Interpreting a field in its Heisenberg group

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January 19, 2021

Abstract

We improve on and generalize a 1960 result of Maltsev. For a field F, we denote by $H(F)$ the Heisenberg group with entries in F. Maltsev showed that there is a copy of F defined in $H(F)$, using existential formulas with an arbitrary non-commuting pair of elements as parameters. We show that F is interpreted in $H(F)$ using computable Σ_1 formulas with no parameters. We give two proofs. The first is an existence proof, relying on a result of Harrison-Trainor, Melnikov, R. Miller, and Montalbán. This proof allows the possibility that the elements of F are represented by tuples in $H(F)$ of no fixed arity. The second proof is direct, giving explicit finitary existential formulas that define the interpretation, with elements of F represented by triples in $H(F)$. Looking at what was used to arrive at this parameter-free interpretation of F in $H(F)$, we give general conditions sufficient to eliminate parameters from interpretations.

1 Introduction

The Heisenberg group of a field F is the upper-triangular subgroup of $GL_3(F)$ in which all matrices have 1's along the diagonal and 0's below it. Maltsev showed that there are existential formulas with parameters, which, for every field F, define F in its Heisenberg group $H(F)$. In this article we will produce existential formulas without parameters, which, for every field F, interpret F in $H(F)$. Observing what is used to obtain this result, we will then formulate a general result on removing parameters from an interpretation.

Languages are assumed to be computable, and structures are assumed to have universe a subset of ω. For a given structure $\mathcal{A}$, the atomic diagram $D(\mathcal{A})$ may be identified, via Gödel numbering, with a subset of ω. We then identify

*The first, second, fourth, fifth, sixth, seventh and ninth authors are grateful for support from NSF grant DMS #1600625. The first, third, and fifth authors also acknowledge support from NSF grant DMS #1800692. The fourth author was partially supported by Grant # 429466 from the Simons Foundation. The seventh author was partially supported by Grant # 581896 from the Simons Foundation and by the City University of New York PSC-CUNY Research Award Program. The eighth author was partially supported by BNSF, DN 02/16, and SU Science Fund, 80-10-128/16.04.2020.

$\mathcal{A}$ itself with the characteristic function of $D(\mathcal{A})$. Classes of structures have a fixed language, and are closed under isomorphism. The following notion, of "Turing computable embedding," is from [1], based on the earlier notion of "Borel embedding" from [2].

Definition 1.1. *For classes K, K', we say that K is* Turing computably embedded *in K', and we write $K \leq_{tc} K'$, if there is a Turing operator $\Theta : K \to K'$ such that for all $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \cong \mathcal{B}$ iff $\Theta(\mathcal{A}) \cong \Theta(\mathcal{B})$.*

Medvedev reducibility is used to compare "problems," where a *problem* is a subset of ω^ω. The problems that concern us have the form "build a copy of $\mathcal{A}$."

Definition 1.2. *For structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is* Medvedev reducible *to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$, if there is a Turing operator Φ that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.*

We are interested in "uniform" Medvedev reductions, which, for a given Turing computable embedding Θ, take any copy of a structure in the range of Θ to a copy of its pre-image.

Definition 1.3. *Let Θ be a Turing computable embedding of a class K to a class K'. We say that the structures in K are* uniformly Medvedev reducible *to their Θ-images in K', if there is a Turing operator Φ such that for all $\mathcal{A} \in K$, Φ serves as a Medvedev reduction of $\mathcal{A}$ to $\Theta(\mathcal{A})$.*

Often, when we have a Turing computable embedding $\Theta : K \to K'$ with a uniform Medvedev reduction of the structures in K to their Θ-images, it is because there are simple formulas that define, for all $\mathcal{A} \in K$, an interpretation of $\mathcal{A}$ in $\Theta(\mathcal{A})$. Montalbán defined a very general kind of interpretation of $\mathcal{A}$ in $\mathcal{B}$ that yields a uniform Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$. In this definition, the tuples from $\mathcal{B}$ that represent elements of $\mathcal{A}$ may have arbitrary arity. The interpretation is defined by formulas that have no specific arity. Here, the arity of a formula is the number of its free variables. As usual, we often write $\mathcal{B}$ both for the structure and its domain.

Definition 1.4 (Generalized computable Σ_1-definition)**.** *Let $R \subseteq \mathcal{B}^{<\omega}$, and let $\varphi_n(\bar{x}_n)_{n\in\omega}$ be a computable sequence of computable Σ_1 formulas, where $\varphi_n(\bar{x}_n)$ has arity n. If for each n, $\varphi_n(\bar{x}_n)$ defines $R \cap \mathcal{B}^n$, then we say that $\bigvee_n \varphi_n(\bar{x}_n)$ is a* generalized computable Σ_1 definition *of R.*

Since a generalized computable Σ_1 formula allows consideration of tuples of all finite arities, it is technically not in $L_{\omega_1\omega}$; however, it is a computable disjunction, over all $n \in \omega$, of $L_{\omega_1\omega}$ formulas φ_n with free variables $x_1, \dots, x_n$. Generalized computable Σ_1 formulas are involved in the following definition.

Definition 1.5 (Montalbán)**.** *For a relational structure $\mathcal{A} = (A, (R_i)_{i\in I})$ and a structure $\mathcal{B}$, we say $\mathcal{A}$ is* effectively interpreted *in $\mathcal{B}$ if there exist a set $D \subseteq \mathcal{B}^{<\omega}$ and relations $\sim$ and R_i^* on D such that*

1. *$(D, (R_i^*)_{i\in I})/_\sim \cong \mathcal{A}$,*

2. *there is a computable sequence of generalized computable Σ_1 formulas, with no parameters, defining the set D and the following relations on D: $\sim$ and the complementary relation $\not\sim$, and for each i, the relation R_i^* and the complementary relation $\neg R_i^*$.*

Notation and terminology: We may later simply write $\pm\sim$ (or $\pm R_i^*$) for the complementary pair of relations $\sim$ and $\not\sim$ (or R_i^* and $\neg R_i^*$). We may think of the pair of generalized computable Σ_1 formulas that define the complementary pair pair $\pm\sim$ (or $\pm R_i^*$) as a generalized Δ_1 definition of $\sim$ (or R_i^*).

Remark: The concept of Σ-definability, which is a staple of logic in the Russian tradition, is closely related to effective interpretability.

Below, we illustrate the use of tuples of arbitrary arity.

Proposition 1.1. *If $\mathcal{A}$ is computable, then it is effectively interpreted in all structures $\mathcal{B}$.*

Proof. Let $D = \mathcal{B}^{<\omega}$. Let $\bar{b} \sim \bar{c}$ if $\bar{b}, \bar{c}$ are tuples of the same length. For simplicity, suppose $\mathcal{A} = (\omega, R)$, where R is binary. If $\mathcal{A} \models R(m, n)$, let $R^*(\bar{b}, \bar{c})$ for all $\bar{b}$ of length m and $\bar{c}$ of length n. Then $(D, R^*)/_\sim \cong \mathcal{A}$. □

If $\mathcal{A}$ is a computable linear ordering and $\mathcal{B}$ is a structure for the empty language, then Proposition 1.1 says that $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, but there is clearly no interpretation of the usual kind, with relations of fixed arity, defined by $L_{\omega_1\omega}$ formulas.

