers.

**Lemma 4.4** *Suppose that  $p$  and  $q$  are odd primes and  $p$  is  $q$ -appropriate. Then there is a primitive solution  $(a, b, p^k)$  to  $X^2 + qY^2 = Z^2$  with  $k \geq 1$ . Hence there is a nontrivial solution to  $X^2 + qY^2 = 1$  in the ring  $\mathbb{Z}[\frac{1}{p}]$ .*

*Proof.* Since  $-q$  is a square mod  $p$ , the ideal  $(p)$  in the ring of integers  $\mathcal{O}$  of  $K = \mathbb{Q}[\sqrt{-q}]$  splits into distinct prime factors,  $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ . If  $d$  is the order of  $\mathfrak{p}$  in the ideal class group of  $\mathcal{O}$ , then  $\mathfrak{p}^d = (\alpha)$  is a principal ideal. The norm  $N: K \rightarrow \mathbb{Q}$ , given by  $N(x + y\sqrt{-q}) = x^2 + qy^2$  for  $x, y \in \mathbb{Q}$ , is multiplicative and so  $N(\alpha) = p^d$ . In case  $q \equiv 1 \pmod{4}$ , the ring of integers  $\mathcal{O} = \mathbb{Z}[\sqrt{-d}]$ . Then  $\alpha = a + b\sqrt{-q}$  with  $a, b \in \mathbb{Z}$  and we are done.

If  $q \equiv 3 \pmod{4}$ , then  $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{-q}}{2}]$ . Suppose that  $\alpha = \frac{a+b\sqrt{-q}}{2}$  for odd integers  $a, b$ . We have  $a^2 + b^2q = 4p^d$ , so  $1 + q \equiv 4 \pmod{8}$  and thus  $q \equiv 3 \pmod{8}$ . An elementary computation then shows that  $\alpha^3 \in \mathbb{Z}[\sqrt{-q}]$ , so we replace  $\alpha$  by  $\alpha^3$ . In either case, note that  $\alpha$  is not divisible by  $p$  in  $\mathcal{O}$ , so we obtain a primitive solution to  $X^2 + qY^2 = Z^2$  of the required form.

To get fancier, one can use the Chebotarev density theorem to choose  $p$  so that it splits completely in the Hilbert class field of  $\mathbb{Q}[\sqrt{-q}]$ . Then the ideal  $\mathfrak{p} = (\alpha)$  already is principal and we can take  $d = 1$  above. (However, if  $q \equiv 3 \pmod{8}$ , we might still need  $\alpha^3$  to get rid of the 2 in the denominator.) We refer the reader to §5 and §9 of [2] for more information. ■

## 5 Turing Inequivalence

Now we apply Lemmas 4.2 and 4.4 to study the  $HTP$  operator. It is already known that two Turing-inequivalent sets  $U$  and  $V$  can have  $HTP(R_U) \equiv_T HTP(R_V)$ . For a standard example, let  $U = \emptyset$  and  $V = \emptyset'$ : then  $HTP(R_U)$  has Turing degree  $\mathbf{0}'$ , by the result of Matiyasevich, Davis, Putnam, and Robinson; whereas  $HTP(R_V)$  is a c.e. set (hence  $\leq_T \emptyset'$ ) which computes  $V$ , and thus also has degree  $\mathbf{0}'$ . The situation here, with  $V \equiv_T U'$ , is the maximum possible difference between sets  $U$  and  $V$  with Turing-equivalent  $HTP$ 's, since one always has  $V \leq_T HTP(R_V) \equiv_T HTP(R_U) \leq_T U'$  and vice versa. The situation with the jump operator is similar in that Turing-inequivalent sets  $U$  and  $V$  can have Turing-equivalent jumps, as with the low noncomputable sets discussed in Section 3. However, the difference between the sets  $U$  and  $V$  could not be a full jump. (That is,  $U' \not\leq_T V$  and  $V' \not\leq_T U$  whenever  $U' \equiv_T V'$ .)

The jump operator does preserve Turing reducibility, as discussed earlier. In fact,  $U \leq_T V$  if and only if there is a 1-reduction from  $U'$  to  $V'$ , by the standard computability result known as the Jump Theorem (see e.g. [13, Theorem III.2.3]). In contrast, we now prove that the  $HTP$  operator does not preserve Turing reducibility. Indeed, we will construct a set  $U$  for which  $HTP(R_{\bar{U}})$  is Turing-equivalent to the jump  $HTP(R_U)'$ . Once again, this is the maximum possible difference, since  $V \leq_T U$  implies

$$HTP(R_V) \leq_T V' \leq_T U' \leq_T HTP(R_U)',$$

with the final reduction holding because  $U \leq_T HTP(R_U)$ . The strong equivalence between our set  $U$  and its complement  $\bar{U}$  makes this all the more striking: the two sets are Turing-equivalent via a bounded-truth-table reduction of norm 1. (One might pursue this further, asking whether computably isomorphic sets  $U$  and  $V$  must have  $HTP(R_U) \equiv_T HTP(R_V)$ . The set  $U$  we build will not be computably isomorphic to its complement, and this question remains open.)

