Local Computability and Uncountable Structures

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1 Introduction

Turing computability has always been restricted to maps on countable sets. This restriction is inherent in the nature of a Turing machine: a computation is performed in a finite length of time, so that even if the available input was a countable binary sequence, only a finite initial segment of that sequence was actually used in the computation. The *Use Principle* then says that an input of any other infinite sequence with that same initial segment will result in the same computation and the same output. Thus, while the domain might have been viewed as the (uncountable) set of infinite binary sequences, the countable domain containing all finite initial segments would have sufficed.

To be sure, there are approaches that have defined natural notions of computable functions on uncountable sets. The bitmap model, detailed in [3] and widely used in computable analysis, is an excellent model for computability on Cantor space 2^ω. On the real numbers $\mathbb{R}$, however, it fails to compute even the simplest discontinuous functions, which somewhat limits its utility. The Blum-Shub-Smale model (see [2]) expands the set of functions which we presuppose to be computable. Having done so, it gives an elegant account of computable functions on the reals, with nice analogies to computability on ω, but the initial assumption immediately distances it from Turing's original concept of computability.

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Nevertheless, mathematicians are hardly daunted by the prospect of doing actual computations on $\mathbb{R}$. When faced with a real number whose binary expansion is not immediately accessible, they do not flinch; they simply call that real "x." All field operations can then be performed with ease within the subfield of $\mathbb{R}$ generated by x; the mathematician only needs to know whether x is algebraic or transcendental, and, in the former case, what its minimal polynomial over $\mathbb{Q}$ is. Similar devices handle the situation of several unknown reals at once. The binary expansions of these reals are not required for the algebraic operations.

In this chapter we formalize this process. Starting with the notion of a *computable model*, which is entirely in keeping with Turing's notion of computability, we will view the real numbers and other fields as locally computable structures. No claim is made that the real numbers can be presented globally, as a single structure with programs for the arithmetic operations on its entire domain, but we develop a definition in which a countable collection of countable objects is used to describe all finitely generated substructures of a (potentially uncountable) structure $\mathcal{S}$. Then the *local computability* of $\mathcal{S}$ is determined by the computability of the countable objects. In cases such as the field $\mathbb{R}$, where every finitely generated substructure is computably presentable, we will say that we have a *computable cover* of the structure. Indeed, for $\mathbb{R}$, a single algorithm can list out all elements of this cover.

The term "cover" is borrowed from the definition of a manifold, and the analogy, while imprecise, can be useful for intuitions about our definitions. For instance, for a topological space M, being a manifold does not just require the existence of a cover by open subsets of $\mathbb{R}^n$, but also that the charts within M given by the cover should fit together in a nice way: the transition functions between open subsets of $\mathbb{R}^n$, defined whenever two charts in M intersect, should be continuous (or differentiable, or C^∞, depending on how nicely we wish the manifold to behave). In short, it is not sufficient just to describe the local behavior of M; one must ensure that where the descriptions overlap, they agree with one another in a reasonable way.

For us, it will certainly be true that finitely generated substructures of a structure $\mathcal{S}$ can overlap. Therefore, our description of finitely generated substructures of $\mathcal{S}$ will include an account of which such substructures extend to others. Since any two finitely generated substructures of $\mathcal{S}$ lie within a single larger finitely generated substructure, it is sufficient for our purposes to consider the question of extensions among them. Topological notions do not fit our setting very well, but embeddings among finitely generated computable structures are themselves inherently computable, since they are determined by their values on the generators of the domain. (This is our main reason for considering only finitely generated substructures of $\mathcal{S}$, in fact, rather than all countable substructures.) In order for a structure to be called *locally computable*, we will require not only that the finitely generated substructures be computably presentable in a uniform way, but also that there be a computable enumeration of the embeddings among them corresponding to extensions in the structure $\mathcal{S}$. Various strengthenings of this requirement, mostly in Section 4, will allow us to prove stronger theorems about certain of the structures.

The technical content of this chapter is not especially high, but when computability-theoretic notions arise, we refer the reader to [12], the standard source, for notation and definitions. A good overview of the field of computable model theory is given in [5]. Certain examples arise in each section to illustrate the concepts discussed, but several other examples are grouped together in Section 8, and it may be useful for the reader to work back and forth between this section and the others.

2 Local Computability

Let T be a $\forall$-axiomatizable theory in a finite language. We first consider simple covers of a model $\mathcal{S}$ of T. These describe only the finitely generated substructures of $\mathcal{S}$, with no attention paid to any relations between those substructures.

Definition 2.1 A *simple cover of* $\mathcal{S}$ is a (finite or countable) collection $\mathfrak{A}$ of finitely generated models $\mathcal{A}_0, \mathcal{A}_1, \ldots$ of T, such that:

- every finitely generated substructure of $\mathcal{S}$ is isomorphic to some $\mathcal{A}_i \in \mathfrak{A}$; and
- every $\mathcal{A}_i \in \mathfrak{A}$ embeds isomorphically into $\mathcal{S}$.

If T is finitely axiomatizable but not $\forall$-axiomatizable, we can Skolemize to give it a set of $\forall$-axioms, while keeping the language finite. The theory of fields is a natural example: one makes the axioms for inverses universal by adding a unary function symbol for negation and another for reciprocation, with 0 defined to be its own reciprocal. In this expanded language, every substructure of a model of T is also a model of T, since the axioms, being universal, hold in all substructures.

Definition 2.1 could allow uncountable covers, of course, but since we will mostly be interested in the possibility of presenting all $\mathcal{A}_i$ computably, the uncountable case is irrelevant for our purposes. We often write $(\mathcal{A}_i; \vec{a}_i)$ to denote that $\vec{a}_i = \langle a_i^1, \ldots, a_i^{k_i} \rangle$ is a finite tuple of generators for $\mathcal{A}_i$. The intention is that $\mathcal{S}$ itself should not be finitely generated, of course, although the definition is still valid in this case. Indeed, $\mathcal{S}$ is not at all required to be countable, since a single $\mathcal{A}_i$ may be isomorphic to many substructures of $\mathcal{S}$. For countable structures $\mathcal{S}$, a related notion is Fraïssé's concept of the *age* of $\mathcal{S}$, i.e. the set of all finitely generated substructures of $\mathcal{S}$. All elements of $\mathfrak{A}$ must be models of T, by the $\forall$-axiomatizability of T. The existence of a cover does mean that in some sense only countably many different things can happen within $\mathcal{S}$. (Model theorists would say that the atomic type space of $\mathcal{S}$ is countable.) Similarly, in our next definition, a computable simple cover suggests that all parts of $\mathcal{S}$ are computably presentable.

Definition 2.2 A simple cover $\mathfrak{A}$ is *computable* if every $\mathcal{A}_i \in \mathfrak{A}$ is a computable structure whose domain is an initial segment of ω. $\mathfrak{A}$ is *uniformly computable* if the sequence $\langle (\mathcal{A}_i, \vec{a}_i) \rangle_{i \in \omega}$ can be given uniformly: there must exist a single computable function which, on input i, outputs a tuple of elements $\langle e_1, \ldots, e_n, \langle a_0, \ldots, a_{k_i} \rangle \rangle \in \omega^n \times \mathcal{A}_i^{<\omega}$ such that $\mathcal{A}_i$ is generated by $\{a_0, \ldots, a_{k_i}\}$ and φ_{e_j} computes the j-th function, relation, or constant of the language in $\mathcal{A}_i$. (Here n is the cardinality of the language, which we assumed to be finite.)

Notice that the definition requires that the generators of $\mathcal{A}_i$ be given as a tuple $\langle a_0, \ldots, a_{k_i} \rangle$, so that k_i is computable uniformly in i and we know how many values from $\mathcal{A}_j$ are needed to define an embedding in $I_{ij}^{\mathfrak{A}}$. (In the language of [12, II.2.4], the definition requires that we compute the *canonical index* for the set $\{a_0, \ldots, a_{k_i}\}$.)

As an example, we show that the best-known uncountable structure in mathematics is locally computable. The pieces of the proof have long been established, but for completeness we repeat the details.

Proposition 2.3 *The field $\mathcal{R} = (\mathbb{R}, +, \cdot, -, r, 0, 1)$ of real numbers is locally computable.*

Notice that we have added the unary operations of negation and inversion (r, for *reciprocal*) to the usual language of fields, in order to get a Π_1 axiom set. For definiteness we set $r(0) = 0$.

Proof. We construct a uniformly computable cover $\mathfrak{A}$ of the field $\mathcal{R}$. For this purpose we will use a computable list $\langle n_0, p_0\rangle, \langle n_1, p_1\rangle, \ldots$ of the set

$$\{\langle n, p\rangle : n \in \omega \ \&\ p \in \mathbb{Q}[X_1, \ldots, X_n, Y]\}.$$

Sublemma 2.4 *There is an algorithm which decides, for an arbitrary $\langle n, p\rangle$ in this set, whether or not p is irreducible in $\mathbb{Q}[X_1, \ldots, X_n, Y]$ and has a solution $(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1}$ with $\{x_1, \ldots, x_n\}$ algebraically independent over $\mathbb{Q}$.*

Proof. The algorithm for deciding the irreducibility of p in $\mathbb{Q}(X_1, \ldots, X_n)[Y]$, uniformly in n, was developed by Kronecker in [8]; and p is irreducible there iff it is irreducible in $\mathbb{Q}[\vec{X}, Y]$. (Details can be found in [4] and in Lemma 2 on p. 92 of [13].) We immediately rule out the reducible polynomials p. For the second part of the sublemma, we show that the set of those p which have such a solution (and are irreducible) is both Σ^0_1 and Π^0_1.

We claim first that there exists a solution as required iff there exist $q_1, \ldots, q_n, q', q'' \in \mathbb{Q}$ such that $p(\vec{q}, q') > 0 > p(\vec{q}, q'')$. If such rational numbers exist, then there exist algebraically independent real numbers $x_1, \ldots x_n$, with each x_i sufficiently close to q_i that $p(\vec{x}, q') > 0 > p(\vec{x}, q'')$ still holds. But then the Intermediate Value Theorem yields the $y \in \mathbb{R}$ with $p(\vec{x}, y) = 0$. Conversely, if we have the solution $(\vec{x}, y)$ as required, then since the set $\{\vec{x}\}$ is algebraically independent, the real polynomial $p(\vec{x}, Y) \in \mathbb{R}[Y]$ is irreducible, and so its derivative $\frac{dp}{dY}$ is nonzero at $(\vec{x}, y)$. Therefore there exist y' and y'' with $p(\vec{x}, y') > 0 > p(\vec{x}, y'')$. But the density of $\mathbb{Q}$ in $\mathbb{R}$ then shows that the rationals $\vec{q}, q', q''$ exist.

Next, we claim that the required solution fails to exist iff there exist $k \in \omega$, polynomials $g_1, \ldots, g_k \in \mathbb{Q}[\vec{X}, Y]$, and rational numbers $c_1, \ldots, c_k \geq 0$ such that $p(\vec{X}, Y) = \pm \sum_{i=1}^{k} c_i \cdot (g_i(\vec{X}, Y))^2$. If these elements exist, then clearly $p(\vec{X}, Y)$ is either positive semidefinite (i.e. takes on only values ≥ 0 on $\mathbb{R}^{n+1}$) or negative semidefinite. But we saw above that the existence of a solution $(\vec{x}, y)$ (with $\vec{x}$ algebraically independent) implies that $p(\vec{x}, Y) \in \mathbb{R}[Y]$ is neither positive nor negative semidefinite. Therefore no solution exists in which $\vec{x}$ is algebraically independent.

For the converse, suppose that no solution with $\vec{x}$ algebraically independent exists. Sublemma 2.6 below, applied with $F = \mathbb{Q}$, shows that in this case $p(\vec{X}, Y)$ must be either positive or negative semidefinite, and the existence of the required k, g_i, and c_i then follows from Artin's Theorem. (For details, we refer the reader to [9, VIII.1.12].)

Theorem 2.5 (Artin's Theorem.) *Let F be an ordered field. A polynomial $f \in F[Y_1, \ldots, Y_m]$ is positive semidefinite iff there exist $g_1, \ldots, g_k \in F[\vec{Y}]$ and $c_1, \ldots c_k \geq 0$ in F such that $f = \sum_{i=1}^{k} c_i \cdot g_i^2$.*

Sublemma 2.6 *Fix any $n \in \omega$, and let $F \subset \mathbb{R}$ be a finitely generated field extension of $\mathbb{Q}$. Let $p \in F[X_1, \ldots, X_n, Y]$ be a polynomial, irreducible in this polynomial ring, which assumes both positive and negative values on $\mathbb{R}^{n+1}$. Then there exists a solution $(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1}$ to the equation $p = 0$ such that the set $\{x_1, \ldots x_n\}$ is algebraically independent over F.*

Proof of Sublemma 2.6. We induct on n, with the statement for $n = 0$ following from the Intermediate Value Theorem. Fix $n > 0$.