The following definition was first presented as [9, Defn. 3.1].

Definition 1.6. *A* computable functor *from $\mathcal{B}$ to $\mathcal{A}$ is a pair (Φ, Ψ) of Turing operators such that:*

1. *Φ takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$,*

2. *Ψ takes each triple $(\mathcal{B}_1, f, \mathcal{B}_2)$ such that $\mathcal{B}_i \cong \mathcal{B}$ for $i = 1, 2$ and $\mathcal{B}_1 \cong_f \mathcal{B}_2$ to a function g such that $\Phi(\mathcal{B}_1) \cong_g \Phi(\mathcal{B}_2)$. Moreover, Ψ preserves identity and composition.*

Harrison-Trainor, Melnikov, Miller, and Montalbán proved the following in [3]. A subsequent generalization appears in [4].

Theorem 1.2. *For a pair of structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:*

1. *$\mathcal{A}$ is effectively interpreted in $\mathcal{B}$,*

2. *there is a computable functor from $\mathcal{B}$ to $\mathcal{A}$.*

Remarks: In the proof of Theorem 1.2, it is important that D consist of tuples of arbitrary arity. Proposition 1.1 said that a computable structure $\mathcal{A}$ can be effectively interpreted in an arbitrary structure $\mathcal{B}$. We proved this by specifying an interpretation, in which D was the set of all tuples from $\mathcal{B}$. There is an alternative proof of Proposition 1.1, using Theorem 1.2. We define a computable functor (Φ, Ψ) from $\mathcal{B}$ to $\mathcal{A}$ in which Φ ignores the oracle and simply computes $\mathcal{A}$, while Ψ always computes the identity function.

In this article we will consider uniform effective interpretations and uniform computable functors.

Definition 1.7. *Suppose $K \leq_{tc} K'$ via Θ. The structures in K are* uniformly effectively interpreted *in their Θ-images if there is a fixed collection of generalized computable Σ_1 formulas (without parameters) that, for all $\mathcal{A} \in K$, define an interpretation of $\mathcal{A}$ in $\Theta(\mathcal{A})$.*

Definition 1.8. *Suppose $K \leq_{tc} K'$ via Θ. Turing operators Φ and Ψ form a* uniform computable functor *from the structures in the range of Θ to their pre-images provided that for all $\mathcal{A} \in K$, (Φ, Ψ) serves as a computable functor from $\Theta(\mathcal{A})$ to $\mathcal{A}$.*

Here is a uniform version of Theorem 1.2, also from [3].

Theorem 1.3. *For classes K, K' with $K \leq_{tc} K'$ via Θ, the following are equivalent:*

1. *there is a uniform effective interpretation of the structures $\mathcal{A} \in K$ in the corresponding structures $\Theta(\mathcal{A})$,*
2. *there is a uniform computable functor (Φ, Ψ) from the structures $\Theta(\mathcal{A})$ in the range of Θ to their pre-images $\mathcal{A}$.*

It is natural to ask whether, when $\mathcal{A} \leq_s \mathcal{B}$, there must be an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$. It is also natural to ask whether, when $\mathcal{A}$ is effectively interpreted in $(\mathcal{B}, \bar{b})$ with parameters $\bar{b}$, it must be effectively interpreted in $\mathcal{B}$ without parameters. Kalimullin [6] gave examples providing negative answers to both questions.

In [7], Maltsev defined a Turing computable embedding of fields – indeed, of all rings – into 2-step nilpotent groups. The embedding takes each field F to its *Heisenberg group* $H(F)$. To show that the embedding preserves isomorphism, Maltsev gave uniform existential formulas defining a copy of F in $H(F)$. The definitions involved a pair of parameters, whose orbit is defined by an existential formula (in fact, the formula is quantifier-free). In Section 2, we recall Maltsev's definitions. In Section 3, we describe a uniform computable functor that, for all countable F, takes copies of $H(F)$, with their isomorphisms, to copies of F, with corresponding isomorphisms. By Theorem 1.3, it follows that there is a uniform effective interpretation of countable fields F in their Heisenberg groups $H(F)$ with no parameters. In Section 4, we give explicit finitary existential formulas that define such an interpretation, and also show that parameter-free interpretations necessarily involve an equivalence relation $\sim$ distinct from equality. (Thus, while one can interpret F in $H(F)$ without parameters, one cannot define F in $H(F)$ without parameters.) In Section 5, we note that although F is effectively interpretable in $H(F)$ and $H(F)$ is effectively interpretable in F, we do not, in general, have effective bi-interpretability. In Section 6, we generalize our process of passing from Maltsev's definition, with parameters, to the uniform effective interpretation, with no parameters.

2 Defining F in $H(F)$

In this section, we recall Maltsev's embedding of fields in 2-step nilpotent groups, and his formulas that define a copy of the field in the group. Recall that for a field F, the Heisenberg group $H(F)$ is the set of matrices of the form

$$h(a,b,c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with entries in F. Note that $h(0,0,0)$ is the identity matrix. We are interested in non-commuting pairs in $H(F)$. One such pair is $(h(1,0,0), h(0,1,0))$. For $u = h(u_1,u_2,u_3)$ and $v = h(v_1,v_2,v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.$$

For a group G, we write $Z(G)$ for the center. For group elements x, y, the *commutator* is $[x,y] = x^{-1}y^{-1}xy$. The following technical lemma provides much of the information we need to show that F is defined, with parameters, in $H(F)$.

Lemma 2.1.

1. *(a) For u and v, the commutator, $[u,v]$, is $h(0,0,\Delta_{(u,v)})$, and*

   (b) $[u,v] = 1$ iff $\Delta_{(u,v)} = 0$.