**Theorem 5.1** *There is a computably enumerable subset  $U$  of  $\mathbb{P}$ , with complement  $\bar{U}$  in  $\mathbb{P}$ , for which  $HTP(R_{\bar{U}})$  computes the set  $\mathbf{Fin} = \{e : W_e \text{ is finite}\}$ , and therefore  $HTP(R_U)' \equiv_T HTP(R_{\bar{U}})$ .*

*Proof.* With  $U$  c.e., we will have  $HTP(R_U) \leq_T \emptyset'$ , and the Jump Theorem then shows that  $HTP(R_U)' \leq_T \emptyset''$ . The second jump  $\emptyset''$  is well known to

satisfy  $\emptyset'' \equiv_T \mathbf{Fin}$  (see, e.g., [13, Theorem IV.3.2]). Conversely,  $HTP(R_{\overline{U}}) \leq HTP(R_U)'$ , as discussed above. Therefore, it is sufficient for us to enumerate a set  $U$ , effectively, so that  $\mathbf{Fin} \leq_T HTP(R_{\overline{U}})$ .

We accomplish this by using the polynomials  $X^2 + q_e Y^2 - 1$ , where  $q_e$  is the  $e$ -th odd prime. Of course, it is desirable to exclude the trivial solutions  $(\pm 1, 0)$ , so in fact we define

$$f_e(X, Y, \vec{Z}, \vec{T}) = (X^2 + q_e Y^2 - 1)^2 + (Y(Z_1^2 + \dots + Z_4^2 + 1) - (T_1^2 + \dots + T_4^2 + 1))^2.$$

The second square forces  $Y$  to be a quotient of positive rationals, hence positive. (With a slightly more complicated polynomial, we could allow negative values of  $Y$  as well, but this is unnecessary here.) Conversely, the Four Squares Theorem shows that every positive rational  $y$  can be expressed as such a quotient with all  $z_i$  and  $t_i$  lying in  $\mathbb{Z}$ . Therefore, for every subring  $R$  of  $\mathbb{Q}$ ,  $f_e$  lies in  $HTP(R)$  just if  $R$  contains elements  $x$  and  $y > 0$  for which  $x^2 + q_e y^2 = 1$ .

As seen in Section 4, it is decidable which odd primes  $p$  can be factors of the common denominator of an  $x$  and a  $y$  with  $x^2 + q_e y^2 = 1$ . Lemma 4.2 showed that every such  $p$  has  $\left(\frac{-q_e}{p}\right) = 1$ . (Recall that we defined such a prime  $p$  to be  $q_e$ -appropriate.) Lemma 4.4 proved the converse in a strong way, establishing that whenever  $\left(\frac{-q_e}{p}\right) = 1$ , the ring  $\mathbb{Z}[\frac{1}{p}]$  already contains such an  $x$  and  $y$ . We will enumerate 2 into our set  $U$  immediately, leaving all odd primes  $p$  as candidates for  $\overline{U}$ . Then we will enumerate the  $q_e$ -appropriate primes  $p$  into  $U$  (that is, out of  $\overline{U}$ ) one by one, as we discover new elements of the c.e. set  $W_e$ . The goal is that, if  $W_e$  is infinite, then all  $q_e$ -appropriate primes should be removed from  $\overline{U}$ , so that  $HTP(R_{\overline{U}})$  will not contain  $f_e$ . (This goal will not quite be fully achieved, but we will come close enough to make our proof work.) However, at any given stage  $s$ , some particular  $q_e$ -appropriate prime  $p_{e,s}$  will be *protected by  $q_e$* , meaning that for the sake of  $q_e$ , we will keep  $p_{e,s}$  in  $\overline{U}$  as long as  $W_e$  does not acquire any more elements. If  $W_e$  is indeed finite, then some particular  $p_{e,s}$  will be protected from some stage  $s_0$  onwards, and therefore will lie in  $\overline{U}$ , forcing  $f_e$  to lie in  $HTP(R_{\overline{U}})$ . This will allow our decision procedure for  $\mathbf{Fin}$  below an oracle for  $HTP(R_{\overline{U}})$  to succeed.

We use the standard computable enumeration  $\langle W_{e,s} \rangle_{e,s \in \mathbb{N}}$  of all computably enumerable sets. This enumeration has the property that for each  $s$ , there is exactly one  $e$  with  $W_{e,s+1} \neq W_{e,s}$ , and that for this one  $e$ ,  $W_{e,s+1}$  contains all elements of  $W_{e,s}$  and exactly one more element as well.