Write $p(\vec{X}, Y)$ as a polynomial in $X_2, \ldots, X_n, Y$, with coefficients $q_i(X_1)$ in $F[X_1]$, and suppose for a contradiction that for every $x_1 \in \mathbb{R}$, the polynomial $p(x_1, X_2, \ldots, X_n, Y)$ is either positive definite or negative semidefinite. By the assumption of the sublemma, each of these possibilities (positive and negative semidefinite) does hold for some value of x_1. By completeness of $\mathbb{R}$, there must be an x_1 such that $p(x_1, X_2, \ldots, X_n, Y)$ is identically 0. In particular, choosing some $\{x_2, \ldots, x_n, y\}$ algebraically independent over $F(x_1)$, we have $p(\vec{x}, y) = 0$, so $p(x_1, X_2, \ldots, X_n, Y)$ must be the zero polynomial. Thus every $q_i(x_1) = 0$, yet since $p(\vec{X}, Y)$ was nonzero, some $q_i(X_1)$ is nonzero. So x_1 is algebraic over F, and its minimal polynomial in $F[X_1]$ is a factor of every $q_i(X_1)$, hence divides $p(\vec{X}, Y)$, contradicting irreducibility.

Therefore there exists an $x_1' \in \mathbb{R}$ such that $p(x_1', X_2, \ldots, X_n, Y)$ assumes both positive and negative values as a function on $\mathbb{R}^n$. But then there exists an x_1 transcendental over F and sufficiently close to x_1' that the polynomial $q = p(x_1, X_2, \ldots, X_n, Y)$ also assumes both positive and negative values. Moreover, since p is irreducible in F, we see from [13, Lemma 2, p. 92] that q is irreducible in the field $K = F(x_1)$. So we apply the inductive hypothesis with the field K and the polynomial q to get the required $x_2, \ldots, x_n, y$. This completes the proof of Sublemma 2.6, and also of Sublemma 2.4. ∎

Hence we may enumerate finitely generated fields $\mathcal{A}_i$ uniformly in i, by considering pairs $\langle n, p\rangle$ as above until we find the least one which has not been used for any $\mathcal{A}_j$ with $j < i$ and which has a solution $(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1}$ with $\vec{x}$ algebraically independent. When we find such a pair, it is straightforward to build a computable field $\mathcal{A}_i$, with domain ω, isomorphic to the quotient field of $\mathbb{Q}[X_1, \ldots, X_n, Y]/(p(\vec{X}, Y))$. These fields $\mathcal{A}_i$ will be the elements of our cover $\mathfrak{A}$. Clearly the uniform computability conditions on the $\mathcal{A}_i$ themselves are satisfied. (Below we consider the embeddings in $I^{\mathfrak{A}}$.) It is also clear that every such field is isomorphic to the subfield $\mathbb{Q}(\vec{x}, y)$ of $\mathcal{R}$. Conversely, we have the following.

Sublemma 2.7 *Every finitely generated subfield of $\mathcal{R}$ is isomorphic to some $\mathcal{A}_i \in \mathfrak{A}$ given by this process.*

Proof. Let F be a finitely generated subfield of $\mathcal{R}$. By the Noether Normalization Lemma, F is an algebraic extension of a purely transcendental extension K of $\mathbb{Q}$. Since F is finitely generated, K must have a finite (possibly empty) transcendence basis $\{x_1, \ldots, x_n\}$ over $\mathbb{Q}$, so $K = \mathbb{Q}(\vec{x})$. Finite generation also implies that F is a finite algebraic extension of K. Since we are in characteristic 0, the Theorem of the Primitive Element applies, showing that $F = K(y)$ for some $y \in F$ algebraic over K. Choose $p \in \mathbb{Q}[X_1, \ldots, X_n, Y]$ such that $p(\vec{x}, Y)$, when divided by its lead coefficient, is the minimal polynomial of y over K. Thus p is irreducible in $K[Y]$, and by dividing by the content of p in the ring $\mathbb{Q}[\vec{X}]$, we may assume that p is also irreducible in $\mathbb{Q}[X_1, \ldots, X_n, Y]$. p still has the solution $(\vec{x}, y) \in \mathbb{R}^{n+1}$, with $\vec{x}$ algebraically independent over $\mathbb{Q}$, so F is isomorphic to that $\mathcal{A}_i$ which we enumerated into $\mathfrak{A}$ when we reached the pair $\langle n, p\rangle$. This completes the proofs of Sublemma 2.7 and Proposition 2.3. ∎

It is also useful to see a negative example. Although the real numbers form a locally computable field, adding the usual $<$ relation to the structure destroys local computability.

Proposition 2.8 *The ordered field $(\mathcal{R}, <)$ of real numbers, with $\mathcal{R}$ as in Proposition 2.3, has no computable simple cover, uniform or otherwise.*

Proof. Let b be any noncomputable real number. (That is, the Dedekind cut of b should be a noncomputable subset of $\mathbb{Q}$.) We claim that the ordered subfield $\mathcal{B}$ of $\mathcal{R}$ generated by b has no computable presentation. Clearly this implies the proposition.

Suppose $\mathcal{A}$ were a computable presentation of $\mathcal{B}$, with $a \in \mathcal{A}$ the image of b under the isomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Then just from knowing the additive and multiplicative identity elements in $\mathcal{A}$, we could compute the representation in $\mathcal{A}$ of any rational number $\frac{p}{q}$. But then we could compute the Dedekind cut of b, just by using the computable relation $<$ in $\mathcal{A}$ to compare a to each rational. Therefore no such $\mathcal{A}$ can exist. ■

We will be concerned mainly with the full definition of a cover, in which we also describe how the substructures of $\mathcal{S}$ fit together.

Definition 2.9 A *cover of* $\mathcal{S}$ consists of a simple cover $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$ of $\mathcal{S}$, along with sets $I_{ij}^{\mathfrak{A}}$ (for all $\mathcal{A}_i, \mathcal{A}_j \in \mathfrak{A}$) of injective homomorphisms $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that:

- for all substructures $\mathcal{B} \subseteq \mathcal{C}$ of $\mathcal{S}$, there exist $i, j \in \omega$ and $f \in I_{ij}^{\mathfrak{A}}$ and isomorphisms $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ and $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{C}$ with $\beta = \gamma \circ f$; and

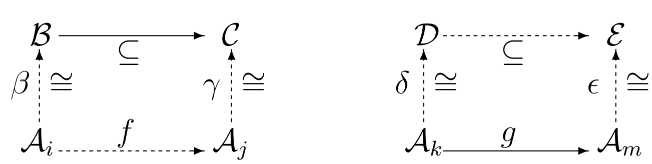
- for every k and m and every $g \in I_{km}^{\mathfrak{A}}$, there exist substructures $\mathcal{D} \subseteq \mathcal{E}$ of $\mathcal{S}$ and isomorphisms $\delta : \mathcal{A}_k \twoheadrightarrow \mathcal{D}$ and $\epsilon : \mathcal{A}_m \twoheadrightarrow \mathcal{E}$ with $\delta = \epsilon \circ g$.

This cover is *uniformly computable* if $\mathfrak{A}$ is a uniformly computable simple cover of $\mathcal{S}$ and there exists a c.e. set W such that for all $i, j \in \omega$,

$$I_{ij}^{\mathfrak{A}} = \{\varphi_e \restriction \mathcal{A}_i : \langle i, j, e \rangle \in W\}.$$

A structure $\mathcal{B}$ is *locally computable* if it has a uniformly computable cover.

Diagrams of the situation are often useful. Solid arrows represent given maps, dotted arrows represent maps whose existence is required by the definitions. Definition 2.9 demands that the following diagrams both commute.



If $\mathfrak{A}$ is a computable simple cover, then every embedding of any $\mathcal{A}_i$ into any $\mathcal{A}_j$ is determined by its values on the generators of $\mathcal{A}_i$. Since $\mathcal{A}_i$ must be finitely generated, all such embeddings are computable, and therefore it is reasonable to call $\mathfrak{A}$ a computable cover without any further requirements on the sets $I_{ij}^{\mathfrak{A}}$. (Our main reason for considering only the finitely generated substructures of $\mathcal{S}$, rather than countable ones, is that embeddings among such structures are always computable.) For a uniformly computable cover, on the other hand, the sets $I_{ij}^{\mathfrak{A}}$ will play a key role in our development of the subject, and it should be kept in mind that $I_{ij}^{\mathfrak{A}}$ need not contain every possible embedding of $\mathcal{A}_i$ into $\mathcal{A}_j$.

It is an easy exercise to see that the second condition of Definition 2.9 follows trivially from the definition of a simple cover, for any embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$. We include this second condition here because it is the dual of the first, and in the rest of our study of local computability, this duality between inclusion maps within $\mathcal{S}$ and embeddings among structures in $\mathfrak{A}$ will appear repeatedly.

In the uniformly computable simple cover $\mathfrak{A}$ of the reals built in Proposition 2.3, we now consider the embeddings between the structures $\mathcal{A}_i$.

Proposition 2.10 *There exists a uniformly computable cover of the field $\mathcal{R}$ of real numbers in which every set $I_{ij}^{\mathfrak{A}}$ is not only c.e., but actually computable uniformly in i and j.*

Proof. The simple cover is the same $\mathfrak{A}$ built in Proposition 2.3. We enumerate into $I_{ij}^{\mathfrak{A}}$ all possible embeddings of the field $\mathcal{A}_i$ into $\mathcal{A}_j$. Fix i and j, and suppose that $\mathcal{A}_i$ was built from the pair $\langle n, p\rangle$ as above. Embeddings are given simply by naming the images of the elements $x_1, \ldots, x_n, y$ in $\mathcal{A}_j$ and then extending the embedding to the rest of $\mathcal{A}_i$ using the function symbols of the language. Of course, though, not all choices of images extend to an embedding, so we need the following sublemma.

Sublemma 2.11 *Let $\mathcal{A}_i = \mathbb{Q}(z_1, \ldots, z_m, w)$ and $\mathcal{A}_j = \mathbb{Q}(x_1, \ldots, x_n, y)$ with $\vec{x}$ and $\vec{z}$ each algebraically independent over $\mathbb{Q}$, and let $q \in \mathbb{Q}(\vec{z})[W]$ and $p \in \mathbb{Q}(\vec{x})[Y]$ be the minimal polynomials of w and y over $\mathbb{Q}(\vec{z})$ and $\mathbb{Q}(\vec{x})$ respectively. Then there exists an algorithm, uniform in n, m, p and q, which decides for any $(a_1, \ldots a_m, b) \in \mathcal{A}_j^{m+1}$ whether the map f with $f(z_i) = a_i$ and $f(w) = b$ extends to an injective homomorphism of $\mathcal{A}_i$ into $\mathcal{A}_j$.*

Proof. We immediately check whether $q(\vec{a}, b) = 0$ and whether $n \geq m$. If either of these fails, then of course f does not extend to an embedding, so assume that they do both hold. Then f extends to an embedding iff $\vec{a}$ is algebraically independent over $\mathbb{Q}$ in $\mathcal{A}_j$.

On one hand, we can search for a nonzero polynomial in $\mathbb{Q}[A_1, \ldots, A_m]$ for which $\vec{a}$ is a solution in $\mathcal{A}_j$. Such a polynomial exists iff $\vec{a}$ is algebraically dependent over $\mathbb{Q}$, so clearly this outcome is Σ^0_1.

On the other hand, knowing that $n \geq m$, we search for $a_{m+1}, \ldots a_n \in \mathcal{A}_j$ and polynomials $p_1, \ldots p_n \in \mathbb{Q}[A_1, \ldots, A_n, X]$ such that for every $i \leq n$ we have

$$p_i(a_1, \ldots, a_n, x_i) = 0 \quad \& \quad p_i(a_1, \ldots, a_n, X) \neq 0.$$

If we find such elements and polynomials, then $\{a_1, \ldots, a_n\}$ is a transcendence basis for $\mathbb{Q}(x_1, \ldots, x_n)$, since it spans $\mathbb{Q}(x_1, \ldots, x_n)$ algebraically and has the minimum possible size for such a spanning set. In this case $\{a_1, \ldots, a_m\}$ is an algebraically independent set. Conversely, if this set really is algebraically independent over $\mathbb{Q}$ in $\mathcal{A}_j$, then it does extend to a transcendence basis, and so such elements and polynomials must exist. Therefore algebraic independence is also a Σ^0_1 condition. This proves Sublemma 2.11. ∎

Since $I^{\mathfrak{A}}_{ij}$ includes every possible embedding of $\mathcal{A}_i$ into $\mathcal{A}_j$, the first condition of Definition 2.9 is immediate, and we have already noted that the second condition is trivial. Hence we have proven Proposition 2.10. ∎

The point of Proposition 2.10 is the computability of the sets $I^{\mathfrak{A}}_{ij}$. That they can be computably enumerated would have followed immediately from our next result.

Lemma 2.12 *A structure $\mathcal{S}$ has a uniformly computable cover (i.e. is locally computable) iff $\mathcal{S}$ has a uniformly computable simple cover.*

Proof. Assume that $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$ is a uniformly computable simple cover of $\mathcal{S}$. The domain $\{a_{i,0}, a_{i,1}, \ldots\}$ of each $\mathcal{A}_i$ is enumerable uniformly in i, and so it is straightforward to enumerate the domain of the substructure

$$\mathcal{B}_{\langle i,j \rangle} = \langle \{a_{i,k} \in \mathcal{A}_i : k \in D_j\} \rangle \subseteq \mathcal{A}_i$$

where the generating set is defined using the finite set D_j with canonical index j, as defined in [12]. Now every A_i is equal to some $\mathcal{B}_{\langle i,j \rangle}$, so $\{\mathcal{B}_{\langle i,j \rangle}\}_{i,j}$ is another uniformly computable simple cover of $\mathcal{S}$. Next, define the set $I_{\langle i,j \rangle, \langle i',j' \rangle}$ to be empty if $i' \neq i$. For $i = i'$, we wait until all elements of D_j

have appeared in $\mathcal{B}_{\langle i,j'\rangle}$. If this ever happens, we enumerate the identity map into $I_{\langle i,j\rangle,\langle i,j'\rangle}$; if not, then $I_{\langle i,j\rangle,\langle i,j'\rangle}$ is empty.