2. *Let $u = h(u_1,u_2,u_3)$, and let $v = h(v_1,v_2,v_3)$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $u \in Z(H(F))$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $[u,v] = 1$ iff there exists α such that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.*

3. *The center $Z(H(F))$ consists of the elements of the form $h(0,0,c)$.*

4. *If $[u,v] \neq 1$, then $x \in Z(H(F))$ iff $[x,u] = [x,v] = 1$.*

Proof. For Part 1, (a) is proved by direct computation, and (b) follows from (a). Parts 2 and 3 are easy consequences of Part 1. We prove Part 4. Suppose $[u,v] \neq 1$. If $x \in Z(H(F))$, then it commutes with both u and v. We must show that if x commutes with both u and v, then $x \in Z(H(F))$. Let $u = h(u_1,u_2,u_3)$,

$v = h(v_1, v_2, v_3)$, and $x = h(x_1, x_2, x_3)$. By Part 2, since $[x, u] = 1$, there exists α such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Similarly, since $[x, v] = 1$, there exists β such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Since the vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are linearly independent, this implies that $\alpha = \beta = 0$. It follows that $x_1 = x_2 = 0$, so $x \in Z(H)$. □

Corollary 2.2. *If $x \in H(F)$ is fixed by all automorphisms of $H(F)$, then $x = 1$.*

Proof. Write $x = h(a, b, c)$. Lemma 2.1(3) shows $a = b = 0$, since all conjugations fix x. But the automorphism of $H(F)$ mapping $h(x, y, z)$ to $h(y, x, xy - z)$, which interchanges $h(1, 0, 0)$ with $h(0, 1, 0)$, maps $h(0, 0, c)$ to $h(0, 0, -c)$, hence shows that $c = 0$ as well. □

The next lemma tells us how, for any non-commuting pair u, v in the group $(H(F), *)$, we can define operations $+$ and $\cdot$, and an isomorphism f from F to $(Z(H(F)), +, \cdot)$.

Lemma 2.3. *Let $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$ be a non-commuting pair. Assume that $\alpha, \beta, \gamma \in F$. Let $x = h(0, 0, \alpha \cdot \Delta_{(u,v)})$, $y = h(0, 0, \beta \cdot \Delta_{(u,v)})$, and $z = h(0, 0, \gamma \cdot \Delta_{(u,v)})$. Then*

1. *$\alpha + \beta = \gamma$ iff $x * y = z$, where $*$ is the matrix multiplication.*
2. *$\alpha \cdot \beta = \gamma$ iff there exist x' and y' such that $[x', u] = [y', v] = 1$, $[u, y'] = y$, $[x', v] = x$, and $z = [x', y']$.*

Proof. For Part 1, matrix multiplication yields the fact that

$$h(0, 0, a) * h(0, 0, b) = h(0, 0, a + b) .$$

Then $\alpha + \beta = \gamma$ iff

$$x * y = h(0, 0, \alpha \cdot \Delta_{(u,v)}) * h(0, 0, \beta \cdot \Delta_{(u,v)}) = h(0, 0, \gamma \cdot \Delta_{(u,v)}) = z .$$

For Part 2, first suppose that $\alpha \cdot \beta = \gamma$. We take $x' = h(\alpha \cdot u_1, \alpha \cdot u_2, 0)$, and $y' = h(\beta \cdot v_1, \beta \cdot v_2, 0)$. Then $\Delta_{(x',u)} = 0$, so $[x', u] = h(0, 0, 0) = 1$. Similarly, $[y', v] = 1$. Also, $\Delta_{(x',v)} = \alpha \cdot \Delta_{(u,v)}$, so $[x', v] = h(0, 0, \alpha \cdot \Delta_{(u,v)}) = x$. Similarly, $\Delta_{(u,y')} = \beta \cdot \Delta_{(u,v)}$, so $[u, y'] = h(0, 0, \beta \cdot \Delta_{(u,v)}) = y$. Finally, $\Delta_{(x',y')} = \alpha \cdot \beta \cdot \Delta_{(u,v)} = \gamma \cdot \Delta_{(u,v)}$, so $[x', y'] = h(0, 0, \gamma \cdot \Delta_{(u,v)}) = z$.

Now, suppose we have x' and y' such that $[x', u] = [y', v] = 1$, $[u, y'] = y$, $[x', v] = x$, and $[x', y'] = z$. Say that $x' = h(x'_1, x'_2, x'_3)$ and $y' = h(y'_1, y'_2, y'_3)$. Since $[x', v] = x$, $\Delta_{(x',v)} = \alpha \cdot \Delta_{(u,v)}$, so $\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Since $[u, y'] = y$, $\Delta_{(u,y')} = \beta \cdot \Delta_{(u,v)}$, so $\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \beta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Combining these facts, we see that $\Delta_{(x',y')} = \begin{vmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{vmatrix} = \begin{vmatrix} \alpha \cdot u_1 & \beta \cdot v_1 \\ \alpha \cdot u_2 & \beta \cdot v_2 \end{vmatrix} = \alpha \cdot \beta \cdot \Delta_{(u,v)}$. Since $[x', y'] = z$, $\Delta_{(x',y')} = \gamma \cdot \Delta_{(u,v)}$. Since u and v do not commute, $\Delta_{(u,v)} \neq 0$. Therefore, $\alpha \cdot \beta = \gamma$. □

The main result of the section follows directly from Lemmas 2.1 and 2.3.

Theorem 2.4 (Maltsev, Morozov)**.** *For an arbitrary non-commuting pair (u, v) in $H(F)$, we get $F_{(u,v)} = (Z(H(F)), \oplus, \otimes_{(u,v)})$ where*

1. *$x \in Z(H(F))$ iff $[x, u] = [x, v] = 1$,*
2. *$\oplus$ is the group operation from $H(F)$,*
3. *$\otimes_{(u,v)}$ is the set of triples (x, y, z) such that there exist x', y' with $[x', u] = [y', v] = 1$, $[x', v] = x$, $[u, y'] = y$, and $[x', y'] = z$,*
4. *the function $g_{(u,v)}$ taking $\alpha \in F$ to $h(0, 0, \alpha \cdot \Delta_{(u,v)}) \in H(F)$ is an isomorphism between F and $F_{(u,v)}$.*

Note: From Part 4, it is clear that $h(0, 0, \Delta_{(u,v)})$ is the multiplicative identity in $F_{(u,v)}$—we may write $1_{(u,v)}$ for this element.

Proposition 2.5. *There is a uniform Medvedev reduction Φ of F to $H(F)$.*

Proof. Given $G \cong H(F)$, we search for a non-commuting pair (u, v) in G, and then use Maltsev's definitions to get a copy of F computable from G. □

It turns out that the Medvedev reduction Φ is half of a computable functor. In the next section, we explain how to get the other half.

3 The computable functor

In the previous section, we saw that for any field F and any non-commuting pair (u, v) in $H(F)$, there is an isomorphic copy $F_{(u,v)}$ of F defined in $H(F)$ by finitary existential formulas with parameters (u, v). The defining formulas are the same for all F. Hence, there is a uniform Turing operator Φ that, for all fields F, takes copies of $H(F)$ to copies of F. In this section, we describe a companion operator Ψ so that Φ and Ψ together form a uniform computable functor. For any field F, and any triple (G_1, p, G_2) such that G_1 and G_2 are copies of $H(F)$ and p is an isomorphism from G_1 onto G_2, the function $\Psi(G_1, p, G_2)$ must be an isomorphism from $\Phi(G_1)$ onto $\Phi(G_2)$, and, moreover, the isomorphisms given by Ψ must preserve identity and composition. We saw in the previous section that for any field F, and any non-commuting pair (u, v) in $H(F)$, the function $g_{(u,v)}$ taking α to $h(0, 0, \alpha \cdot \Delta_{(u,v)})$ is an isomorphism from F onto $F_{(u,v)}$. We use this $g_{(u,v)}$ below.