At stage 0, we set  $U_0 = \{2\}$ , and (for every  $e \geq 0$ ) define  $q_e$  to be the  $e$ -th odd prime and  $p_{e,0}$  to be the least  $q_e$ -appropriate prime which does not lie in  $\{p_{0,0}, \dots, p_{e-1,0}\}$ . At each stage  $s$ , the prime  $p_{e,s}$  is said to be *protected* by  $q_e$  at stage  $s$ , although the choice of a protected prime may change from one stage to the next. This  $p_{e,s}$  is the prime which  $q_e$  currently desires to keep in  $\overline{U}$ .

At stage  $s + 1$ , we find the unique  $e$  with  $W_{e,s+1} \neq W_{e,s}$ . This stage is evidence that this particular  $W_e$  may be infinite, and so we enumerate  $p_{e,s}$  into  $U_{s+1}$ . For each  $i < e$ , we keep  $p_{i,s+1} = p_{i,s}$ . For each  $j \geq e$  (in increasing order), we choose  $p_{j,s+1}$  to be the least  $q_j$ -appropriate prime which is not in  $U_{s+1}$  and which does not lie in  $\{p_{0,s+1}, \dots, p_{j-1,s+1}\}$ . This clearly makes  $p_{e,s+1} \neq p_{e,s}$ , since  $p_{e,s} \in U_{s+1}$ , but it may leave many subsequent  $p_{j,s+1}$  equal to  $p_{j,s}$ : the only reason why  $p_{j,s+1}$  might not equal  $p_{j,s}$  (for  $j > e$ ) is if  $p_{j,s}$  has now been chosen as  $p_{k,s+1}$  for some  $e \leq k < j$ , i.e., if  $p_{j,s}$  is now protected by a higher-priority  $q_k$ . This completes stage  $s + 1$ , and we define the c.e. set  $U = \cup_s U_s$ .

We can describe the arc of a single odd prime  $p$  through this process. Certain primes  $p$  might never be protected by any  $q_e$ : such a  $p$  will lie in  $\overline{U}$ . If at some stage  $p$  becomes protected, say  $p = p_{e,s}$ , then three things can happen. If no higher-priority  $q_i$  (that is, with  $i < e$ ) subsequently decides to protect  $p$ , then either  $W_e$  eventually receives a new element and enumerates  $p$  into  $U$ , or else  $W_e$  never receives any new elements and  $p$  stays in  $\overline{U}$ . The third possibility is that some higher-priority  $q_i$  does subsequently protect  $p$ , at a stage  $s' > s$ , so that  $p = p_{i,s'}$  but now  $p \neq p_{e,s'}$ . In this case, the same analysis now applies with  $q_i$  in place of  $q_e$ . This  $p$  could subsequently be protected by yet another  $q_{i'}$  with  $i' < i$ , but the protecting index can only change finitely often, of course, since it decreases every time. Thus, if  $p$  ever becomes protected by any  $q_e$ , it will either wind up in  $U$  or else become the limiting value  $p_i$  for some  $i \leq e$ .

We now prove, by induction on  $e$ , that the sequence  $\langle p_{e,s} \rangle_{s \in \mathbb{N}}$  stabilizes on a limit  $p_e \notin U$  if  $W_e$  is a finite set, but increases without bound if  $W_e$  is infinite. Let  $F = \{i < e : W_i \text{ is finite}\}$ . First suppose that  $W_e$  is finite. By induction, we may fix a stage  $s_0$  such that for all  $i \in F$ , the value of  $p_{i,s_0}$  never changes at stages  $s > s_0$ , and we may also assume  $s_0$  to be sufficiently large that  $W_{e,s_0} = W_e$ . By Corollary 4.3, there exist infinitely many primes which are  $q_e$ -appropriate but are  $q_i$ -inappropriate for all  $i < e$ . Such primes will never be enumerated into  $U$  by any  $q_i$  with  $i < e$  at any stage, and only finitely many of them can lie in the finite set  $U_{s_0}$ . Since the prime  $p_{e,s}$

is always chosen to be the least available  $q_e$ -appropriate prime not already protected by a higher-priority  $q_i$ , one of these infinitely many primes will eventually be chosen as  $p_{e,s}$  (unless the sequence  $\langle p_{e,s} \rangle_{s \in \mathbb{N}}$  stabilizes on some other prime), and from that stage on,  $q_e$  will continue to protect that same prime  $p_{e,s}$ : no higher-priority requirement enumerates it into  $U$  because it is  $q_i$ -inappropriate for all  $i < e$ ;  $q_e$  itself will not enumerate it into  $U$  because  $W_e$  never again receives a new element; and once this  $p_{e,s}$  has been selected, no  $q_j$  with  $j > e$  ever again chooses it as  $p_{j,s}$ , so no lower-priority  $q_j$  ever enumerates it into  $U$ . Therefore the limiting value  $p_e$  exists as required and lies in  $\bar{U}$ .