Clearly this uniformly enumerates the sets $I_{\langle i,j\rangle,\langle i',j'\rangle}$, and we claim that with these sets, $\mathfrak{B} = \{\mathcal{B}_{\langle i,j\rangle} : i, j \in \omega\}$ forms a cover of $\mathcal{S}$. First, if $f \in I_{\langle i,j\rangle,\langle i',j'\rangle}$, then $i = i'$ and $\mathcal{B}_{\langle i,j\rangle} \subseteq \mathcal{B}_{\langle i,j'\rangle} \subseteq A_i$. Since $\mathcal{A}_i$ is isomorphic to a substructure $\mathcal{B} \subseteq \mathcal{S}$, we can simply lift f to the identity map on the corresponding substructures of this $\mathcal{B}$. Conversely, if $\mathcal{B} \subseteq \mathcal{C}$ are finitely generated substructures of $\mathcal{S}$, then $\mathcal{C}$ is isomorphic to some $\mathcal{A}_i$, and there are some j and j' such that $\mathcal{B}_{\langle i,j'\rangle} = \mathcal{A}$ and $\mathcal{B}_{\langle i,j\rangle}$ is the substructure of $\mathcal{A}_i$ corresponding to $\mathcal{B}$ within $\mathcal{C}$. But then the identity map from $\mathcal{B}_{\langle i,j\rangle}$ into $\mathcal{B}_{\langle i,j'\rangle}$ must have appeared in $I_{\langle i,j\rangle,\langle i,j'\rangle}$, and matches the inclusion map from $\mathcal{B}$ into $\mathcal{C}$. ■

In light of this lemma, one naturally asks why we bothered to give Definition 2.9. The answer is that local computability will be the $\theta = 0$ case in the definition of *θ-extensionally computable structures*, which appears in the next section as Definition 3.6 and which uses the enumeration of the sets $I^{\mathfrak{A}}_{ij}$ extensively. Indeed, it is the enumeration of the embeddings, rather than that of the finitely generated substructures of $\mathcal{S}$, which will be the heart of our study of local computability.

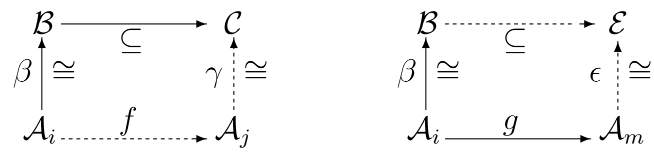
3 Theory of Locally Computable Structures

Definition 3.1 Let $\mathfrak{A}$ be a cover of a structure $\mathcal{S}$. We say that an $\mathcal{A}_i \in \mathfrak{A}$ matches a substructure $\mathcal{B} \subseteq \mathcal{S}$ *extensionally* if there is an isomorphism $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ for which the following hold.

- for every finitely generated $\mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S}$, there exists $j \in \omega$, $f \in I^{\mathfrak{A}}_{ij}$, and an isomorphism γ mapping $\mathcal{A}_j$ onto $\mathcal{C}$ such that $\beta = \gamma \circ f$; and
- for every $m \in \omega$ and every $g \in I^{\mathfrak{A}}_{im}$, there exists a $\mathcal{E} \subseteq \mathcal{S}$ and an isomorphism ϵ mapping $\mathcal{A}_m$ onto $\mathcal{E}$ such that $\mathcal{B} \subseteq \mathcal{E}$ and $\beta = \epsilon \circ g$.

This β is called an *extensional match* between $\mathcal{A}_i$ and $\mathcal{B}$.

Again, diagrams help explain this definition. The two conditions may be expressed as follows.



The difference from the diagrams of Definition 2.9 is that now $\mathcal{A}_i$, $\mathcal{B}$, and the isomorphism β are all fixed: the required j, f, and γ and the required $\mathcal{E}$ and ϵ must all work for this particular $\beta : \mathcal{A}_i \to \mathcal{B}$ on the left edge of the diagram. We refer to β as an *extensional match* between $\mathcal{A}_i$ and $\mathcal{B}$. The idea is that the embeddings in the sets $I^{\mathfrak{A}}_{ij}$ (for all j) correspond precisely to the finitely generated superstructures of $\mathcal{B}$ in $\mathcal{S}$, rather than just to possible extensions of various $\mathcal{B}' \cong \mathcal{B}$ within $\mathcal{S}$. This distinction will be illustrated in the examples below.

Definition 3.2 We say that a uniformly computable cover $\mathfrak{A}$ of $\mathcal{S}$ is *extensionally computable* (and we call $\mathfrak{A}$ an *extensional cover* of $\mathcal{S}$) if every $\mathcal{A}_i \in \mathfrak{A}$ extensionally matches some substructure $\mathcal{B} \subseteq \mathcal{S}$ and every finitely generated substructure $\mathcal{B} \subseteq \mathcal{S}$ extensionally matches some $\mathcal{A}_i \in \mathfrak{A}$.

If such a cover exists, we say that $\mathcal{S}$ is *extensionally locally computable.*

The point of this definition is that the extra conditions strengthen the idea that each finitely generated substructure of $\mathcal{S}$ is represented by some $\mathcal{A}_i \in \mathfrak{A}$: not only are they isomorphic, but the embeddings (enumerated effectively by $I^{\mathfrak{A}}$) of $\mathcal{A}_i$ into other structures in $\mathfrak{A}$ coincide exactly with the extensions of $\mathcal{B}$ to larger finitely generated substructures of $\mathcal{S}$. This point is best illustrated by the negative example of Proposition 3.5 below. However, we first show that the definition holds for the field of complex numbers.

Proposition 3.3 *Every algebraically closed field* $\mathcal{C} = (C, +, \cdot, -, r, 0, 1)$ *of characteristic* 0 *is extensionally locally computable. In particular, the field* $\mathbb{C}$ *of complex numbers is extensionally locally computable.*

Proof. The construction of a uniformly computable cover $\mathfrak{A}$ of $\mathcal{C}$ is largely the same as that for the field of real numbers in Proposition 2.3. Of course, we no longer need worry whether a given polynomial of degree > 0 has a root in $\mathcal{C}$, and so Sublemma 2.4 in the construction of $\mathfrak{A}$ is not needed, except for the remark about Kronecker's algorithm. Also, in case the transcendence degree d of $\mathcal{C}$ over $\mathbb{Q}$ is finite, we make sure that $\mathfrak{A}$ only contains fields with transcendence degree $\leq d$ over $\mathbb{Q}$. Once again we define $I_{ij}^{\mathfrak{A}}$ to contain every embedding of $\mathcal{A}_i$ into $\mathcal{A}_j$, and the same analysis from the proof of Sublemma 2.11 shows that the sets $I_{ij}^{\mathfrak{A}}$ are computably enumerable (indeed computable) uniformly in i and j, and that $\mathfrak{A}$ is a uniformly computable cover of $\mathcal{C}$. (In fact, if two algebraically closed fields of characteristic 0 are isomorphic, or if they both have infinite transcendence degree, then we have built the same $\mathfrak{A}$ for both of them.)

It remains to show that $\mathfrak{A}$ is an extensionally computable cover of $\mathcal{C}$. Indeed, we will show more.

Lemma 3.4 *In this situation, every embedding of any $\mathcal{A}_i \in \mathfrak{A}$ into $\mathcal{C}$ is extensional.*

Proof. Fix any $\mathcal{A}_i \in \mathfrak{A}$, with generators $x_1, \ldots, x_m, y$ given in the construction of $\mathfrak{A}$ and with an irreducible polynomial $q \in \mathbb{Q}[\vec{X}, Y]$ such that $q(\vec{x}, y) = 0$, and let $\mathcal{B} \subseteq \mathcal{C}$ be any subfield isomorphic to $\mathcal{A}_i$, via any isomorphism β. We claim that β is an extensional match.

Now for any $g \in I_{ij}^{\mathfrak{A}}$, we start with the embedding $\gamma_0 = \beta \circ g^{-1}$ of $g(\mathcal{A}_i)$ into $\mathcal{C}$ and extend it one-by-one to the generators $z_1, \ldots z_l, z_{l+1} = w$ of $\mathcal{A}_j$ given by the construction of $\mathfrak{A}$. For each $k \leq l+1$, if z_k is transcendental over the domain of γ_{k-1} (i.e. the subfield of $\mathcal{A}_j$ generated by $g(\mathcal{A}_i)$ and $z_1, \ldots, z_{k-1}$), then we choose $\gamma_k(z_k)$ to be any element of $\mathcal{C}$ transcendental over the image of γ_{k-1}. (Such an element of $\mathcal{C}$ must exist, since we ensured that $\mathcal{A}_j$ cannot have transcendence degree $> d$ over $\mathbb{Q}$.) Otherwise z_k is algebraic over $\mathrm{dom}(\gamma_{k-1})$, so we let $p(Z)$ be its minimal polynomial over that subfeld. and choose $\gamma_k(z_k)$ to be any root in $\mathcal{C}$ of the polynomial $\overline{p}(Z) \in \mathcal{C}[Z]$ gotten by applying γ_{k-1} to the coefficients of p. In either case, γ_k then extends to an embedding into $\mathcal{C}$ of the subfield of $\mathcal{A}_j$ generated by $g(\mathcal{A}_i)$ and $z_1, \ldots, z_k$. By induction, the map $\gamma = \gamma_{k+1}$ is an isomorphism from $\mathcal{A}_j$ onto a finitely generated substructure of $\mathcal{C}$, and since γ extends $\gamma_0 = \beta \circ g^{-1}$, it is clear that $\beta = \gamma \circ g$.

The converse is quicker. Fix any finitely generated subfield $\mathcal{B}$ of $\mathcal{C}$ with $g(\mathcal{A}_i) \subseteq \mathcal{B}$. Extend the transcendence basis $\{\beta(x_1), \ldots, \beta(x_m)\}$ to a (finite) transcendence basis X for $\mathcal{B}$ over $\mathbb{Q}$, and pick a primitive element generating $\mathcal{B}$ over $\mathbb{Q}(X)$, using the Primitive Element Theorem. Then $\mathcal{B}$ is isomorphic to that field $\mathcal{A}_j \in \mathfrak{A}$ with the same transcendence degree over $\mathbb{Q}$ and the same minimal polynomial for its primitive element. Let γ be this isomorphism. Then $\gamma^{-1} \circ \beta$ is an embedding of $\mathcal{A}_i$ into $\mathcal{A}_j$, hence lies in $I_{ij}^{\mathfrak{A}}$, and we take this embedding as our g. Thus this $\mathcal{B}$ is an extensional match for $\mathcal{A}_i$. ■

Since $\mathfrak{A}$ was shown to be a cover of $\mathcal{C}$, every $\mathcal{A}_i \in \mathfrak{A}$ is isomorphic to some subfield $\mathcal{B} \subset \mathcal{C}$, and every finitely generated subfield of $\mathcal{C}$ is isomorphic to an element of $\mathfrak{A}$. By Lemma 3.4, the isomorphism in each case is an extensional match. Thus $\mathfrak{A}$ is an extensional cover of $\mathcal{C}$. ■

To make the meaning of Definition 3.2 more obvious, we now give an instance where it does not apply.

Proposition 3.5 *The field $\mathcal{R}$ of real numbers is not extensionally locally computable.*

Proof. Suppose that $\mathfrak{A}$ were an extensionally computable cover of $\mathcal{R}$. Fix any noncomputable real number $t \in \mathbb{R}$. Definition 3.2 gives an $\mathcal{A}_i \in \mathfrak{A}$ which extensionally matches (via some isomorphism β) the subfield $\mathcal{B}$ of $\mathcal{R}$ generated by t, and we may assume we know i and $\beta^{-1}(t)$, since they constitute finitely much information.