Lemma 3.1. *For all F and all non-commuting pairs (u, v), (u', v') in $H(F)$, there is a natural isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ onto $F_{(u',v')}$. Moreover, the family of isomorphisms $f_{(u,v),(u',v')}$ is functorial; i.e.,*

1. *for every non-commuting pair (u, v), the function $f_{(u,v),(u,v)}$ is the identity,*

2. for any three non-commuting pairs (u,v), (u',v'), and (u'',v''),

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')}.$$

Proof. We let $f_{(u,v),(u',v')} = g_{(u',v')} \circ g^{-1}_{(u,v)}$. This is an isomorphism from $F_{(u,v)}$ onto $F_{(u',v')}$. It is clear that $f_{(u,v),(u,v)}$ is the identity. Consider non-commuting pairs (u,v), (u',v'), and (u'',v''). We must show that $f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')} = f_{(u,v),(u'',v'')}$. We have:

$$\begin{aligned} f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')} &= g_{(u'',v'')} \circ g^{-1}_{(u',v')} \circ g_{(u',v')} \circ g^{-1}_{(u,v)} = \\ &= g_{(u'',v'')} \circ g^{-1}_{(u,v)} = \\ &= f_{(u,v),(u'',v'')}. \end{aligned}$$

□

The next lemma says that there is a uniform existential definition of the family of isomorphisms $f_{(u,v),(u',v')}$.

Lemma 3.2. *There is a finitary existential formula $\psi(u,v,u',v',x,y)$ that, for any two non-commuting pairs (u,v) and (u',v'), defines the isomorphism $f_{(u,v),(u',v')}$ taking $x \in F_{(u,v)}$ to $y \in F_{(u',v')}$.*

Proof. Since the operation $\otimes_{(u,v)}$ and the element $1_{(u',v')}$ are definable by $\exists$–formulas with parameters u,v and u',v' respectively, it suffices to prove the equivalence

$$f_{(u,v),(u',v')}(x) = y \Leftrightarrow x \otimes_{(u,v)} 1_{(u',v')} = y.$$

First assume that $f_{(u,v),(u',v')}(x) = y$, i.e., $y = g_{(u',v')} \circ g^{-1}_{(u,v)}(x)$. Let $\alpha = g^{-1}_{(u,v)}(x)$, i.e., $x = h(0,0,\alpha \cdot \Delta_{(u,v)})$. It follows that $y = h\left(0,0,\alpha \cdot \Delta_{(u',v')}\right)$. Then

$$\begin{aligned} x \otimes_{(u,v)} 1_{(u'v')} &= h\left(0,0,\alpha \cdot \Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\Delta_{(u',v')}\right) = \\ &= h\left(0,0,\alpha \cdot \Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\frac{\Delta_{(u',v')}}{\Delta_{(u,v)}} \cdot \Delta_{(u,v)}\right) = \\ &= h\left(0,0,\alpha \cdot \frac{\Delta_{(u',v')}}{\Delta_{(u,v)}} \cdot \Delta_{(u,v)}\right) = \\ &= h\left(0,0,\alpha \cdot \Delta_{(u',v')}\right) = y. \end{aligned}$$

Assume now that $x \otimes_{(u,v)} 1_{(u',v')} = y$ and let $x = h\left(0,0,\alpha \cdot \Delta_{(u,v)}\right)$. Then

$$\begin{aligned} y &= x \otimes_{(u,v)} 1_{(u',v')} = h\left(0,0,\alpha \cdot \Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\Delta_{(u',v')}\right) = \\ &= h\left(0,0,\alpha \cdot \Delta_{(u',v')}\right) = g_{(u',v')} \circ g^{-1}_{(u,v)}(x) = f_{(u,v),(u',v')}(x). \end{aligned}$$

□

We will use Lemmas 3.1 and 3.2 to prove the following.

Proposition 3.3. *There is a uniform computable functor that, for all fields F, takes $H(F)$ to F.*

Proof. Let Φ be the uniform Medvedev reduction of F to $H(F)$. Take copies G_1, G_2 of $H(F)$ and take p such that $G_1 \cong_p G_2$. We describe $q = \Psi(G_1, p, G_2)$ as follows. Let (u, v) be the first non-commuting pair in G_1, and let (u', v') be the first non-commuting pair in G_2. Now, p takes (u, v) to a non-commuting pair $(p(u), p(v))$, and p maps $F_{(u,v)}$ isomorphically onto $F_{(p(u),p(v))}$. The function $f_{(p(u),p(v)),(u',v')}$ is an isomorphism from $F_{(p(u),p(v))}$ onto $F_{(u',v')}$. We get an isomorphism q from $F_{(u,v)}$ onto $F_{(u',v')}$ by composing p with $f_{(p(u),p(v)),(u',v')}$. For $x \in F_{(u,v)}$, we let $q(x) = f_{(p(u),p(v)),(u',v')}(p(x))$. Since $f_{(p(u),p(v)),(u',v')}$ is defined by an existential formula, with parameters $p(u), p(v), u', v'$, we can apply a uniform effective procedure to compute q from (G_1, p, G_2).

If $G_1 = G_2$ and p is the identity, then $(u, v) = (u', v')$, and by Lemma 3.1, $f_{(u,v),(u',v')}$ is the identity. Consider G_1, G_2, G_3, all copies of G, with functions p_1, p_2 such that $G_1 \cong_{p_1} G_2$ and $G_2 \cong_{p_2} G_3$. Then $p_3 = p_2 \circ p_1$ is an isomorphism from G_1 onto G_3. Let $q_1 = \Psi(G_1, p_1, G_2)$, $q_2 = \Psi(G_2, p_2, G_3)$, and $q_3 = \Psi(G_1, p_3, G_3)$. We must show that $q_3 = q_2 \circ q_1$. The idea is to transfer everything to G_3 and use Lemma 3.1. Let r_1 be the result of transferring q_1 down to G_3, so $r_1 = f_{(p_3(u),p_3(v)),(p_2(u),p_2(v))}$. We have $q_1(x) = y$ if and only if $r_1(p_3(x)) = p_2(y)$. Let r_2 be the result of transferring q_2 down to G_3, so $r_2 = f_{(p_2(u),p_2(v)),(u,v)}$. We have $q_2(y) = z$ if and only if $r_2(p_2(y)) = z$. We let r_3 be the result of transferring q_3 down to G_3, so $r_3 = f_{(p_3(u),p_3(v)),(u,v)}$. We have $q_3(x) = z$ if and only if $r_3(p_3(x)) = z$. By Lemma 3.1, $r_3 = r_2 \circ r_1$. If $q_1(x) = y$ and $q_2(y) = z$, then $r_1(p_3(x)) = p_2(y)$, and $r_2(p_2(y)) = z$. Then $r_3(p_3(x)) = z$, so $q_3(x) = z$, as required. □

Corollary 3.4. *There is a uniform effective interpretation of F in $H(F)$.*

Proof. Apply the result from [3]. □

The result from [3] gives a uniform interpretation of F in $H(F)$, valid for all countable fields F, using computable Σ_1 formulas with no parameters. The tuples from $H(F)$ that represent elements of F may have arbitrary arity. In the next section, we will do better.