On the other hand, suppose  $W_e$  is infinite, and fix a stage  $s_0$  after which, for all  $i \in F$ , the value of  $p_{i,s_0}$  never changes again. Then at each of the infinitely many subsequent stages at which  $W_{e,s+1} \neq W_{e,s}$ ,  $q_e$  will enumerate the current  $p_{e,s}$  into  $U_{s+1}$  and will choose a new  $p_{e,s+1}$ . This  $p_{e,s+1}$  is always the least  $q_e$ -appropriate prime not yet in  $U$  and not currently protected by any  $q_i$  with  $i < e$ . Now each  $i \in F$  has  $p_{i,s} = p_{i,s_0}$  at all these stages, and this limiting value  $p_i$  will lie in  $\bar{U}$ . Some of these finitely many primes may be  $q_e$ -appropriate, and so there may be as many as  $e$   $q_e$ -appropriate primes in  $\bar{U}$ . However, apart from these  $p_i$ , every  $q_e$ -appropriate prime will eventually be enumerated into  $U$ . (If not, then the least  $q_e$ -appropriate prime not among these protected  $p_i$ 's and not in  $U$  can never have been protected by any  $q_i$  with  $i < e$ , since such an  $i$  would not lie in  $F$ , and the least  $i$  such that  $q_i$  ever protected  $p$  would therefore have eventually put that  $p$  into  $U$ . But since  $p$  was not protected by any  $i < e$ , it will have been chosen as  $p_{e,s}$  once all smaller  $q_e$ -appropriate primes not protected by higher-priority requirements have been enumerated into  $U$ , and therefore it too will have been enumerated into  $U$ .) Therefore,  $p_{e,s}$  does increase without bound as  $s \rightarrow \infty$ , as claimed. Moreover, with at most  $e$  exceptions, all  $q_e$ -appropriate primes lie in  $U$ .

Now we give a procedure which uses an  $HTP(R_{\bar{U}})$ -oracle to compute **Fin**. The procedure decides, for each  $e = 0, 1, 2, \dots$  in turn, whether  $W_e$  is infinite or not. For  $e = 0$ , this is easy, since there is no higher-priority  $q_i$  than  $q_0$ . If  $W_0$  is infinite,  $U$  contains all  $q_0$ -appropriate primes, and so  $f_0 \notin HTP(R_{\bar{U}})$ . On the other hand, if  $W_0$  is finite, then  $U$  contains only finitely many  $q_0$ -appropriate primes, and in particular does not contain  $p_0 = \lim_s p_{0,s}$ . By Lemma 4.4, the subring  $\mathbb{Z}[\frac{1}{p_0}]$  of  $R_{\bar{U}}$  contains a solution to  $f_e$ , and so  $f_0 \in HTP(R_{\bar{U}})$ .

The procedure now continues by recursion on  $e$ , having determined the

finite set  $F$  of values  $i < e$  lying in **Fin**. When it reaches  $e$ , it runs the enumeration of  $U$  until it finds a stage  $s$  for which every  $p_{i,s}$  with  $i \in F$  lies in  $\bar{U}$ . (Recall that  $\bar{U} \leq_T \text{HTP}(R_{\bar{U}})$ , so our oracle allows the procedure to determine membership in  $\bar{U}$ .) Such a stage must exist, since every such  $W_i$  is finite. For the least  $i_0 \in F$ ,  $p_{i_0,s}$  can never have become  $p_{i',s'}$  for any  $i' < i_0$  at any  $s' > s$ , since then it would have entered  $U$ ; thus  $p_{i_0,s} = p_{i_0}$ . But then the same argument shows inductively for each  $i \in F$  that  $p_{i,s} = p_i$ . Therefore the procedure uses Proposition 2.1 to find a polynomial  $g_e$  which lies in  $\text{HTP}(R_{\bar{U}})$  if and only if  $f_e \in \text{HTP}(R_{\bar{U}-\{p_i:s:i \in F\}})$ . The oracle then reveals whether  $g_e \in \text{HTP}(R_{\bar{U}})$ , which in turn determines whether  $e \in \mathbf{Fin}$ , since  $f_e$  has a solution in the subring  $R_{\bar{U}-\{p_i:i \in F\}}$  just if  $W_e$  was finite.

It now follows that  $\mathbf{Fin} \leq_T \text{HTP}(R_{\bar{U}})$ , and therefore  $\text{HTP}(R_{\bar{U}})$  has Turing degree at least  $\mathbf{0}''$ , the degree of **Fin**. In fact,  $\text{HTP}(R_{\bar{U}})$  has exactly this degree, since it must be computable from  $\text{HTP}(R_U)'$ , and with  $U$  computably enumerable,  $\text{HTP}(R_U)$  is c.e. as well, forcing  $\text{HTP}(R_U) \leq_T \emptyset'$  and thus  $\text{HTP}(R_U)' \leq_T \emptyset''$ . ■