Now we can enumerate the lower cut $\{q \in \mathbb{Q} : q < t\}$ defined by t, knowing that extensions of $\mathcal{B}$ in $\mathcal{R}$ correspond to embeddings $f \in I_{ij}^{\mathfrak{A}}$ (for all j) in the extensionally computable cover. For any rational $q \in \mathcal{R}$:

$$\begin{aligned}\models_{\mathcal{R}} q < t &\iff \models_{\mathcal{R}} (\exists x) x^2 = t - q \\ &\iff (\exists \text{ f.g. } \mathcal{C})[\mathcal{B} \subseteq \mathcal{C} \subset \mathcal{R} \ \& \ \models_{\mathcal{C}} (\exists x) x^2 = t - q] \\ &\iff (\exists j)(\exists f \in I_{ij}^{\mathfrak{A}}) \models_{\mathcal{A}_j} (\exists x) x^2 = f(\beta^{-1}(t - q)) \\ &\iff (\exists j)(\exists f \in I_{ij}^{\mathfrak{A}})(\exists a \in \mathcal{A}_j) \models_{\mathcal{A}_j} a^2 = f(\beta^{-1}(t)) - f(\beta^{-1}(q)).\end{aligned}$$

A similar argument holds for the upper cut $\{q \in \mathbb{Q} : q > t\}$, using square roots of $(q - t)$ in $\mathcal{R}$. So the lower cut is both Σ_1^0 and Π_1^0, contradicting the noncomputability of t. (Of course, β and f fix the rationals, so $f(\beta^{-1}(q)) \in \mathcal{A}_j$ is just the element of $\mathcal{A}_j$ representing q. The domain element of $\mathcal{A}_j$ representing any particular rational q can easily be computed from the numerator and denominator of q, uniformly in j, by using the functions of $\mathcal{A}_j$.) ■

So the extensional local computability of $\mathcal{C}$ does not follow solely from the existential closure of the structure; after all, $\mathcal{R}$, viewed as a real closed field, is also existentially closed. The difficulty for $\mathcal{R}$ is that real closed fields have an implicit order on their elements, whether it is included in the language of the structure or not, and as we saw in Proposition 2.8, adding the order relation to $\mathcal{R}$ destroys local computability. $\mathcal{R}$ itself can still be locally computable, because the relation $<$ cannot be defined in $\mathcal{R}$ without quantifiers (even though it is both Σ_1-definable and Π_1-definable!) and existential questions about $\mathcal{R}$ can be left unanswered by a uniformly computable cover. An extensional cover, on the other hand, answers all such questions, as will be seen in Proposition 3.8 and Theorem 3.10.

Next we entend Definition 3.2. Isomorphisms that were called extensional matches will now be called 1-extensional, according to the following.

Definition 3.6 Let $\mathfrak{A}$ be a cover of a structure $\mathcal{S}$. Every isomorphism β between any $\mathcal{A}_i \in \mathfrak{A}$ and any substructure $\mathcal{B} \subseteq \mathcal{S}$ will be called 0-*extensional*. For any ordinal $\theta > 0$, we say that such an isomorphism β is θ-*extensional* if:

- for every finitely generated $\mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S}$, and every ordinal $\zeta < \theta$, there exists $j \in \omega$, $f \in I_{ij}^{\mathfrak{A}}$, and a ζ-extensional γ mapping $\mathcal{A}_j$ onto $\mathcal{C}$ such that $\beta = \gamma \circ f$; and

- for every $m \in \omega$ and every $g \in I_{im}^{\mathfrak{A}}$, and every ordinal $\zeta < \theta$, there exists an $\mathcal{E} \subseteq \mathcal{S}$ and a ζ-extensional ϵ mapping $\mathcal{A}_m$ onto $\mathcal{E}$ such that $\mathcal{B} \subseteq \mathcal{E}$ and $\beta = \epsilon \circ g$.

A uniformly computable cover $\mathfrak{A}$ of $\mathcal{S}$ is θ-*extensional* if every $\mathcal{A}_i \in \mathfrak{A}$ θ-extensionally matches some substructure of $\mathcal{S}$ (i.e. there exists a θ-extensional isomorphism between them) and every finitely generated substructure of $\mathcal{S}$ θ-extensionally matches some $\mathcal{A}_i \in \mathfrak{A}$. If such a cover exists, we say that $\mathcal{S}$ is θ-*extensionally locally computable*. Often we will abbreviate this and just call $\mathcal{S}$ itself θ-extensional.

The diagram here is exactly the same as that for Definition 3.1. The only difference is the stronger requirement about the isomorphisms γ and ϵ being ζ-extensional.

Notice that $\mathcal{S}$ is 0-extensional iff $\mathcal{S}$ is locally computable, iff $\mathcal{S}$ has a uniformly computable simple cover (by Lemma 2.12). In Section 4 we will add one more version of extensionality, even stronger than θ-extensional local computability. The rest of this section is devoted to generalizing results such as Proposition 3.5 and extending them to results about the complexity of the theory of $\mathcal{S}$ and of various fragments of its elementary diagram.

Knowing that a structure $\mathcal{S}$ is globally computable gives information about the decidability of the atomic diagram and the Σ_n-diagram of $\mathcal{S}$ for each $n > 0$: the former is computable, and each of the rest is 1-reducible to $\emptyset^{(n)}$. (Indeed, if we allow computable infinitary formulas from the hyperarithmetic hierarchy, then for any computable ordinal θ, the Σ_θ diagram is likewise Σ^0_θ, i.e. 1-reducible to $\emptyset^{(\theta)}$, under reasonable definitions.) When $\mathcal{S}$ is computably presentable, these observations still hold for the quantifier-free theory and the Σ_θ-theory of $\mathcal{S}$, although they may not hold for the actual diagrams. (The *Σ_θ-diagram of $\mathcal{S}$* refers to the Σ_θ-theory of the structure $\mathcal{S}_S$ in the extended language with a constant for every $s \in \mathcal{S}$.) If $\mathcal{S}$ is uncountable, then it is pointless to talk about the Σ_θ-diagrams in terms of Turing computability, since the diagrams themselves are uncountable. However, we can still prove analogous results about the Σ_θ-theory for any θ-extensionally locally computable structure.

For these purposes, a first-order formula is Σ_n if it can be written in prenex normal form with n blocks of like quantifiers, beginning with an existential. For even n, this means:

$$(\exists x_1^1 \cdots \exists x_1^{k_1})(\forall x_2^1 \cdots \forall x_2^{k_2}) \cdots (\forall x_n^1 \cdots \forall x_n^{k_n})\varphi(\vec{x})$$

where φ is quantifier-free, and similarly for odd n, with $(\exists x_n^1 \cdots \exists x_n^{k_n})$. These notions generalize with computable ordinals $\theta \geq \omega$ in place of n; we refer the reader to [1] for details. We begin with simple results which do not require extensional local computability.

Proposition 3.7 *If $\mathcal{S}$ has a computable cover, then the quantifier-free theory of $\mathcal{S}$ is computable. If $\mathcal{S}$ is locally computable, then the Σ_1-theory of $\mathcal{S}$ is computably enumerable.*

Proof. The truth of a quantifier-free sentence φ in $\mathcal{S}$ can be checked just by determining whether φ holds in $\mathcal{A}_i$, for any fixed $\mathcal{A}_i$ in a computable cover of $\mathcal{A}$. If $\mathfrak{A}$ is a uniformly computable cover of $\mathcal{S}$, then we can enumerate the Σ_1-theory of $\mathcal{S}$ by enumerating all existential sentences true in each $\mathcal{A}_i \in \mathfrak{A}$. ∎

This is as much as we can say in general about locally computable structures, but with extensional or perfect local computability we can develop results for more complex sentences. For simplicity we stick to finitary formulas in Proposition 3.8 and its proof. The subsequent Theorem 3.10 will generalize to computable infinitary formulas, as well as to parameters from $\mathcal{S}$.

Proposition 3.8 *For $m \in \omega$, any m-extensionally locally computable structure $\mathcal{S}$, and any $n \leq m+1$, the Σ_n-theory of $\mathcal{S}$,*

$$\{\varphi \in Th(\mathcal{S}) : \varphi \text{ is a } \Sigma_n \text{ sentence}\},$$

is itself a Σ^0_n set in the arithmetic hierarchy. (For $n > 0$, this means that the Σ_n-theory is 1-reducible to $\emptyset^{(n)}$, and for $n = 0$, the Σ_0-theory is computable.)

Proof. Let $\langle \mathcal{A}_i \rangle_{i \in \omega}$ be an m-extensionally computable cover of $\mathcal{S}$. Proposition 3.7 already proved the result for in case $m \leq 1$. For arbitrary m, the key fact is simply that for any formula $\varphi(x_1, \ldots, x_j)$,

$$\models_{\mathcal{S}} (\exists \vec{x})\varphi(\vec{x}) \text{ iff } (\exists \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\exists\, \vec{b} \in \mathcal{B}^j) \models_{\mathcal{S}} \varphi(\vec{b}).$$

We restate this fact:

$$\models_{\mathcal{S}} (\forall \vec{x})\varphi(\vec{x}) \text{ iff } (\forall \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\forall \vec{b} \in \mathcal{B}^j) \models_{\mathcal{S}} \varphi(\vec{b}).$$

When we have alternating quantifiers, we need to take superstructures at each step. For an arbitrary formula $\varphi(\vec{x}, \vec{y})$,

$$\begin{aligned} & \models_{\mathcal{S}} (\exists \vec{x})(\forall \vec{y})\varphi(\vec{x}, \vec{y}) \\ \text{iff } & (\exists \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\exists\, \vec{x} \in \mathcal{B}^k) \models_{\mathcal{S}} (\forall \vec{y})\varphi(\vec{x}, \vec{y}) \\ \text{iff } & (\exists \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\exists\, \vec{x} \in \mathcal{B}^k)(\forall \text{ f.g. } \mathcal{C} \text{ s.t. } \mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S})(\forall \vec{y} \in \mathcal{C}^p) \models_{\mathcal{S}} \varphi(\vec{x}, \vec{y}) \end{aligned}$$

If the original sentence was Σ_2, then the matrix (after all the quantifiers) will be the truth in $\mathcal{S}$ of the quantifier-free formula $\varphi(\vec{x}, \vec{y})$. In this case, $\varphi(\vec{x}, \vec{y})$ holds in $\mathcal{S}$ iff it holds in $\mathcal{C}$, so we can continue as follows:

$$\begin{aligned} \text{iff } & (\exists \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\exists\, \vec{x} \in \mathcal{B}^k)(\forall \text{ f.g. } \mathcal{C} \text{ s.t. } \mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S})(\forall \vec{y} \in \mathcal{C}^p) \models_{\mathcal{C}} \varphi(\vec{x}, \vec{y}) \\ \text{iff } & (\exists i)(\exists\, \vec{b} \in \mathcal{A}_i^k)(\forall j)(\forall f \in I_{ij}^{\mathfrak{A}})(\forall \vec{c} \in \mathcal{A}_j^p) \models_{\mathcal{A}_j} \varphi(f(\vec{b}), \vec{c}). \end{aligned}$$

The definition of 1-extensional cover shows these last two lines to be equivalent. Specifically, if the last line holds, then the witness $\mathcal{A}_i$ has a 1-extensional match β onto some $\mathcal{B} \subseteq \mathcal{S}$, and Definition 3.6, applied to any $\mathcal{A}_j$ and $f \in I_{ij}^{\mathfrak{A}}$, provides a 0-extensional match ϵ from $\mathcal{A}_j$ onto some $\mathcal{E} \supseteq \mathcal{B}$ such that $\epsilon \circ f = \beta$. Then $\varphi(\epsilon(f(\vec{b})), \epsilon(\vec{c}))$ must hold in $\mathcal{C}$, since $\varphi(f(\vec{b}), \vec{c})$ holds in $\mathcal{A}_j$ and ϵ is an isomorphism. Conversely, if the next-to-last line holds, then there is some 1-extensional match β onto the witness $\mathcal{B}$ from some $\mathcal{A}_i \in \mathfrak{A}$, and a similar argument applies to any $\mathcal{C}$ extending $\mathcal{B}$, yielding the j, f, and γ required by the last line. This completes the proof of the result on 1-extensionally locally computable structures.

The obvious iteration of this process, applied to any Σ_n sentence about $\mathcal{S}$, yields a statement consisting of a Σ_n-sequence of quantifiers over structures in $\mathfrak{A}$, their elements, and the sets $I_{ij}^{\mathfrak{A}}$, followed by a quantifier-free statement about an $\mathcal{A}_j \in \mathfrak{A}$. The argument requires that each $\mathcal{A}_i$ correspond to some $\mathcal{B}$ via an $(n-1)$-extensional map, so that the extensions must then correspond via $(n-2)$-extensional maps, and so on down to 0-extensional maps once all the quantifiers have been moved outside the turnstile $\models$. Therefore, for an m-extensionally locally computable $\mathcal{S}$ with $m \geq (n-1)$, the Σ_n^0 statement yielded by iterating the process holds iff the original Σ_n sentence held in $\mathcal{S}$. Since the structures in $\mathfrak{A}$, the sets $I_{ij}^{\mathfrak{A}}$ and the atomic diagram of such an $\mathcal{A}_j$ are all computable uniformly in i and j, the truth of the original Σ_n-sentence in $\mathcal{S}$ is itself a Σ_n^0 fact. Moreover, this process is entirely uniform in n. ∎

Notice that for 1-extensionally locally computable structures, the equivalence of a Σ_3 sentence in $\mathcal{S}$ with the corresponding Σ_3 statement about $\mathfrak{A}$ would not follow from this argument. Although the initial $\mathcal{A}_i$ would correspond to some $\mathcal{B}$ via a 1-extensional map, the isomorphism between the $\mathcal{A}_j$ and the $\mathcal{C}$ might be only 0-extensional, and so with a third quantifier, embeddings from $\mathcal{A}_j$ into various $\mathcal{A}_k$ would not necessarily correspond to extensions of $\mathcal{C}$. The same applies to other values of m, and for $m = 0$ a specific counterexample appears in Proposition 3.5. More counterexamples can be derived from Proposition 8.2 and Theorem 8.7, where the Σ_2-theory and the Σ_6-theory, respectively, can be arbitrarily complex.