We note here that the uniform interpretation of F in $H(F)$ given in this section allows one to transfer computable-structure-theoretic properties (in particular, computable dimension) of any graph G to a 2-step nilpotent group, without introducing any constants. This is not a new result: in [8], Mekler gave a related coding of graphs into 2-step-nilpotent groups, which, in concert with the completeness of graphs for such properties (see [5]), yields the same fact. Mekler's goal was to transfer model-theoretic (stability) properties, not the completeness from computable structure theory. In [5], Hirschfeldt, Khoussainov, Shore, and Slinko used Maltsev's interpretation of an integral domain in its Heisenberg group with two parameters, along with the completeness of integral domains, to re-establish it. More recently, [9] demonstrated the completeness of fields, by coding graphs into fields, From that result, along with

Corollary 3.4 and the usual definition of $H(F)$ as a matrix group given by a set of triples from F, we achieve a coding of graphs into 2-step nilpotent groups, different from Mekler's coding, with no constants required.

4 Defining the interpretation directly

Our goal in this section is to give explicit finitary existential formulas that define a uniform effective interpretation of a field in its Heisenberg group. We discovered this interpretation by thinking of the computable functor and recalling the formulas that were used in proving Proposition 3.3 and Corollary 3.4. These formulas—Maltsev's formulas defining copies of the field using parameters, and Morozov's formula that defines isomorphisms between the copies—were all existential.

Theorem 4.1. *There are finitary existential formulas that, uniformly for every field F, define an effective interpretation of F in $H(F)$, with elements of F represented by triples of elements from $H(F)$.*

We offer intuition before giving the formal proof. The domain D of the interpretation will consist of those triples (u, v, x) from $H(F)$ with $uv \neq vu$ and x in the center: for each single (u, v), we apply Maltsev's definitions, with u, v as parameters, to get $F_{(u,v)} \cong F$. We view the triples arranged as follows:

$F_{(u,v)}$	$F_{(u',v')}$	$F_{(u'',v'')}$	$\cdots$
(u, v, x_0)	(u', v', x_0)	(u'', v'', x_0)	
(u, v, x_1)	(u', v', x_1)	(u'', v'', x_1)	
(u, v, x_2)	(u', v', x_2)	(u'', v'', x_2)	
(u, v, x_3)	(u', v', x_3)	(u'', v'', x_3)	
$\vdots$	$\vdots$	$\vdots$	

Here each column can be seen as $F_{(u,v)}$ for some non-commuting pair (u, v). Now the system of isomorphisms from Lemma 3.1 will allow us to identify each element in one column with a single element from each other column, and modding out by this identification will yield a single copy of F.

Proof. Let H be a group isomorphic to $H(F)$. Recalling the natural isomorphisms $f_{(u,v),(u',v')}$ defined in Lemma 3.1 for non-commuting pairs (u, v) and (u', v'), we define $D \subseteq H$, a binary relation $\sim$ on D, and ternary relations $\oplus$, $\odot$ (which are binary operations) on D, as follows.

1. D is the set of triples (u, v, x) such that $uv \neq vu$ and $xu = ux$ and $xv = vx$. (Notice that, no matter which non-commuting pair (u, v) is chosen, the set of corresponding elements x is precisely the center $Z(H)$, by Theorem 2.4.)

2. $(u,v,x) \sim (u',v',x')$ holds if and only if the isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ to $F_{(u',v')}$ maps x to x'.

3. $\oplus((u,v,x),(u',v',y'),(u'',v'',z''))$ holds if there exist $y,z \in H$ such that $(u,v,y) \sim (u',v',y')$ and $(u,v,z) \sim (u'',v'',z'')$, and $F_{(u,v)} \models x+y=z$.

4. $\odot((u,v,x),(u',v',y'),(u'',v'',z''))$ holds if there exist $y,z \in H$ such that $(u,v,y) \sim (u',v',y')$ and $(u,v,z) \sim (u'',v'',z'')$, and $F_{(u,v)} \models x \cdot y = z$.

Lemma 3.2 yielded a finitary existential formula defining the relation $(u,v,x) \sim (u',v',x')$. Moreover, the field addition and multiplication were defined in $F_{(u,v)}$ by finitary existential formulas using u and v, which were parameters there but here are elements of the triples in D. Finally, we must consider the negations of the relations. First, $(u,v,x) \not\sim (u',v',x')$ if and only if some y' commuting with u' and v' satisfies $(u,v,x) \sim (u',v',y')$ and $y' \neq x'$ – that is, just if $f_{(u,v),(u',v')}$ maps x to some element different from x'. Likewise, since $+$ is a binary operation in $F_{(u,v)}$, the negation of $\oplus((u,v,x),(u',v',y'),(u'',v'',z''))$ is defined by saying that some $w'' \neq z''$ is the sum:

$$\exists w''([w'',u''] = 1 = [w'',v''] \;\&\; w'' \neq z'' \;\&\; \oplus((u,v,x),(u',v',y'),(u'',v'',w''))),$$

which is also existential, and similarly for the negation of $\odot$. Therefore, all of these sets have finitary existential definitions in the language of groups, with no parameters, as do the negations of $\sim$, $\oplus$, and $\odot$. (The complement of D, as a relation on triples, also has a finitary quantifier-free definition, like D itself.)

The functoriality of the system of isomorphisms $f_{(u,v),(u',v')}$ (across all pairs of pairs of noncommuting elements) ensures that $\sim$ will be an equivalence relation. Lemma 3.1 showed that $f_{(u,v),(u,v)}$ is always the identity, giving reflexivity. Transitivity follows from the functorial property in that same lemma:

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')},$$

and with $(u'',v'') = (u,v)$, this property also yields the symmetry of $\sim$.

The definitions of $\oplus$ and $\odot$ essentially say to convert all three triples into $\sim$-equivalent triples with the same initial coordinates u and v, and then to check whether the final coordinates satisfy Maltsev's definitions of $+$ and $\cdot$ in the field $F_{(u,v)}$. Understood this way, they clearly respect the equivalence $\sim$. Finally, by fixing any single noncommuting pair (u,v), we see that the set $\{(u,v,x) \ : \ x \in Z(H)\}$ contains one element from each $\sim$-class and, under $\oplus$ and $\odot$, is isomorphic to the field $F_{(u,v)}$ defined by Maltsev, which in turn is isomorphic to the original field F. □

It should be noted that, although this interpretation of F in $H(F)$ was developed using computable functors on countable fields F, it is valid even when F is uncountable (or finite). A full proof requires checking that the system of isomorphisms $f_{(u,v),(u',v')}$ remains functorial and existentially definable even in the uncountable case, but this is straightforward.