In the proof of Theorem 5.1, our only concern was to make  $\text{HTP}(R_{\bar{U}})$  compute  $\emptyset''$ , while keeping  $U$  c.e. Now we consider ways to augment this construction. First, it is not be difficult to use a further finite-injury procedure to enhance the construction so that it satisfies requirements to ensure that the c.e. set  $U$  be *HTP-generic*, in addition to satisfying  $\text{HTP}(R_U) \leq_T \emptyset'$  and  $\emptyset'' \leq \text{HTP}(R_{\bar{U}})$ . The concept of HTP-genericity is defined and fully explained in [9, Defn. 2]. Roughly, it means that for every polynomial  $f$ , there is a finite initial segment of  $U$  which either ensures that  $f \in \text{HTP}(R_U)$  or else ensures that  $f \notin \text{HTP}(R_U)$ . It then follows that  $\text{HTP}(R_U) \equiv_T U \oplus \text{HTP}(\mathbb{Q})$ . However, it would not be possible to make the complement  $\bar{U}$  HTP-generic: each  $e \notin \mathbf{Fin}$  yields a polynomial  $f \notin \text{HTP}(R_{\bar{U}})$  that has solutions in infinitely many rings  $\mathbb{Z}[\frac{1}{p}]$ , and therefore no finite subset of the complement of  $\bar{U}$  (i.e., of  $U$ ) suffices to guarantee that  $f \notin \text{HTP}(R_{\bar{U}})$ . (In the construction above, the  $f$  in question is the polynomial  $g_e$  for this  $e$ , since it is necessary to rule out the finitely many primes  $p_i$  with  $i < e$  and  $i \in \mathbf{Fin}$ .) Indeed,  $\bar{U}$  must be HTP-nongeneric in order to satisfy  $\text{HTP}(R_{\bar{U}}) >_T \emptyset'$  and  $\bar{U} \leq_T \emptyset'$ .

Computability theorists familiar with *high permitting* will see a further enhancement to Theorem 5.1: one can make  $U$  Turing-reducible to any given high c.e. set  $C$ . This is more delicate, and we explain it in Corollary 5.2 below, although the proof will be intelligible mainly to those already familiar

with permitting arguments. Finally, it is not difficult to make  $C \leq_1 U$ , by a coding argument that requires only a straightforward finite-injury process. Therefore, the fully decorated version of Theorem 5.1 is as follows.

**Corollary 5.2** *For every c.e. set  $C$  of high degree (i.e., with  $\emptyset'' \leq_T C'$ ), there exists a c.e. set  $U \subseteq \mathbb{P}$  with  $U \equiv_T C$  such that*

$$HTP(R_U) \equiv_T U \oplus HTP(\mathbb{Q}) \leq_T \emptyset' \quad \& \quad HTP(R_{\bar{U}}) \equiv_T \emptyset''.$$

*Proof.* Without rewriting the entire construction, we give reasonable details about the new condition of high permitting, referring the reader to [1, Lemma 12.7.5] for background. Given any high c.e. degree  $\mathbf{c}$ , we can use high permitting below  $\mathbf{c}$  to guarantee that the set  $U$  constructed by the theorem satisfies  $\deg(U) \leq_T \mathbf{c}$ . Indeed, there must be a c.e. set  $C \in \mathbf{c}$  with a computable enumeration  $\langle C_s \rangle_{s \in \mathbb{N}}$  such that the computation function of  $C$  using this enumeration dominates every total computable function. High permitting requires that we allow a prime  $p$  to enter  $U$  only at a stage  $s + 1$  such that  $C_{s+1} \upharpoonright p \neq C_s \upharpoonright p$ . So, when  $W_{e,s+1} \neq W_{e,s}$  we may not be allowed to enumerate the current  $p_{e,s}$  into  $C$  immediately. Instead, we *mark* the prime for  $U$ , meaning that we put it on a waiting list to enter  $U$ . Whenever a number  $m$  appears in  $C_{t+1} - C_t$ , all primes  $> m$  currently on the waiting list are enumerated into  $U_{t+1}$ . This guarantees that  $U \leq_T C$ , since  $p \in U$  just if  $p \in U_s$ , where  $s$  is the least stage for which  $C_s \upharpoonright p = C \upharpoonright p$  (and this stage  $s$  can be computed given a  $C$ -oracle).