Corollary 3.9 *Any two structures with the same m-extensionally locally computable cover are elementarily $(m+1)$-equivalent. Specifically, any two structures with the same uniformly computable cover are elementarily 1-equivalent, and any two structures with the same computable cover are elementarily 0-equivalent.*

Proof. Given any sentence φ, the proof of Proposition 3.8 shows that $\models_{\mathcal{S}} \varphi$ is equivalent to a statement about the cover of $\mathcal{S}$. In fact, this would hold even if the computability-theoretic requirements were dropped from the definition of perfect cover. The other results of the corollary are proven similarly. ∎

Proposition 3.8 is ostensibly only a result about the theory of the structure $\mathcal{S}$, not about its elementary diagram: the sentences considered there are not allowed to contain constants from the domain of $\mathcal{S}$. However, the intention of local computability is to talk about individual elements of the structure $\mathcal{S}$, not just about the theory, and it is not hard to adapt Proposition 3.8 to do so. We view the following result as the crux of our discussion of extensionality. We also extend it to include hyperarithmetical formulas.

Theorem 3.10 *For any computable ordinal θ, any θ-extensionally locally computable structure $\mathcal{S}$, any finite tuple $\vec{p}$ of parameters from $\mathcal{S}$, and any $\zeta \leq \theta$, the Σ_ζ-theory of $\mathcal{S}$ over $\vec{p}$,*

$$\{\varphi \in Th(\mathcal{S}, \vec{p}) : \varphi \text{ is a } \Sigma_\zeta \text{ sentence}\},$$

is itself a Σ^0_ζ in the hyperarithmetic hierarchy. For a fixed computable presentation of a single θ, this holds uniformly in ζ and in an appropriate description of the parameters (as discussed after the proof).

Proof. Let $\mathcal{B}$ be generated by $\vec{p}$ in $\mathcal{S}$, and fix an θ-extensional match $\beta : \mathcal{A}_l \twoheadrightarrow \mathcal{B}$ for some $\mathcal{A}_l \in \mathfrak{A}$. As before, we give an example by evaluating the truth in $\mathcal{S}$ of an arbitrary Σ_2 sentence with the parameters $\vec{p}$, assuming now that $\theta \geq 2$. By an argument similar to that in the proof of Proposition 3.8, the Σ_2 sentence $(\exists \vec{x})(\forall \vec{y})\varphi(\vec{p}, \vec{x}, \vec{y})$ holds in $\mathcal{S}$ iff

$$\begin{aligned}&(\exists i)(\exists h \in I^{\mathfrak{A}}_{li})(\exists \vec{b} \in \mathcal{A}^k_i)(\forall j)(\forall f \in I^{\mathfrak{A}}_{ij})(\forall \vec{c} \in \mathcal{A}^m_j)\\ &\models_{\mathcal{A}_j} \varphi(f(h(\vec{a})), f(\vec{b}), \vec{c}),\end{aligned}$$

which is a Σ^0_2 condition, uniformly in $\vec{a}$ and l. The obvious iteration works for any $\zeta \leq \theta$, but no longer applies when $\zeta = \theta + 1$, whereas Proposition 3.8 did hold when $n = m + 1$. In the example above, as long as $\mathcal{S}$ is 2-extensional, we may assume that β is 2-extensional, that the h we find lifts to an inclusion in $\mathcal{S}$ via β and some 1-extensional $\gamma : \mathcal{A}_i \hookrightarrow \mathcal{S}$, and therefore that every inclusion of $\gamma(\mathcal{A}_i)$ into any larger finitely generated substructure of $\mathcal{S}$ must lift some f in some $I^{\mathfrak{A}}_{ij}$. If β were only 1-extensional, this argument would not suffice. Adding the parameters forces us to start by fixing an $\mathcal{A}_l \in \mathfrak{A}$ and a β, whereas in Proposition 3.8 we were allowed simply to search for any $\mathcal{A}_i$ and a single embedding into an $\mathcal{A}_j$. Hence parameters require one more level of extensionality.

The extension of this argument to hyperarithmetical formulas (of complexity Σ_ζ with $\zeta \leq \theta$, of course) is intuitively reasonable; the argument is by induction on ordinals, with the paragraph above covering the case of a successor ordinal. When ζ is a limit ordinal, the natural argument applies, using the uniformity of the disjuncts in the Σ_ζ formula. Writing out the details becomes very messy, and we leave that task to the reader. Notice that it is *not* necessary to quantify over the isomorphisms β, γ, etc.; one simply fixes an appropriate θ-extensional β at the beginning of the argument, and then translates the formula into a new hyperarithmetic statement in which all quantification is over effectively given objects such as domains $\mathcal{A}_i$ or sets $I_{ij}^{\mathfrak{A}}$ of maps.

Of course, knowing an original parameter $p_i \in \mathcal{B}$ is useless to us; we need to know l and the value $a_i = \beta^{-1}(p_i)$ in $\mathcal{A}_l$. For finitely many parameters, this constitutes only finitely much information, but we also wish to consider uniformity. Of course, it does not make sense to ask for parameters from a potentially uncountable structure $\mathcal{S}$ to be given uniformly. Instead, our formal statement of uniformity is that if we are given an l and finitely many parameters $\vec{a}$ from $\mathcal{A}_l$, then for any $\zeta \leq \theta$ and any ζ-extensional match β mapping $\mathcal{A}_l$ into $\mathcal{S}$, the Σ_ζ-theory of $\mathcal{S}$ over the parameters $\beta(a_1), \ldots, \beta(a_n)$ is a Σ_ζ^0 set in the hyperarithmetical hierarchy, uniformly in l and $\vec{a}$. ∎

We view Theorem 3.10 as the strongest argument yet that local computability, and in particular these ordinal levels of extensional local computability, form the correct analogue in uncountable structures to computable presentability in countable structures. The point of a computable presentation of a structure is not just that it allows us to compute the atomic theory and enumerate the Σ_1-theory and so on, but that it actually allows us to do over specific elements of the structure: the atomic *diagram* is computable, and the Σ_θ diagram is Σ_θ^0, uniformly in (a presentation of) θ. For an uncountable $\mathcal{S}$, of course, there is no effective way to name all individual elements, so it is hopeless to expect the entire atomic diagram to be computable. A θ-extensional cover, however, gives us a way of describing individual elements and tuples of them: using the cover, we name an $\mathcal{A}_l$ which θ-extensionally matches the substructure of $\mathcal{S}$ generated by the tuple, and specify which elements of $\mathcal{A}_l$ correspond to the tuple.

To state the same fact differently, having a θ-extensional cover tells us exactly what information we need about the tuple $\vec{p}$ from $\mathcal{S}$ in order to compute the atomic theory of $\mathcal{S}$ over $\vec{p}$, or to enumerate its Σ_1 theory over $\vec{p}$, etc.

For the field of complex numbers, for instance, an $\mathcal{A}_i$ is given by its transcendence degree and the minimal polynomial of a single additional element generating the rest of $\mathcal{A}_i$ over a transcendence basis. If we can determine this information for the subfield $\mathbb{Q}(\vec{p}) \subset \mathbb{C}$, and know which elements correspond to $\vec{p}$, then without further information we can give a Σ^0_ζ description of the Σ_ζ-theory of $(\mathbb{C}, \vec{p})$. Each θ-extensional cover of any $\mathcal{S}$ says, "if you tell me this particular information about your tuple $\vec{p}$ from $\mathcal{S}$, I will give you a Σ^0_ζ-presentation of the Σ_ζ facts about $\vec{p}$ in $\mathcal{S}$, for each $\zeta \leq \theta$."

For a useful example of the foregoing (rather abstract) remarks, we urge the reader to examine the discussion in Section 8 of the lexicographic order on Cantor space 2^ω. Some further philosophical discussion takes place there as well, in the context of that example.

4 Perfect Local Computability

Since local computability is conceived as a generalization of the notion of computable structure, it is natural to ask about the relationship between local computability and computable presentability for countable structures. All computably presentable structures are readily seen to be locally computable, and indeed θ-extensionally locally computable for all $\theta < \omega_1^{CK}$; a proof appears below. However, the converse fails. The attempt to find a version of extensionality equivalent to computable presentability for countable structures leads one to the following definition, which can be viewed as a kind of ∞-extensionality, stronger than θ-extensionality for every computable θ.

Definition 4.1 Let $\mathfrak{A}$ be a uniformly computable cover for a structure $\mathcal{S}$. A set M is a *correspondence system* for $\mathfrak{A}$ and $\mathcal{S}$ if it satisfies the following five conditions:

(1). Each element of M is an embedding of some $\mathcal{A}_i \in \mathfrak{A}$ into $\mathcal{S}$; and

(2). For every $\mathcal{A}_i \in \mathfrak{A}$, there exists a $\beta \in M$ with domain $\mathcal{A}_i$; and

(3). For every finitely generated substructure $\mathcal{B}$ of $\mathcal{S}$, there exists a $\beta \in M$ with image $\mathcal{B}$; and

(4). For every $\mathcal{A}_i \in \mathfrak{A}$, every $\beta \in M$ with domain $\mathcal{A}_i$, and every finitely generated $\mathcal{C} \subseteq \mathcal{S}$ such that $\beta(\mathcal{A}_i) \subseteq \mathcal{C}$, there exists an $\mathcal{A}_j \in \mathfrak{A}$, a $\gamma \in M$ with domain $\mathcal{A}_j$ and image $\mathcal{C}$, and an $f \in I^{\mathfrak{A}}_{ij}$ such that $\beta = \gamma \circ f$; and

(5). For every $\mathcal{A}_i \in \mathfrak{A}$, every $\beta \in M$ with domain $\mathcal{A}_i$, every $\mathcal{A}_m \in \mathfrak{A}$, and every $g \in I^{\mathfrak{A}}_{im}$, there exists an $\epsilon \in M$ with domain $\mathcal{A}_m$ such that $\beta = \epsilon \circ g$ (and hence $\beta(\mathcal{A}_i) \subseteq \epsilon(\mathcal{A}_m)$).

If $\mathcal{S}$ has a uniformly computable cover $\mathfrak{A}$ with a correspondence system M, then we say $\mathcal{S}$ is ∞-*extensionally locally computable*, and refer to elements of M as ∞-*extensional matches.*

A correspondence system M is *perfect* if it also satisfies:

(6). For every finitely generated $\mathcal{B} \subseteq \mathcal{S}$, if $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ and $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{B}$ both lie in M, then $\gamma^{-1} \circ \beta \in I^{\mathfrak{A}}_{ij}$.

If a perfect correspondence system exists, then its elements are called *perfect matches* between their domains and their images. The uniformly computable cover $\mathfrak{A}$ is then called a *perfect cover* for $\mathcal{S}$, and $\mathcal{S}$ itself is said to be *perfectly locally computable.*

Once again, the diagrams for conditions (4) and (5) are exactly those from Definition 3.1; the only difference is that now the isomorphisms γ and ϵ are required to lie in M.

This concept is related to extensionality, clearly, and any correspondence system M is quickly seen (by induction on θ) to contain only θ-extensional matches, for every ordinal θ. So every ∞-extensionally locally computable structure is θ-extensionally locally computable for every θ. This justifies the terminology and, using Corollary 3.9, also yields:

Corollary 4.2 *Any two structures with the same perfect cover are elementarily equivalent, and indeed have the same hyperarithmetical theory.* ■

However, the definition of ∞-extensionality is stronger than that of θ-extensionality. For the map β to be a θ-extensional match, we only needed the existence of ζ-extensional matches γ (with $\zeta < \theta$) to relate the embeddings $f \in I^{\mathfrak{A}}_{ij}$ (for all j) to the finitely generated extensions of the image of β in $\mathcal{S}$, and for different values of ζ, we could use different maps γ. Here Conditions (4) and (5) require that the isomorphisms γ be in M themselves, hence that they satisfy the same conditions.

For perfect covers, Condition (6) creates a second difference, which will be important in Theorem 6.3, but is not related to Definition 3.2. In fact a converse of Condition (6) follows from the first five conditions: if $f \in I^{\mathfrak{A}}_{ij}$ is an isomorphism of $\mathcal{A}_i$ onto $\mathcal{A}_j$, then M contains maps β with domain $\mathcal{A}_i$ and γ

with domain $\mathcal{A}_j$, with the same image in $\mathcal{S}$ and with $f = \gamma^{-1} \circ \beta$. Indeed, the $\beta : \mathcal{A}_i \hookrightarrow \mathcal{S}$ given by Condition (2) and the $\gamma : \mathcal{A}_j \hookrightarrow \mathcal{S}$ subsequently given by Condition (4) have the same image in $\mathcal{S}$, since if $y \in \gamma(\mathcal{A}_j)$, then $y = \gamma(f(a))$ for some $a \in \mathcal{A}_i$, but $\gamma(f(a)) = \beta(a)$ lies in the image of β. Thus Condition (6) is essentially a characterization of the isomorphisms in $I^{\mathfrak{A}}$: there is an isomorphism $f \in I^{\mathfrak{A}}_{ij}$ iff $\mathcal{A}_i$ and $\mathcal{A}_j$ describe the same substructure of $\mathcal{S}$ under the perfect correspondence system M, in which case f factors through the relevant maps in M.