In Theorem 4.1, to eliminate parameters from Maltsev's definition of F in $H(F)$, we gave an interpretation of F in $H(F)$, rather than another definition. (Recall that a definition is an interpretation in which the equivalence relation on the domain is simply equality.) We now demonstrate the impossibility of strengthening the theorem to give a parameter-free definition of F in $H(F)$.

Proposition 4.2. *There is no parameter-free definition of any field F in its Heisenberg group $H(F)$ by finitary formulas.*

Proof. Suppose that there were such a definition, and let $D \subseteq (H(F))^n$ be its domain. By Corollary 2.2, the only $(x_1, \dots, x_n) \in (H(F))^n$ that is fixed by all automorphisms of $H(F)$ is the tuple where every x_i is the identity element of $H(F)$. So, for every $\vec{x} \in D$ except this identity tuple, there would be an $\alpha_{\vec{x}} \in \mathrm{Aut}(H(F))$ that does not fix $\vec{x}$. With equality of n-tuples as the equivalence relation on D, $\alpha_{\vec{x}}$ yields an automorphism of the field F (viewed as D under the definable addition and multiplication) that does not fix $\vec{x}$. However, both identity elements 0 and 1 in F must be fixed by every automorphism of F. □

5 Question of bi-interpretability

If $\mathcal{B}$ is interpreted in $\mathcal{A}$, we write $\mathcal{B}^{\mathcal{A}}$ for the copy of $\mathcal{B}$ given by the interpretation of $\mathcal{B}$ in $\mathcal{A}$. The structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are uniformly relatively computable isomorphisms f from $\mathcal{A}$ onto $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and g from $\mathcal{B}$ onto $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. In general, the isomorphism f would map each element of $\mathcal{A}$ to an equivalence class of equivalence classes of tuples in $\mathcal{A}$. We would represent f by a relation R_f that holds for $a, \bar{a}_1, \dots, \bar{a}_r$ if f maps a to the equivalence class of the tuple of equivalence classes of the $\bar{a}_i$'s. Similarly, the isomorphism g would be represented by a relation R_g that holds for $b, \bar{b}_1, \dots, \bar{b}_r$ if g maps b to the equivalence class of the tuple of equivalence classes of the $\bar{b}_i$'s. Saying that f and g are uniformly relatively computable is equivalent to saying that the relations R_f and R_g have generalized computable Σ_1 definitions without parameters.

For a field F and its Heisenberg group $H(F)$, when we define $H(F)$ in F, the elements of $H(F)$ are represented by triples from F, and we have finitary formulas, quantifier-free or existential, that define the group operation (as a relation). When we interpret F in $H(F)$, the elements of F are represented by triples from $H(F)$, and we have finitary existential formulas that define the field operations and their negations (as ternary relations). Thus, in $F^{H(F)^F}$ (the copy of F interpreted in the copy of $H(F)$ that is defined in F), the elements are equivalence classes of triples of triples. In $H(F)^{F^{H(F)}}$ (the copy of $H(F)$ defined in the copy of F that is interpreted in $H(F)$), the elements are triples of equivalence classes of triples. So, an isomorphism f from F to $F^{H(F)^F}$ is represented by a 10-ary relation R_f on F, and an isomorphism g from $H(F)$ to $H(F)^{F^{H(F)}}$ is represented by a 10-ary relation R_g on $H(F)$.

For a Turing computable embedding Θ of K in K' we have *uniform effective bi-interpretability* if there are (generalized) computable Σ_1 formulas with no parameters that, for all $\mathcal{A} \in K$ and $\mathcal{B} = \Theta(\mathcal{A})$, define isomorphisms from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and from $\mathcal{B}$ to $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. After a talk by the fifth author, Montalbán asked the following very natural question.

Question 5.1. *Do we have uniform effective bi-interpretability of F and $H(F)$?*

The answer to this question is negative. In particular, $\mathbb{Q}$ and $H(\mathbb{Q})$ are not effectively bi-interpretable. One way to see this is to note that $\mathbb{Q}$ is rigid, while $H(\mathbb{Q})$ is not—in particular, for any non-commuting pair, $u, v \in H(\mathbb{Q})$, there is a group automorphism that takes (u, v) to (v, u). The negative answer to Question 5.1 then follows from [10, Lemma VI.26(4)], which states that if $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable, then their automorphism groups are isomorphic.

Morozov's result shows which half of effective bi-interpretability causes the difficulties.

Proposition 5.1 (Morozov)**.** *There is a finitary existential formula that, for all F, defines in F a specific isomorphism k from F to $F^{H(F)^F}$.*

Proof. In F, we have the copy of $H(F)$, consisting of triples (a, b, c) (representing $h(a, b, c)$), for $a, b, c \in F$. The group operation, derived from matrix multiplication, is $(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab')$. The definitions of the universe and the operation are quantifier-free, with no parameters. We have seen how to interpret F in $H(F)$ using finitary existential formulas with no parameters. There is a natural isomorphism k from F onto $F^{H(F)^F}$ obtained as follows. In $H(F)$, let $u = h(1, 0, 0)$ and $v = h(0, 1, 0)$. Then $\Delta_{(u,v)} = 1$. We have an isomorphism mapping F to $F_{(u,v)}$ that takes α to $h(0, 0, \alpha)$. We let $k(\alpha)$ be the $\sim$-class of $(u, v, h(0, 0, \alpha))$. The isomorphism k is defined in F by an existential formula. The complement of k is defined by saying that $k(\alpha)$ has some other value. □

The other half of what we would need for uniform effective bi-interpretability is sometimes impossible, as remarked above in the case $F = \mathbb{Q}$. We do not know of any examples where F and $H(F)$ are effectively bi-interpretable: the obstacle for $\mathbb{Q}$ might hold in all cases.

Problem 5.1. *For which fields F, if any, are the automorphism groups of F and $H(F)$ isomorphic?*

Even if there are fields F such that $\mathrm{Aut}(F) \cong \mathrm{Aut}(H(F))$, we suspect that F and $H(F)$ are not effectively bi-interpretable, simply because it is difficult to see how one might give a computable Σ_1 formula in the language of groups that defines a specific isomorphism from $H(F)$ to $H(F)^{F^{H(F)}}$.

6 Generalizing the method

Our first general definition and proposition follow closely the example of a field and its Heisenberg group.