The principal difference from Theorem 5.1 is that with high permitting, in the case where  $e \in \mathbf{Inf}$ , we do not know exactly how many  $q_e$ -appropriate primes will have to be left out of  $U$ , although the total number left out will be finite. Therefore, instead of knowing exactly which question to ask about  $HTP(R_{\bar{U}})$  to determine whether an  $e$  lies in  $\mathbf{Inf}$ , we will need to resort to a search for a finite set  $A \subseteq \mathbb{P}$  such that  $f_e \notin HTP(R_{\bar{U}-A})$ , employing Corollary 2.2. This will yield an enumeration of  $\mathbf{Inf}$ , and we will then apply Lemma 5.4 (below). In turn, we will need to ensure, whenever  $e \in \mathbf{Fin}$ , not just that one particular limit prime  $p_e$  remains in  $\bar{U}$ , but that infinitely many  $q_e$ -appropriate primes lie in  $\bar{U}$ ; otherwise we would mistakenly enumerate  $e$  into  $\mathbf{Inf}$ . The full requirement is:

$$\mathcal{R}_e : e \in \mathbf{Inf} \iff \bar{U} \text{ contains only finitely many } q_e\text{-appropriate primes.}$$

Therefore, for each  $e$  at stage  $s$ , we do not define just one protected prime  $p_{e,s}$ , but rather make a list  $p_{e,0,s} < p_{e,1,s} < \dots$  of all  $q_e$ -appropriate primes

not yet marked for  $U$  nor protected by any higher-priority requirement at that stage. The basic rule of protection is that  $\mathcal{R}_{e+1}$  cannot mark  $p_{e,0,s}$  for  $U$  (but can mark any other  $p_{e,j,s}$ ),  $\mathcal{R}_{e+2}$  cannot mark  $p_{e,0,s}$  nor  $p_{e,1,s}$ , and in general  $\mathcal{R}_{e+k}$  cannot mark any  $p_{e,j,s}$  with  $j < k$  for  $U$ . By the same token,  $\mathcal{R}_e$  itself cannot mark  $p_{e-1,0,s}$ , nor  $p_{e-2,0,s}$  nor  $p_{e-2,1,s}$ , etc., so these primes will not be chosen as  $p_{e,i,s}$ . This protection rule ensures that  $\mathcal{R}_e$  will not force any lower-priority  $\mathcal{R}_i$  to leave infinitely many  $q_i$ -appropriate primes in  $\overline{U}$ , but also that  $\mathcal{R}_e$  will have a choice of infinitely many  $q_e$ -appropriate primes to keep in  $\overline{U}$  if necessary, since at some stage  $s$  some  $q_{e+1}$ -inappropriate prime will be chosen as  $p_{e,1,s}$ , and later some  $p_{e,2,s}$  will be chosen which is both  $q_{e+1}$ -inappropriate and  $q_{e+2}$ -inappropriate, and so on. Indeed, if  $e \in \mathbf{Fin}$ , then every limit  $p_{e,i} = \lim_s p_{e,i,s}$  will exist and all these limits  $p_{e,i}$  will be  $q_e$ -appropriate and will lie in  $\overline{U}$ . On the other hand, if  $e \in \mathbf{Inf}$ , then all but finitely many  $q_e$ -appropriate primes will be chosen as  $p_{e,i,s}$  at some stage (or else enumerated into  $U$  by a higher-priority requirement), and thus all but finitely many  $q_e$ -appropriate primes will eventually be marked for  $U$ , since the current  $p_{e,0,s}$  is marked for  $U$  whenever  $W_e$  gets a new element.

Next, it is necessary to see that for each  $e \in \mathbf{Inf}$ , only finitely many of the  $q_e$ -appropriate primes ever marked for  $U$  fail to enter  $U$ . To see this, notice that if  $p$  is marked for  $U$  at a stage  $s$  but never enters  $U$ , then  $C_s \upharpoonright p = C \upharpoonright p$ . That is,  $s \geq C_C(p)$ , where  $C_C$  is the computation function defined (as in [1, p. 230]) so that  $C_C(x)$  is the least  $s$  with  $C_s \upharpoonright x = C \upharpoonright x$ . By hypothesis, this  $C_C$  is not only noncomputable, but dominates every total computable function. If  $e \in \mathbf{Inf}$ , then as seen above, cofinitely many  $q_e$ -appropriate primes are eventually marked for  $U$  by  $q_e$ . Let these primes be  $p_0 < p_1 < \dots$ , which is a computable infinite sequence, and define the computable function  $f(n)$  to equal the stage at which  $p_n$  is marked for  $U$ . Since  $C_C$  dominates  $f$ , there are only finitely many  $n$  with  $f(n) \geq C_C(n)$ , and for all other  $n$  than these, we have  $C_{f(n)} \upharpoonright n \neq C \upharpoonright n$ , by the definition of  $C_C$ . But  $p_n > n$ , so also  $C_{f(n)} \upharpoonright p_n \neq C \upharpoonright p_n$ , and therefore, at some stage  $s$  after the stage  $f(n)$  at which  $p_n$  is marked for  $U$ ,  $C$  will permit  $p_n$  to enter  $U$ . Therefore  $\mathcal{R}_e$  is indeed satisfied.