Condition (6) is most easily met by requiring the map β in Condition (3) to be unique. We refer to this as the *Uniqueness Condition*:

(3′). For every finitely generated substructure $\mathcal{B}$ of $\mathcal{S}$, there exists a unique $\beta \in M$ with image $\mathcal{B}$.

Then Condition (6) only requires that each $I^{\mathfrak{A}}_{ii}$ contain the identity map; we will discuss this further in Section 5. If the Uniqueness Condition holds, then each finitely generated $\mathcal{B} \subseteq \mathcal{S}$ has a unique $\mathcal{A}_i \in \mathfrak{A}$ to describe it.

Condition (6) itself does not quite require this uniqueness, but it comes close to doing so. If we build the equivalence relation $\equiv$ on $\{\mathcal{A}_i : i \in \omega\}$ generated by $\{\langle \mathcal{A}_i, \mathcal{A}_j \rangle : (\exists f \in I^{\mathfrak{A}}_{ij}) \text{ range}(f) = \mathcal{A}_j\}$, and define the new cover $\mathfrak{A}/\equiv$, with an appropriate adjustment to $I^{\mathfrak{A}}$, then we would have uniqueness. Of course, in a uniformly computable cover $\mathfrak{A}$ of $\mathcal{S}$, it is Σ_1 but not necessarily decidable whether the image of an $f \in I^{\mathfrak{A}}_{ij}$ contains all of $\mathcal{A}_j$ or not, and therefore $\mathfrak{A}/\equiv$ might not be a uniformly computable cover.

We note that it is not reasonable to replace Condition (2) in Definition 4.1 by any uniqueness condition dual to Condition (3′) above. Since $\mathfrak{A}$ is countable, the uniqueness of the maps β in Condition (2) would force $\mathcal{S}$ also to be countable, and of course we wish our analysis to apply to uncountable structures as well as countable ones.

Corollary 4.3 *Every algebraically closed field of characteristic* 0 *is perfectly locally computable.*

Proof. We refer to the proof of Proposition 3.3. The necessary correspondence system M is the set of all embeddings of structures of $\mathfrak{A}$ into $\mathcal{C}$. Lemma 3.4 showed that every such isomorphism is an extensional match, and so the first five conditions of Definition 4.1 are quickly satisfied. Finally, if β and γ are as in Condition (6), then $\gamma^{-1} \circ \beta$ is an embedding of $\mathcal{A}_i$ into $\mathcal{A}_j$, and therefore must lie in $I^{\mathfrak{A}}_{ij}$. ∎

The situation of Corollary 4.3 generalizes to a further result:

Proposition 4.4 *Suppose that $\mathfrak{A}$ is a perfect cover for $\mathcal{S}$, with a correspondence system M such that for all $\mathcal{A}_i \in \mathfrak{A}$, every embedding of $\mathcal{A}_i$ into $\mathcal{S}$ is an element of M. Then $\mathcal{S}$ is ω-homogeneous.*

This result is purely model-theoretic: the same proof would hold for any cover $\mathfrak{A}$ for which such a correspondence system M exists, regardless of whether $\mathfrak{A}$ were computable. For an introduction to ω-homogeneity, see p. 212 of [7].

Proof. Let $\vec{x}, \vec{y} \in \mathcal{S}^m$ be two finite sequences of elements of $\mathcal{S}$ such that $(\mathcal{S}, \vec{x}) \equiv (\mathcal{S}, \vec{y})$. (The notation $\equiv$ denotes elementary equivalence, as usual.) We must show that for every $z \in \mathcal{S}$ there exists $w \in \mathcal{S}$ such that $(\mathcal{S}, \vec{x}, z) \equiv (\mathcal{S}, \vec{y}, w)$. To see this, notice that the substructures $\mathcal{B}_x$ and $\mathcal{B}_y$ of $\mathcal{S}$ generated by $\vec{x}$ and $\vec{y}$ must be isomorphic, say via an isomorphism $g : \mathcal{B}_x \twoheadrightarrow \mathcal{B}_y$. Now there exists an $\mathcal{A}_i \in \mathfrak{A}$ and a $\beta \in M$ mapping $\mathcal{A}_i$ isomorphically onto $\mathcal{B}_x$, and also an $\mathcal{A}_j \in \mathfrak{A}$, an $f \in I^{\mathfrak{A}}_{ij}$, and a $\gamma \in M$ mapping $\mathcal{A}_j$ isomorphically onto the substructure $\mathcal{B}_z$ generated by $\mathcal{B}_x$ and z, such that $\gamma \circ f = \beta$. But the isomorphism $g \circ \beta$ from $\mathcal{A}_i$ onto $\mathcal{B}_y$ must lie in M, by the assumption of the Proposition. So there also exists some $\mathcal{C} \subseteq \mathcal{S}$ and some isomorphism $\alpha : \mathcal{A}_j \twoheadrightarrow \mathcal{C}$ such that $\alpha \circ f = g \circ \beta$. Then $\alpha \circ \gamma^{-1}$ is an isomorphism from $\mathcal{B}_z$ onto $\mathcal{C}$ extending β, and we let $w = \alpha(\gamma^{-1}(z))$. Definition 4.1 shows that every extension of $\mathcal{B}_z$ in $\mathcal{S}$ corresponds to an embedding of $\mathcal{A}_j$ into some other element of $\mathfrak{A}$, which in turn corresponds to an extension of $\mathcal{C}$, and conversely. Hence $(\mathcal{S}, \vec{x}, z) \equiv (\mathcal{S}, \vec{y}, w)$ as required.

(Alternatively, having found $\mathcal{A}_j$ and w as above, work in the language augmented by constants $\vec{x}$ and z. Apply Corollary 3.9 to the structures $(\mathcal{S}, \vec{x}, z)$ and $(\mathcal{S}, \vec{y}, w)$, for each of which $\mathfrak{A}' = \{A_m \in \mathfrak{A} : I^{\mathfrak{A}}_{jm} \neq \emptyset\}$ is a perfect cover, with all $I^{\mathfrak{A}'}_{mn} = I^{\mathfrak{A}}_{mn}$.) ∎

5 A Dash of Category Theory

Here we define the *natural cover* of a structure $\mathcal{B}$. If $\mathcal{B}$ is a (globally) computable structure, then its natural cover will be uniformly computable and perfect. We give the definition under the assumption that $\mathcal{B}$ is computable.

Let $\mathcal{B}_i$ be the substructure of $\mathcal{B}$ generated by the i-th tuple $(\vec{b})_i$ of $\omega^{<\omega}$. Then the domain of $\mathcal{B}_i$ is a computably enumerable set. Enumerate its elements, and let A_i be the domain of the enumeration. (That is, A_i is an initial segment of ω from which we have a bijection β_i onto the domain of $\mathcal{B}_i$, all computable uniformly in i.) Then we may define computable structures $\mathcal{A}_i$, uniformly in i and each with domain A_i, isomorphic to each $\mathcal{B}_i$ via the map β_i. Next, we define $I_{ij}^{\mathfrak{A}}$ to be the set of functions

$$\{\beta_j^{-1} \circ \beta_i : \mathcal{B}_i \hookrightarrow \mathcal{B}_j\},$$

i.e. the inclusion maps in $\mathcal{B}$, lifted to the cover. Clearly these are embeddings, and the condition $(\vec{b})_i \subseteq \mathcal{B}_j$ is Σ_1, so this is a c.e. set, uniformly in i and j.

If $\mathcal{B}$ is countable but not computable, then the $\mathfrak{A}$ built here is still a cover of $\mathcal{B}$, but may fail to be uniformly computable, or even to be computable at all. If $\mathcal{B}$ is uncountable, then the indices i range over the uncountable set containing all finite tuples of elements of the domain of $\mathcal{B}$, and so the cover $\mathfrak{A}$ itself is uncountable. In this case, of course, effectiveness considerations do not apply.

Modulo the pullback to the domain ω, the natural cover of $\mathcal{B}$ really just the category **FGSub**($\mathcal{B}$) of all finitely generated substructures of $\mathcal{S}$, known long before now to model theorists and category theorists. The objects of **FGSub**($\mathcal{B}$) are precisely the finitely generated substructures of $\mathcal{B}$, and the morphisms are the inclusions among these substructures. (Fraïssé referred to the set of objects of **FGSub**($\mathcal{B}$) as the *age of* $\mathcal{B}$.) Our definition of the natural cover pulls each substructure back to the domain ω, or an initial segment thereof, since we are concerned with issues of computability, but in pure category theory one normally uses just the substructure itself.

This raises the question of whether all covers are themselves categories. As collections of objects with maps among them, covers seem ripe for consideration as categories. Definition 2.9 does come up short in two respects. The first is trivial: for a cover $\mathfrak{A}$, it does not require that $I_{ii}^{\mathfrak{A}}$ contain the identity map from $\mathcal{A}_i$ to itself. The second is less so: it does not require that the sets $I_{ij}^{\mathfrak{A}}$ of embeddings be closed under composition. This may be rectified in the case of an ∞-extensional cover.

Lemma 5.1 *Let $\mathfrak{A}$ be a uniformly computable cover of a structure $\mathcal{S}$.*

1. *Define $\mathfrak{A}_{id}$ to consist of the same simple cover as $\mathfrak{A}$, with*

$$I_{ij}^{\mathfrak{A}_{id}} = \begin{cases} I_{ij}^{\mathfrak{A}}, & \text{if } i \neq j \\ I_{ii}^{\mathfrak{A}} \cup \{id_{\mathcal{A}_i}\}, & \text{if } j = i. \end{cases}$$

Then $\mathfrak{A}_{id}$ is also a uniformly computable cover of $\mathcal{S}$. Moreover, each of these covers is θ-extensional (resp. ∞-extensional) iff the other is. If $\mathfrak{A}$ was perfect, then $\mathfrak{A}_{id} = \mathfrak{A}$.

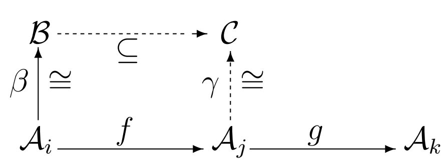
2. *Define $\overline{\mathfrak{A}}$ to consist of the same simple cover as $\mathfrak{A}$, with $I_{ij}^{\overline{\mathfrak{A}}}$ containing all compositions*

$$\mathcal{A}_i = \mathcal{A}_{i_0} \hookrightarrow \mathcal{A}_{i_1} \hookrightarrow \cdots \hookrightarrow \mathcal{A}_{i_n} \hookrightarrow \mathcal{A}_{i_{n+1}} = \mathcal{A}_j$$

with the intermediate maps each from the appropriate $I_{i_k i_{k+1}}^{\mathfrak{A}_{id}}$. The $\overline{\mathfrak{A}}$ is another uniformly computable cover of $\mathcal{S}$. If $\mathfrak{A}$ was ∞-extensional (resp. perfect), then so is $\overline{\mathfrak{A}}$, and if $\mathfrak{A}$ has the Amalgamation Property (as described in Definition 6.1 below), then so does $\overline{\mathfrak{A}}$.

Proof. Item (1) is immediate from the definitions, since Condition (6) of Definition 4.1 shows that in a perfect cover, each $I_{ii}^{\mathfrak{A}}$ already contains the identity embedding. Likewise, it is readily seen that $\overline{\mathfrak{A}}$ is a uniformly computable cover of $\mathcal{S}$: one need only remark that if $f \in I_{ij}^{\mathfrak{A}}$ and $g \in I_{jk}^{\mathfrak{A}}$, then g lifts to an inclusion $\mathcal{B} \subseteq \mathcal{C}$ via some β and γ, and $(g \circ f)$ lifts to the inclusion $\beta(f(\mathcal{A}_i)) \subseteq \mathcal{C}$. Now suppose that M is a correspondence system for $\mathfrak{A}$. Only Condition (5) of Definition 4.1 warrants consideration, and it is not difficult: if $\beta \in M$ with domain $\mathcal{A}_i$ and $g \in I_{im}^{\overline{\mathfrak{A}}}$, then g is the composition of a finite chain of maps from sets $I_{i_k i_{k+1}}^{\mathfrak{A}}$, and the result follows by induction on the length of this chain. Condition (6) is also immediate, giving the result for perfect covers. A similar induction shows the the Amalgamation Property holds for $\overline{\mathfrak{A}}$ as well, assuming it held for $\mathfrak{A}$. ∎

For a θ-extensional cover $\mathfrak{A}$ with $\theta > 0$, our $\overline{\mathfrak{A}}$ can fail to be θ-extensional. With $\theta = 1$, for example, suppose that β is a 1-extensional match with respect to $\mathfrak{A}$, and fix $f \in I_{ij}^{\mathfrak{A}}$ and $g \in I_{jk}^{\mathfrak{A}}$. 1-extensionality guarantees the existence of the $\mathcal{C}$ and γ in the diagram below, and $\mathcal{A}_j$ in turn will have a 1-extensional match onto some substructure of $\mathcal{S}$, but not necessarily onto $\mathcal{C}$, let alone via γ. There may not exist any $\mathcal{D} \supseteq \mathcal{B}$ and 0-extensional $\delta : \mathcal{A}_k \to \mathcal{D}$ corresponding to $(g \circ f)$ and β, in which case $(g \circ f)$ shows β not to be 1-extensional with respect to $\overline{\mathfrak{A}}$.