Definition 6.1. *Let $\mathcal{A}$ be a structure for a computable relational language. Assume that its basic relations are R_i, where R_i is k_i-ary. We say that $\mathcal{A}$ is* effectively defined in $\mathcal{B}$ with parameters $\bar{b}$ *if there exist $D(\bar{b}) \subseteq \mathcal{B}^{<\omega}$, and relations $R_i(\bar{b})$ and $\neg R_i(\bar{b})$ on $D(\bar{b})^{k_i}$, defined by a uniformly computable sequence of generalized computable Σ_1 formulas with parameters $\bar{b}$.*

Proposition 6.1. *Suppose $\mathcal{A}$ is effectively defined in $\mathcal{B}$ with parameters $\bar{b}$. For $\bar{c}$ in the orbit of $\bar{b}$, let $\mathcal{A}_{\bar{c}}$ be the copy of $\mathcal{A}$ defined by the same formulas, but with parameters $\bar{c}$ replacing $\bar{b}$. Then the following conditions together suffice to give an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters:*

1. *The orbit of $\bar{b}$ is defined by a computable Σ_1 formula $\varphi(\bar{u})$;*
2. *There is a generalized computable Σ_1 formula $\psi(\bar{u}, \bar{v}, \bar{x}, \bar{y})$ such that for all $\bar{c}, \bar{d}$ in the orbit of $\bar{b}$, the formula $\psi(\bar{c}, \bar{d}, \bar{x}, \bar{y})$ defines an isomorphism $f_{\bar{c},\bar{d}}$ from $\mathcal{A}_{\bar{c}}$ onto $\mathcal{A}_{\bar{d}}$; and*
3. *The family of isomorphisms $f_{\bar{c},\bar{d}}$ preserves identity and composition.*

Proof. We write $D(\bar{b})$, $\pm R_i(\bar{b})$ for the set and relations that give a copy of $\mathcal{A}$ and for the defining formulas (with parameters $\bar{b}$). We obtain a parameter-free interpretation of $\mathcal{A}$ in $\mathcal{B}$ as follows:

1. Let D consist of the tuples $(\bar{c}, \bar{x})$ such that $\bar{c}$ is in the orbit of $\bar{b}$ and $\bar{x}$ is in $D(\bar{c})$. This is defined by a generalized computable Σ_1 formula.
2. Let $\sim$ be the set of pairs $((\bar{c}, \bar{x}), (\bar{d}, \bar{y}))$ in D^2 such that $f_{\bar{c},\bar{d}}(\bar{x}) = \bar{y}$. This is defined by a generalized computable Σ_1 formula. For pairs $(\bar{c}, \bar{x})$, $(\bar{d}, \bar{y})$ from D, it follows that $(\bar{c}, \bar{x}) \not\sim (\bar{d}, \bar{y})$ if and only if

$$(\exists \bar{y}')((\bar{d}, \bar{y}') \in D \ \& \ f_{\bar{c},\bar{d}}(\bar{x}) = \bar{y}' \ \& \ \bar{y}' \neq \bar{y}).$$

   Hence the negation of $\sim$ is also defined by a generalized computable Σ_1 formula.

3. We let R_i^* be the set of k_i-tuples $((\bar{b}_1, \bar{x}_1), \ldots, (\bar{b}_{k_i}, \bar{x}_{k_i}))$ in D^{k_i} such that for the tuple $(\bar{y}_1, \ldots, \bar{y}_{k_i})$ with $f_{\bar{b}_j,\bar{b}_1}(\bar{x}_j) = \bar{y}_j$, we have $(\bar{y}_1, \ldots, \bar{y}_{k_i}) \in R_i(\bar{b}_1)$. This is defined by a generalized computable Σ_1 formula. The complementary relation $\neg R_i^*$ is the set of tuples $((\bar{b}_1, \bar{x}_1), \ldots, (\bar{b}_{k_i}, \bar{x}_{k_i}))$ such that for $\bar{y}_1, \ldots, \bar{y}_{k_i}$ as above, $(\bar{y}_1, \ldots, \bar{y}_{k_i}) \in \neg R_i(\bar{b}_1)$. This is also defined by a generalized computable Σ_1 formula.

The verification is identical to that of Theorem 4.1. □

Corollary 6.2. *In the situation of Proposition 6.1, if $D(\bar{b})$ is contained in $\mathcal{B}^n$ for some single $n \in \omega$, then the ψ in item (2) and the formulas in Definition 6.1 will simply be computable Σ_1 formulas (as opposed to generalized computable Σ_1 formulas) and the interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters will also be by computable (as opposed to generalized) Σ_1 formulas.* □

The reader will have noticed that we only produced an *interpretation* of $\mathcal{A}$ in $\mathcal{B}$, even though we originally had a *definition* (with parameters) of $\mathcal{A}$ in $\mathcal{B}$. Proposition 4.2 shows that in general this is the best that can be done. On the other hand, we may extend Proposition 6.1 and remove parameters even in the case where $\mathcal{A}$ is interpreted (as opposed to being defined) with parameters in $\mathcal{B}$. Here we use $\equiv$ to denote the equivalence relation on the domain of this interpretation with parameters. (In Proposition 6.3 we will use it to build a new interpretation without parameters, whose domain will have the equivalence $\sim$, just as before.)

Definition 6.2 (Effective Interpretation with Parameters)**.** *We say that $\mathcal{A}$, with basic relations R_i, k_i-ary, is* effectively interpreted with parameters $\bar{b}$ *in $\mathcal{B}$ if there exist $D \subseteq \mathcal{B}^{<\omega}$, $\equiv \subseteq D^2$, and $R_i^* \subseteq D^{k_i}$ such that*

1. *$(D, (R_i^*)_i)/_{\equiv} \cong \mathcal{A}$,*
2. *D, $\pm \equiv$, and $\pm R_i^*$ are defined by a computable sequence of generalized computable Σ_1 formulas, with a fixed finite tuple of parameters $\bar{b}$.*

Again, in the case where $D \subseteq \mathcal{B}^n$ for some fixed n, the formulas defining the effective interpretation are computable Σ_1 formulas of the usual kind, with parameters $\bar{b}$.

Proposition 6.3. *Suppose that $\mathcal{A}$ (with basic relations R_i, k_i-ary) has an effective interpretation in $\mathcal{B}$ with parameters $\bar{b}$. For $\bar{c}$ in the orbit of $\bar{b}$, let $\mathcal{A}_{\bar{c}}$ be the copy of $\mathcal{A}$ obtained by replacing the parameters $\bar{b}$ by $\bar{c}$ in the defining formulas, with domain $D_{\bar{c}}/\equiv_{\bar{c}}$ containing $\equiv_{\bar{c}}$-classes $[\bar{a}]_{\equiv_{\bar{c}}}$. Then the following conditions suffice for an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ (without parameters):*

1. *The orbit of $\bar{b}$ is defined by a computable Σ_1 formula $\varphi(\bar{x})$;*
2. *There is a relation $F \subseteq \mathcal{B}^{<\omega}$, with a generalized computable Σ_1-definition, such that for every $\bar{c}$ and $\bar{d}$ in the orbit of $\bar{b}$, the set of pairs $(\bar{x}, \bar{y}) \in D_{\bar{c}} \times D_{\bar{d}}$ with $(\bar{c}, \bar{d}, \bar{x}, \bar{y}) \in F$ is invariant under $\equiv_{\bar{c}}$ on $\bar{x}$ and under $\equiv_{\bar{d}}$ on $\bar{y}$, and defines an isomorphism $f_{\bar{c},\bar{d}}$ from $\mathcal{A}_{\bar{c}}$ onto $\mathcal{A}_{\bar{d}}$; and*
3. *The family of isomorphisms $f_{\bar{c},\bar{d}}$ preserves identity and composition.*

Proof. Let the new domain D consist of those tuples $(\bar{c}, \bar{x})$ with $\bar{c}$ in the orbit of $\bar{b}$ and $\bar{x}$ in $D_{\bar{c}}$. This is defined by a generalized computable Σ_1 formula.