This being the case, Proposition 2.1 now allows us to use an oracle for  $HTP(R_{\overline{U}})$  to enumerate the set  $\mathbf{Inf}$ , the complement of  $\mathbf{Fin}$ . To do so, for each  $e \in \mathbb{N}$ , we simply go through the finite initial segments  $A_n = \{2, 3, 5, \dots, p_n\}$  of  $\mathbb{P}$ , one set at a time. (One could speed up this process by considering only initial segments of the set of  $q_e$ -appropriate primes.) For



each  $n$ , we ask the  $HTP(R_{\bar{U}})$ -oracle whether the polynomial

$$(f_e(X, Y))^2 + \sum_{i \leq n} (g_{p_i}(X, Z_1, Z_2, Z_3))^2 + \sum_{i \leq n} (g_{p_i}(Y, T_1, T_2, T_3))^2$$

has a solution in  $R_{\bar{U}}$ . If the answer is ever negative, then we know that  $e \in \mathbf{Inf}$ , because all  $q_e$ -appropriate primes except those in that  $A_n$  must lie in  $U$ . On the other hand, if no  $n$  ever yields a negative answer, then infinitely many  $q_e$ -appropriate primes must lie in  $\bar{U}$ , and so  $e \in \mathbf{Fin}$ . Therefore, Lemma 5.4 below proves that  $\mathbf{Inf} \leq_T HTP(R_{\bar{U}})$ . Adding the coding of  $C$  into  $U$  is standard, as is the addition of requirements for HTP-genericity, since both of these are finitary and blend easily with the permitting. ■

The next result follows directly from Corollary 5.2, but is nevertheless quite striking: the  $HTP$  operator can reverse strict Turing reducibility.

**Corollary 5.3** *There exist subrings  $R$  and  $S$  of  $\mathbb{Q}$  with  $R <_T S$ , yet with  $HTP(S) <_T HTP(R)$ .*

*Proof.* Using an incomplete high c.e. set  $C$  in Corollary 5.2 yields a set  $\bar{U} < \emptyset'$ , yet  $HTP(R_{\emptyset'}) \equiv_T \emptyset'$ , being c.e., while  $\emptyset'' \leq HTP(R_{\bar{U}})$ . So we let  $R = R_{\bar{U}}$  and  $S = R_{\emptyset'}$ . (Alternatively, as the c.e. sets are dense under Turing reducibility, we could fix a c.e. set  $D$  with  $C <_T D <_T \emptyset'$  and apply Corollary 5.2 to  $D$ , getting  $S = R_V$  for a c.e. set  $V \equiv_T D$ , and coding  $\emptyset'$  into  $HTP(R_V)$  as well, thus making  $S <_T HTP(S)$ .) ■

It remains to establish the lemma required for Corollary 5.2, which is a standard computability result.

**Lemma 5.4 (Folklore)** *For subsets  $A \subseteq \mathbb{N}$ ,*

$$\mathbf{Inf} \leq_T A \iff \mathbf{Inf} \text{ is } A\text{-computably enumerable.}$$

*Proof.* The forward direction is immediate. For the converse, we define a computable total function  $f$  so that, for every  $e$ ,

$$\varphi_{f(e)}(s) = \begin{cases} 0, & \text{if } \varphi_{e,s}(e) \uparrow; \\ \uparrow, & \text{if } \varphi_{e,s}(e) \downarrow. \end{cases}$$

Thus  $e$  lies in the complement  $\bar{\emptyset}'$  of the Halting Problem if and only if the domain  $W_{f(e)}$  of  $\varphi_{f(e)}$  is infinite. (That is,  $f$  is an  $m$ -reduction from  $\bar{\emptyset}'$  to  $\mathbf{Inf}$ .) But now, since  $\mathbf{Inf}$  is  $A$ -computably enumerable, so is  $\bar{\emptyset}'$ , and therefore, with an  $A$ -oracle, we can compute  $\emptyset'$ . Since  $\mathbf{Fin}$  is c.e. relative to  $\emptyset'$ , and  $\mathbf{Inf}$  is already  $A$ -c.e., this allows us to compute  $\mathbf{Inf}$  using an  $A$ -oracle. ■

High permitting, as opposed to ordinary c.e. permitting, does appear necessary in Corollary 5.2. With ordinary permitting, one could ensure that infinitely many  $q_e$ -appropriate primes entered  $U$ , in the case where  $e \in \mathbf{Inf}$ , but the requirement that cofinitely many should enter  $U$  requires the strength of high permitting. If we could have enumerated an HTP-generic  $U$  below a non-high  $C$ , while still achieving  $\emptyset'' \leq_T \text{HTP}(R_{\overline{U}})$ , then  $\text{HTP}(R_U)$  would have been high, hence  $>_T U$ , which (with  $U$  HTP-generic) would have established the undecidability of  $\text{HTP}(\mathbb{Q})$ . The fact that this is not possible here essentially means that, while  $\text{HTP}(\mathbb{Q})$  may yet turn out to be undecidable, the polynomials  $X^2 + qY^2 = 1$  are not complex enough to prove it.