It remains open whether, for $0 < \theta < \infty$, a structure $\mathcal{S}$ can have a θ-extensional cover without having a θ-extensional cover closed under composition and identity embeddings.

The following appears as Proposition 4.1 in [11].

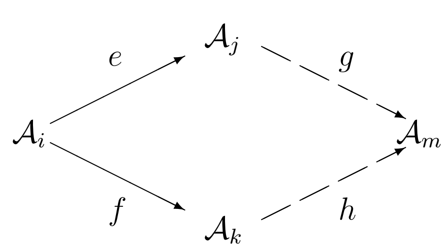
Proposition 5.2 (Miller-Mulcahey) *If $\mathcal{S}$ is perfectly locally computable, then there exists a faithful functor R mapping $\boldsymbol{FGSub}(\mathcal{S})$ into a perfect cover $\mathfrak{A}$ of $\mathcal{S}$ which is closed under composition and identity embeddings. Moreover, there exists a natural isomorphism $\beta : (I_{\mathfrak{A}} \circ R) \to I_{\boldsymbol{FGSub}(\mathcal{S})}$ where the I_{-} denote the appropriate inclusions into the category of all L-structures under embeddings.* ∎

As shown there, this functor $R : \mathbf{FGSub}(\mathcal{S}) \to \mathfrak{A}$ is essentially surjective, although it need not be onto. It also follows that $colim(I_{\mathfrak{A}} \circ R) \simeq \mathcal{S}$.

6 Countable Structures

The general intention of local computability is to apply computability theory to uncountable structures. Nevertheless, we can learn a good deal about our definitions by asking which countable structures satisfy them. In particular, our first result makes clear (especially when seen in concert with Theorem 3.10) that among all the concepts we have defined for uncountable structures, perfect local computability is the most apt generalization of the notion of computable presentability for countable structures. First we require a notion from model theory, adapted to our concept of a cover.

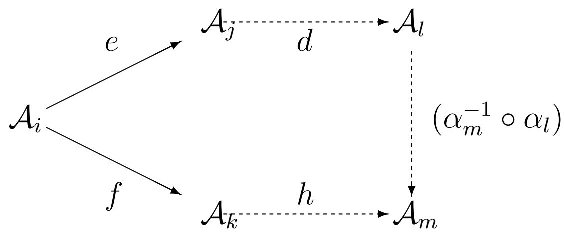
Definition 6.1 A cover $\mathfrak{A}$ has the *Amalgamation Property*, abbreviated AP, if for all $i, j, k \in \omega$ and all maps $e \in I^{\mathfrak{A}}_{ij}$ and $f \in I^{\mathfrak{A}}_{ik}$, there exists an m and maps $g \in I^{\mathfrak{A}}_{jm}$ and $h \in I^{\mathfrak{A}}_{km}$ for which $h \circ f = g \circ e$:



We say that we can *amalgamate* $\mathcal{A}_j$ and $\mathcal{A}_k$ over $\mathcal{A}_i$, relative to the maps e and f.

Lemma 6.2 *Every perfect cover $\mathfrak{A}$ of any structure $\mathcal{S}$ has the Amalgamation Property.*

Proof. Let M be a perfect correspondence system for $\mathcal{S}$ over $\mathfrak{A}$, and fix maps $e \in I_{ij}^{\mathfrak{A}}$ and $f \in I_{ik}^{\mathfrak{A}}$. We use the six conditions from Definition 4.1, and suggest that the reader follow the diagram below (which shows only the maps among structures in the cover, omitting their images in $\mathcal{S}$).



Now there is an $\alpha_i : \mathcal{A}_i \hookrightarrow \mathcal{S}$ in M, by Condition (2), and Condition (5) yields maps $\alpha_j : \mathcal{A}_j \hookrightarrow \mathcal{S}$ and $\alpha_k : \mathcal{A}_k \hookrightarrow \mathcal{S}$ in M with $\alpha_k \circ f = \alpha_j \circ e = \alpha_i$. Let $\mathcal{C} \subseteq \mathcal{S}$ be generated by the images of α_j and α_k together. Then $\mathcal{C}$ is finitely generated, so by Condition (4) we have elements $l, m \in \omega$, embeddings $d \in I_{jl}^{\mathfrak{A}}$ and $h \in I_{km}^{\mathfrak{A}}$, and maps $\alpha_l, \alpha_m \in M$ with domains $\mathcal{A}_l$ and $\mathcal{A}_m$, respectively, and both with image $\mathcal{C}$, such that $\alpha_l \circ d = \alpha_j$ and $\alpha_m \circ h = \alpha_k$. But Condition (6) also holds, since M is perfect, and so $\alpha_m^{-1} \circ \alpha_l \in I_{lm}^{\mathfrak{A}}$. Lemma 5.1 allows us to assume closure of the sets of embeddings under composition, In particular, we set

$$g = \alpha_m^{-1} \circ \alpha_j = (\alpha_m^{-1} \circ \alpha_l) \circ d \in I_{jm}^{\mathfrak{A}}.$$

But then

$$g \circ e = \alpha_m^{-1} \circ (\alpha_j \circ e) = \alpha_m^{-1} \circ \alpha_i = \alpha_m^{-1} \circ (\alpha_k \circ f) = \alpha_m^{-1} \circ (\alpha_m \circ h) \circ f = h \circ f,$$

as required by the Amalgamation Property. ■

Theorem 6.3 *Let $\mathcal{S}$ be any countable structure. Then the following are equivalent.*

1. *$\mathcal{S}$ is computably presentable.*

2. $\mathcal{S}$ is perfectly locally computable.

3. $\mathcal{S}$ is ∞-extensional over a cover $\mathfrak{A}$ with the Amalgamation Property.

Proof. $(1 \Longrightarrow 2)$ is not difficult. Let $\mathfrak{A}$ be the natural cover of a computable presentation $\mathcal{B}$ of $\mathcal{S}$, as defined in Section 5, and fix the maps β_i defined there as well. It is quickly seen that this $\mathfrak{A}$ is a perfect cover of $\mathcal{B}$, under the correspondence system $M = \{\beta_i : i \in \omega\}$. Conditions (1)-(3) of Definition 4.1 are immediate, as is Condition (5), given our definition of $I^{\mathfrak{A}}_{ij}$. For Condition (4), when $\beta_i(\mathcal{A}_i) \subseteq \mathcal{C}$, we know that $\mathcal{C} = \mathcal{B}_j$ for some j, that the map β_j lies in M, and that $(\beta_j^{-1} \circ \beta_i) \in I^{\mathfrak{A}}_{ij}$ has the necessary properties. Finally, if $\beta_i, \beta_j \in M$ have the same image, then $\mathcal{B}_i = \mathcal{B}_j$, and so $(\beta_j^{-1} \circ \beta_i) \in I^{\mathfrak{A}}_{ij}$ as required by Condition (6). Thus $\mathcal{B}$ is perfectly locally computable. To see that $\mathcal{S}$ is perfectly locally computable, take the same cover $\mathfrak{A}$ and use the correspondence system $\{\gamma \circ \beta_i : \beta_i \in M\}$, where γ is an isomorphism from $\mathcal{B}$ onto $\mathcal{S}$.

$(2 \Longrightarrow 3)$ follows from Lemma 6.2, so it remains to prove $(3 \Longrightarrow 1)$. Suppose that $\mathfrak{A}$ is a uniformly computable cover for $\mathcal{S}$, with a correspondence system M satisfying AP. Using Lemma 5.1, we may assume that $\mathfrak{A}$ is closed under composition. In category-theoretic terms, $\mathcal{S}$ is just the inverse limit of the category $\mathfrak{A}$, but we need to build a computable presentation $\mathcal{B}$ of this inverse limit. We will define i_s recursively so that $\mathcal{B}_s = \mathcal{A}_{i_s}$, with a map $g_s : \mathcal{B}_s \hookrightarrow \mathcal{B}_{s+1}$ from $I^{\mathfrak{A}}_{i_s,i_{s+1}}$. To start, we let $\mathcal{B}_0 = \mathcal{A}_0$ and $i_0 = 0$.

Given $\mathcal{B}_s$, let $s = \langle t, u, v\rangle$, and fix $i = i_t$ and $k = i_s$, so that $\mathcal{B}_t = \mathcal{A}_i$ and $\mathcal{B}_s = \mathcal{A}_k$. We begin stage $s+1$ by listing out those maps in $I^{\mathfrak{A}}$ with domain $\mathcal{B}_t$ until we find the u-th element on this list. Let $f \in I^{\mathfrak{A}}_{ij}$ (for some j) be this element. (Of course $t \leq s$, so $\mathcal{B}_t$ is defined. Also, if there were only finitely many maps in $I^{\mathfrak{A}}$ with domain $\mathcal{A}_i$, then Definition 4.1 would imply that $\mathcal{S}$ is itself finitely generated, hence isomorphic to some $\mathcal{A}_i \in \mathfrak{A}$, hence computably presentable. Therefore we may assume that we do find such an f.)

We will incorporate this f into $\mathcal{B}_{s+1}$ as follows. By the Amalgamation Property, there exists some m and embeddings $g \in I^{\mathfrak{A}}_{km}$ and $p \in I^{\mathfrak{A}}_{jm}$ such that

$$p \circ f = g \circ (g_{s-1} \circ \cdots \circ g_t).$$

(Recall that we closed the cover under composition of maps, so that $g_{s-1} \circ \cdots \circ g_t \in I^{\mathfrak{A}}_{ik}$.) We find the least such triple $\langle m, g, p\rangle$, and define $i_{s+1} = m$ and $\mathcal{B}_{s+1} = \mathcal{A}_m$, with $f_s = p \circ f : \mathcal{B}_{t_{s+1}} \hookrightarrow \mathcal{B}_{s+1}$ and $g_s = g : \mathcal{B}_s \hookrightarrow \mathcal{B}_{s+1}$.

This completes stage $s+1$, and we say that f has been incorporated into $\mathcal{B}$ at this stage. Notice that every $f \in I^{\mathfrak{A}}$ whose domain is $\mathcal{A}_{i_t}$ $(= \mathcal{B}_t)$ for any t will be incorporated into $\mathcal{B}$ at infinitely many stages.

The structure $\mathcal{B}$ itself is the union of the chain of structures $\mathcal{B}_s$ under the embeddings $g_{s+1} : \mathcal{B}_s \hookrightarrow \mathcal{B}_{s+1}$. To build this $\mathcal{B}$ computably, at stage $\langle u, v\rangle$ we consider the element t (if any) which is enumerated into the domain of $\mathcal{B}_u$ at stage v. (Note that $\mathcal{B}_u$ might be finite, so we cannot just wait for its next element to appear.) For each of the finitely many elements x which has already entered $\mathcal{B}$ at some stage $\langle u', v'\rangle < \langle u, v\rangle$, we check whether either $x = g'_u \circ g_{u'-1} \circ \cdots \circ g_{u+1}(t)$ (if $u' > u$) or $t = g_u \circ g_{u-1} \circ \cdots \circ g_{u'+1}(x)$ (if $u' < u$). If either of these holds, we do nothing at this stage; if neither holds, then we add a fresh element to $\mathcal{B}$ and identify it with $t \in \mathcal{B}_u$. Iterating this process over all stages $\langle u, v\rangle$ builds the domain of $\mathcal{B}$, and we define the functions and relations on it in the obvious way. Notice that for every s the embeddings of $\mathcal{B}_s$ and $\mathcal{B}_{s+1}$ into $\mathcal{B}$ are compatible with the map $g_s : \mathcal{B}_s \hookrightarrow \mathcal{B}_{s+1}$, and that we can compute these embeddings uniformly in s, so from here on we will view each $\mathcal{B}_s$ as a substructure of $\mathcal{B}$, with $g_s : \mathcal{B}_s \hookrightarrow \mathcal{B}_{s+1}$ as an inclusion.

Next we build the isomorphism $\alpha : \mathcal{B} \to \mathcal{S}$. Of course, α need not be computable. At every stage $s+1$, the image $\mathcal{S}_s$ of the current approximation α_s will be a finitely generated substructure of $\mathcal{S}$, its domain will be some $\mathcal{B}_t$, and α_s will lie in M. We will ensure that the extension of α_s is compatible with the inclusion maps $g_t : \mathcal{B}_t \hookrightarrow \mathcal{B}_{t+1}$, so that at the end of the construction, we may define $\alpha(x)$ for $x \in \mathcal{B}$ simply by finding some s and t with $x \in \mathcal{B}_t = \mathrm{dom}(\alpha_s)$ and letting $\alpha(x) = \alpha_s(x)$. We start by taking $\alpha_0 \in M$ to be any embedding of $\mathcal{B}_0 = \mathcal{A}_{i_0}$ into $\mathcal{S}$.