Let the equivalence relation $\sim$ on D be the set of pairs $((\bar{c}, \bar{x}), (\bar{d}, \bar{y})) \in D^2$ such that $f_{\bar{c},\bar{d}}([\bar{x}]_{\equiv_{\bar{c}}}) = [\bar{y}]_{\equiv_{\bar{d}}}$. This is defined by a generalized computable Σ_1 formula. For $(\bar{c}, \bar{x})$, $(\bar{d}, \bar{y}) \in D$, we have $(\bar{c}, \bar{x}) \not\sim (\bar{d}, \bar{y})$ if and only if

$$(\exists \bar{y}' \in D_{\bar{d}})\ (f_{\bar{c},\bar{d}}([\bar{x}]_{\equiv_{\bar{c}}}) = [\bar{y}']_{\equiv_{\bar{d}}}\ \&\ \bar{y} \not\equiv_{\bar{d}} \bar{y}').$$

Hence $\not\sim$ is also defined by a generalized computable Σ_1 formula.

Let R_i^* be the set of k_i-tuples $((\bar{b}_1, \bar{x}_1), \ldots, (\bar{b}_{k_i}, \bar{x}_{k_i}))$ in D^{k_i} such that for the tuple $(\bar{y}_1, \ldots, \bar{y}_{k_i})$ with $f_{\bar{b}_j,\bar{b}_1}(\bar{x}_j) = \bar{y}_j$, we have $(\bar{y}_1, \ldots, \bar{y}_{k_i}) \in R_i(\bar{b}_1)$. This is defined by a generalized computable Σ_1-formula. The complementary relation $\neg R_i^*$ is the set of tuples $((\bar{b}_1, \bar{x}_1), \ldots, (\bar{b}_{k_i}, \bar{x}_{k_i}))$ such that for $\bar{y}_1, \ldots, \bar{y}_{k_i}$ as above, $(\bar{y}_1, \ldots, \bar{y}_{k_i}) \in \neg R_i(\bar{b}_1)$. This too is defined by a generalized computable Σ_1 formula. Finally, as in the proofs of Theorem 4.1 and Proposition 6.1, it is clear that this yields an interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters. □

A relation $R \subseteq \mathcal{B}^{<\omega}$ may have a definition that is *generalized computable* Σ_α for a computable ordinal α, or *generalized X-computable* Σ_α for an X-computable ordinal α, or *generalized* $L_{\omega_1\omega}$, or *generalized* Σ_α for a countable ordinal α. The definition has the form $\bigvee_n \varphi_n(\bar{x}_n)$, where the sequence of disjuncts (each in $L_{\omega_1\omega}$, but of different arities n) is computable, or X-computable, or just countable. We note that each generalized $L_{\omega_1\omega}$ formula is generalized X-computable Σ_α for an appropriately chosen X and α, and each generalized Σ_α-formula is generalized X-computable Σ_α for an appropriately chosen X.

As computable structure theorists, we have focused here on effective interpretations. Nevertheless, we wish to point out that our results apply not only to effective interpretations, but to all interpretations using generalized $L_{\omega_1\omega}$ formulas. The following theorem generalizes Proposition 6.3 and considers every variation we can imagine.

Theorem 6.4. *Let $\mathcal{A}$ be a relational structure with basic relations R_i that are k_i-ary. Suppose there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by generalized $L_{\omega_1\omega}$ formulas, with parameters $\bar{b}$ from $\mathcal{B}$. For $\bar{c}$ in the orbit of $\bar{b}$, let $\mathcal{A}_{\bar{c}}$ be the copy of $\mathcal{A}$ obtained by the interpretation with parameters $\bar{c}$ replacing $\bar{b}$. Assume that there is a generalized $L_{\omega_1\omega}$-definable relation F defining, for each $\bar{c}$ and $\bar{d}$ in the orbit of $\bar{b}$, an isomorphism $f_{\bar{c},\bar{d}} : \mathcal{A}_{\bar{c}} \to \mathcal{A}_{\bar{d}}$ as in Proposition 6.3, and that this family is closed under composition, with the identity map as $f_{\bar{c},\bar{c}}$ for all $\bar{c}$.*

Then there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by $L_{\omega_1\omega}$ formulas without parameters. Moreover, the new interpretation satisfies all of the following.

- *For each countable ordinal α, if the interpretation in $(\mathcal{B}, \bar{b})$ defines D, $\equiv$, and each R_i using Σ_α formulas from $L_{\omega_1\omega}$, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define these sets.*

- *For each countable ordinal α, if the interpretation in $(\mathcal{B}, \bar{b})$ defines each of D, $\pm\equiv$, and $\pm R_i$ using Σ_α formulas, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define its domain, its equivalence relation $\sim$, the complement $\not\sim$, and its relations $\pm R_i$. (Defining $\not\sim$ and $\neg R_i$ this way is required by the usual notion of effective Σ_α interpretation.)*

- *Let $X \subseteq \omega$. If the interpretation in $(\mathcal{B}, \bar{b})$ used X-computable formulas, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by X-computable formulas, then the parameter-free interpretation also uses X-computable formulas.*

  Of course, for every countable set of $L_{\omega_1\omega}$ formulas, there is an X that computes them all. If the signature of $\mathcal{A}$ is infinite, and the formulas for *the interpretation of $\mathcal{A}$ in $(\mathcal{B}, \bar{b})$ are computable uniformly in X, then so are the formulas for the parameter-free interpretation of $\mathcal{A}$ in $\mathcal{B}$.*

(With $X = \emptyset$, X-computable formulas are simply computable formulas.)

- *If the interpretation in $(\mathcal{B}, \bar{b})$ had domain contained in $\mathcal{B}^n$ for a single n, so that the defining formulas for this interpretation and for F in $\mathcal{B}$ are all in $L_{\omega_1\omega}$ (as opposed to generalized $L_{\omega_1\omega}$), then the parameter-free interpretation also uses (non-generalized) $L_{\omega_1\omega}$ formulas, and its domain is contained in $\mathcal{B}^{n+|\bar{b}|}$.*

- *If the interpretation in $(\mathcal{B}, \bar{b})$ used finitary formulas, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by finitary formulas, then the parameter-free interpretation also uses finitary formulas.*

Proof. We obtain the parameter-free interpretation just as in the proof of Proposition 6.3. Notice that, by a result of Scott in [11], the orbit of $\bar{b}$ must be definable by some $L_{\omega_1\omega}$ formula. Checking the specific claims is simply a matter of writing out the new formulas using the old ones. □

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