As a final side note, it is possible to adjust the construction in Theorem 5.1 to ensure that  $\emptyset'' \leq_1 \text{HTP}(R_{\overline{U}})$ . Since  $\text{HTP}(R_{\overline{U}}) \leq_1 U'$ , this situation is only possible when  $\emptyset' \leq_T U$ , and thus cannot be accomplished using high permitting below an incomplete set  $C$ . The adjustment simply stipulates that, for all  $e < i$ , if  $p$  is the least prime which is  $q_j$ -appropriate for all  $e \leq j \leq i$ , then the only  $q_i$ -appropriate primes which  $q_e$  ever restrains from entering  $U$  should be those which are  $\leq p$ . When  $q_e$  redefines its protected prime  $p_{e,s+1}$ , it takes this stipulation into account, and immediately enumerates into  $U$  all  $q_e$ -appropriate primes in the interval  $[p_{e,s}, p_{e,s+1})$  which are not protected by higher-priority requirements.

## 6 Questions

There is a natural analogy between the *HTP* operator, mapping  $W$  to  $\text{HTP}(R_W)$ , and the *jump operator*, mapping  $W$  to  $W'$ .  $W'$  and  $\text{HTP}(R_W)$  are both  $W$ -computably enumerable, and as noted earlier, the basic situation for Turing reducibility is that

$$W \leq_T \text{HTP}(R_W) \leq_T W',$$

with equality possible at either end, though of course not at both ends simultaneously.

The analogy is strengthened by the parallels between  $\text{HTP}(\mathbb{Q})$  and the Halting Problem  $\emptyset'$ . The class  $\mathbf{GL}_1$  of *generalized low<sub>1</sub> sets*  $W$  is defined by the property

$$W' \equiv_T \emptyset' \oplus W.$$

This class is comeager of measure 1 in Cantor space. Of course,  $\emptyset' \oplus W \leq_1 W'$  always holds. The opposite reduction is trickier: it does fail on a meager set

of measure 0, and even within  $\mathbf{GL}_1$  it is in general only a Turing reduction, not a 1-reduction. This opposite reduction holds uniformly on a comeager class, but not on any class of measure 1. That is, there is a single Turing functional  $\Phi$  such that

$$\{W \subseteq \mathbb{N} : \chi_{W'} = \Phi^{\theta' \oplus W}\}$$

is comeager, but for every  $\Phi$ , this class fails to have measure 1. (Given  $\varepsilon > 0$ , one can choose a  $\Phi$  for which it has measure  $> 1 - \varepsilon$ , and the choice of the program for  $\Phi$  is uniform in  $\varepsilon \in \mathbb{Q}$ .)

For the *HTP* operator, the analogy leads us to the *HTP*-generic sets, introduced at the end of the previous section. All *HTP*-generic sets  $W \subseteq \mathbb{P}$  satisfy

$$HTP(R_W) \equiv_T HTP(\mathbb{Q}) \oplus W.$$

The class of *HTP*-generic sets is comeager in Cantor space, but its measure there is unknown. Again,  $HTP(\mathbb{Q}) \oplus W \leq_1 HTP(R_W)$  always holds, while in the opposite direction, Turing reducibility holds uniformly on a comeager class. (Details about these results appear in [9].)

On the other hand, as the name implies, the *HTP*-generic sets are defined by a genericity property. In this sense, they are analogous to the *1-generic sets*: those  $U \subseteq \mathbb{N}$  such that, for every  $e$ ,

$$\mu(\{V \in 2^{\mathbb{N}} : (\exists n) V \upharpoonright n \in W_e\}) = 1 \implies (\exists n) U \upharpoonright n \in W_e.$$

The *1-generic sets* are precisely the points in  $2^{\mathbb{N}}$  at which the jump operator is continuous, and likewise the *HTP*-generic sets are precisely the points of continuity of the *HTP* operator. All *1-generic sets* lie in  $\mathbf{GL}_1$ , and the *1-generic sets* form a comeager class in  $2^{\mathbb{N}}$ , but of measure 0. Our analogy therefore suggests that the class of *HTP*-generic sets, already known to be comeager, might also have measure 0.

As one can see, the state of knowledge about *HTP*-genericity is strong *vis-à-vis* Baire category, but less so *vis-à-vis* Lebesgue measure. In work yet to appear, the author has shown that if the *HTP*-generic sets have measure 1, then there can be no existential definition of  $\mathbb{Z}$  in the field  $\mathbb{Q}$ . Theorem 5.1 here is an existence theorem, but says nothing about measure; indeed, the construction used in its proof involves (for certain values  $e$ , namely those in  $\mathbf{Inf}$ ) enumerating cofinitely many  $q_e$ -appropriate primes into  $U$ . Clearly, even for a single  $q$ , the class of sets  $U \subseteq \mathbb{P}$  which contain all but finitely many

of the  $q$ -appropriate primes is a class of measure 0, and so Theorem 5.1 yields no conclusions about measure. New results regarding measure would be of real interest for the study of definability and decidability in number theory.

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