Suppose that $\mathrm{dom}(\alpha_s) = \mathcal{B}_t = \mathcal{A}_i$ and that we wish to extend α_s to α_{s+1} by adding a new element $y \in \mathcal{S}$ to $\mathcal{S}_s = \mathrm{range}(\alpha_s)$. Since M is a correspondence system containing α_s, there exists $A_j \in \mathfrak{A}$, $f \in I^{\mathfrak{A}}_{ij}$, and a map $\beta \in M$ such that β maps $\mathcal{A}_j$ onto the substructure of $\mathcal{S}$ generated by y and $\mathcal{S}_s$, with $\beta \circ f = \alpha_s$. Now since $f \in I^{\mathfrak{A}}$, there is some stage $s' > t$ at which f is incorporated into $\mathcal{B}$. When this happened, we defined an embedding $f_{s'} = p \circ f : \mathcal{B}_t \hookrightarrow \mathcal{B}_{s'+1}$, with $p \in I^{\mathfrak{A}}_{jm}$ for some m, and with $f_{s'} \in I^{\mathfrak{A}}_{im}$ because $I^{\mathfrak{A}}$ is closed under composition. But since β lies in the correspondence system M, there exists $\gamma \in M$ with domain $\mathcal{B}_{s'+1}$ such that $\beta = \gamma \circ p$. We define $\alpha_{s+1} = \gamma$, with domain $\mathcal{B}_{s'+1}$ and let $\mathcal{S}_{s+1} = \mathrm{range}(\alpha_{s+1})$, noting that for $x \in \mathrm{dom}(\alpha_s)$,

$$\alpha_{s+1}(g_{s'} \circ \cdots \circ g_t(x)) = \gamma(p \circ f(x)) = \beta(f(x)) = \alpha_s(x).$$

Finally, $y \in \text{range}(\beta) \subseteq \text{range}(\gamma) = \text{range}(\alpha_{s+1})$. This completes the construction. Notice that for all s, $\text{dom}(\alpha_s) = \mathcal{B}_t \supseteq \mathcal{B}_s$, so that $\text{dom}(\alpha)$ is all of $\mathcal{B}$, and $\text{range}(\alpha) = \mathcal{S}$ by construction. Thus $\mathcal{S}$ and $\mathcal{B}$ really are isomorphic, and $\mathcal{S}$ is computably presentable. ■

7 Computable Simulations

We now show that perfectly locally computable structures can be "simulated" in a strong way by globally computable (and hence countable) structures. This can be viewed either as an indictment of perfect local computability, saying that it is such a strong condition that the only uncountable structures satisfying it are those which are very closely related to computable structures; or as further evidence that perfect local computability is the correct analogue, in the uncountable setting, to computable presentability in the countable setting. We leave this judgment to the reader. Work in this section is joint between Dustin Mulcahey and the author.

Definition 7.1 Let $\mathcal{S}$ be any structure. A *simulation of* $\mathcal{S}$ is an elementary substructure $\mathcal{B} \preceq \mathcal{S}$ such that $\mathcal{B}$ and $\mathcal{S}$ realize exactly the same finitary types. We often refer to any $\mathcal{A}$ isomorphic to such a $\mathcal{B}$ as a simulation of $\mathcal{S}$, even if $\mathcal{A}$ is not itself a substructure of $\mathcal{S}$. Hence a *computable simulation of* $\mathcal{S}$ is a computable structure isomorphic to a simulation of $\mathcal{S}$.

Lemma 7.2 *Let $\mathcal{S}$ be locally computable, with a correspondence system N over a uniformly computable cover $\mathfrak{A}$. Then $\mathcal{S}$ has a countable substructure $\mathcal{B}$ with its own correspondence system $M \subseteq N$ over $\mathfrak{A}$. If N was a perfect correspondence system for $\mathcal{S}$, then M is perfect for $\mathcal{B}$ as well.*

Proof. $\mathcal{B}$ will be a countable union of countable substructures $\mathcal{B}_s$ of $\mathcal{S}$. To start, we fix for each $i \in \omega$ one map $\alpha_i \in N$ with domain $\mathcal{A}_i$, Let $M_0 = \{\alpha_i : i \in \omega\}$, and let $\mathcal{B}_0$ be the substructure of $\mathcal{S}$ generated by the union of all the images of these α_i. The conditions for a perfect cover are $\forall\exists$ conditions, so now we will be able to keep $\mathcal{B}$ countable as we close M under those conditions, using the analogous conditions in the correspondence system N.

Assume we have defined a countable $\mathcal{B}_s$ and M_s. First, for every i, every $\alpha \in M_s$ with domain $\mathcal{A}_i$, and every $f \in I_{ij}^{\mathfrak{A}}$ (for any j), there exists some $\gamma \in N$ with domain $\mathcal{A}_j$ such that f lifts via α and γ to the inclusion $\alpha(\mathcal{A}_i) \subseteq \gamma(\mathcal{A}_j)$. Form $M_s' \supseteq M_s$ by adjoining one such γ to M_s for each such i, α and

f. Also, let $\mathcal{B}'_s$ be generated by the union of the images of the maps in M'_s. Clearly both $\mathcal{B}'_s$ and M'_s remain countable.

Next, for every i, every $\alpha \in M_s$ with domain $\mathcal{A}_i$, and every finitely generated $\mathcal{C} \subseteq \mathcal{B}_s$ with $\alpha(\mathcal{A}_i) \subseteq \mathcal{C}$, there exists some j, some $f \in I_{ij}^{\mathfrak{A}}$, and some $\gamma \in N$ with domain $\mathcal{A}_j$ such that f lifts to the inclusion $\alpha(\mathcal{A}_i) \subseteq \mathcal{C}$ via α and γ. Adjoin to M'_s one such γ for each such i, α, and $\mathcal{C}$, to form M''_s.

Finally, for every finitely generated substructure $\mathcal{C} \subseteq \mathcal{B}_s$, there exists a $\gamma \in N$ with image $\mathcal{C}$ (since $\mathcal{C} \subseteq \mathcal{S}$). Form M_{s+1} by adjoining to M''_s one such γ for each such $\mathcal{C}$. Since $\mathcal{B}_s$ was countable, it has only countably many finitely generated substructures, and so M_{s+1} is still countable.

It is clear that the union $\mathcal{B} = \cup_s \mathcal{B}_s$ is a countable substructure of $\mathcal{S}$, with cover $\mathfrak{A}$, and that $M = \cup_s M_s$ is a correspondence system for this $\mathcal{B}$ over $\mathfrak{A}$. Our $\mathcal{B}_0$ already satisfied item (2) of Definition 4.1, and our ensuing adjoinments satisfied (4), (5), and (3), in that order, without ever violating (1). (Of course $\mathfrak{A}$ is still uniformly computable as well; that definition has nothing to do with the structure covered by $\mathfrak{A}$.)

It remains to see that this M is perfect for $\mathcal{B}$ whenever N is perfect for $\mathcal{S}$. But this is easy: if α and γ lie in M and have the same image in $\mathcal{B}$, then they lie in N and have the same image in $\mathcal{S}$. Since N is perfect, $\gamma^{-1} \circ \alpha$ must then lie in the appropriate $I_{ij}^{\mathfrak{A}}$. ■

In this situation, $\mathcal{B}$ will be an elementary substructure of $\mathcal{S}$. The next lemma extends this observation. (If $\mathcal{B}$ is as in Lemma 7.2, and P is empty, then in the proof of Lemma 7.3 we may show that at every step ψ_s is just inclusion.)

Lemma 7.3 *Let $\mathcal{B}$ and $\mathcal{S}$ be two structures, each with a correspondence system over the same uniformly computable cover. Assume that $\mathcal{B}$ is countable. Then $\mathcal{B}$ is a simulation of $\mathcal{S}$. Indeed, for any countable set $P \subseteq \mathcal{S}$ of parameters, we can elementarily embed $\mathcal{B}$ into $\mathcal{S}$ so that its image contains P and realizes the same finitary types as $\mathcal{S}$ over every finite $P_0 \subseteq P$.*

Proof. Let $\mathfrak{A}$ be a common uniformly computable cover of $\mathcal{S}$ and $\mathcal{B}$, with correspondence systems M for $\mathcal{B}$ and N for $\mathcal{S}$. Our embedding is built step by step, so we start by enumerating the domain of $\mathcal{B}$ as $\{b_0, b_1, \ldots\}$, and P as $\{p_0, p_1, \ldots\}$. Fix an $\alpha \in M$ whose image is the substructure $\mathcal{B}_0 \subseteq \mathcal{B}$ generated by b_0, and a $\gamma \in N$ with the same domain as α, and define ψ_0 to be $\gamma \circ \alpha^{-1}$, with $\mathcal{B}_0 = \mathrm{dom}(\psi_0) \subseteq \mathcal{B}$ and $\mathcal{C}_0 = \mathrm{range}(\psi_0) \subseteq \mathcal{S}$.

At stage $t+1 = 2s+1$, we extend ψ_t so that its range contains p_s. By induction $\psi_t = \gamma \circ \alpha^{-1}$ for some $\gamma \in N$ and $\alpha \in M$ with common domain $\mathcal{A}_i$ in $\mathfrak{A}$. Let $\mathcal{C}_{t+1}$ be the substructure of $\mathcal{S}$ generated by $\mathcal{C}_t$ and p_s. By induction $\mathcal{C}_t$ is finitely generated, so there is a $\delta \in N$ with some domain $\mathcal{A}_j \in \mathfrak{A}$, and an $f \in I^{\mathfrak{A}}_{ij}$, such that f lifts via γ and δ to the inclusion $\mathcal{C}_t \subseteq \mathcal{C}_{t+1}$. In turn there is a $\beta \in M$ with domain $\mathcal{A}_j$ such that f lifts via α and β to the inclusion $\mathcal{B}_t \subseteq \beta(\mathcal{A}_j)$. Set $\mathcal{B}_{t+1} = \text{range}(\beta)$ and $\psi_{t+1} = \delta \circ \beta^{-1}$.

At stage $t+1 = 2s+2$, we extend the embedding ψ_t from its current domain $\mathcal{B}_t$ to the structure $\mathcal{B}_{t+1}$ generated by $\mathcal{B}_t$ and b_s. By induction $\mathcal{B}_t$ is finitely generated, and $\psi_t = \gamma \circ \alpha^{-1}$ for some $\gamma \in N$ and $\alpha \in M$ with common domain $\mathcal{A}_i$ in $\mathfrak{A}$. So there is a $\beta \in M$ with some domain $\mathcal{A}_j \in \mathfrak{A}$, and an $f \in I^{\mathfrak{A}}_{ij}$, such that f lifts via α and β to the inclusion $\mathcal{B}_t \subseteq \mathcal{B}_{t+1}$. In turn there is a $\delta \in M$ with domain $\mathcal{A}_j$ such that f lifts via γ and δ to the inclusion $\mathcal{C}_t \subseteq \delta(\mathcal{A}_j)$. Set $\mathcal{C}_{t+1} = \text{range}(\delta)$ and $\psi_{t+1} = \delta \circ \beta^{-1}$.

Now we define $\psi = \cup_t \psi_t$. Clearly ψ has domain $\mathcal{B}$ and range $\subseteq \mathcal{S}$ containing P, and ψ must be an embedding. To see that it is elementary, suppose that $\exists x \theta(\psi(b_0), \ldots, \psi(b_s), x)$ is an existential formula true in $\mathcal{S}$. Now $\psi_s = \gamma \circ \alpha^{-1}$ for some $\alpha \in M$ and $\gamma \in N$ with common domain $\mathcal{A}_i \in \mathfrak{A}$. Since N is perfect, there is a $\delta \in N$ (with some domain $\mathcal{A}_j$) and an $a \in \mathcal{A}_j$ and an $f \in I^{\mathfrak{A}}_{ij}$, such that $\theta(f(\alpha^{-1}(b_0)), \ldots, f(\alpha^{-1}(b_s)), a)$ holds in $\mathcal{A}_j$. But since M is also a perfect cover, there is a $\beta \in M$ with the same domain $\mathcal{A}_j$ such that f lifts to the inclusion $\mathcal{B}_s \subseteq \beta(\mathcal{A}_j)$ via α and β. Therefore $\theta(b_0, \ldots, b_s, \beta(a))$ holds in $\mathcal{B}$. Thus ψ is an elementary embedding.

Finally, given any n-type Γ over any finite parameter set $P_0 \subseteq P$, such that Γ is realized in $\mathcal{S}$ by a tuple $(d_1, \ldots, d_n)$, we start with the substructure $\mathcal{P}_0 \subseteq \mathcal{S}$ generated by P_0. Since $P_0 \subseteq \text{range}(\psi)$, we have a t for which $P_0 \subseteq \text{range}(\psi_t)$. Let $\psi_t = \gamma \circ \alpha^{-1}$ with $\alpha \in M$ and $\gamma \in N$. There must be a $\delta \in N$ with some domain $\mathcal{A}_j \in \mathfrak{A}$ and an $f \in I^{\mathfrak{A}}_{ij}$ such that f lifts via γ and δ to the inclusion of $\mathcal{P}_0$ into the substructure generated by $\mathcal{P}_0$ and $d_0, \ldots, d_n$. But now there is also some $\beta \in M$ with domain $\mathcal{A}_j$ such that f lifts via α and β to the inclusion $\mathcal{B}_t \subseteq \beta(\mathcal{A}_j)$, and we set $b_i = \beta(\delta^{-1}(d_i))$ and $c_i = \psi(b_i)$ for each i. Then $(c_1, \ldots, c_n)$ is an n-tuple within the image of ψ which realizes the type Γ over P_0, by standard arguments using M and N. ■

When we have a parameter set P as in Lemma 7.3, we refer to the image of $\mathcal{B}$ as a *simulation of* $\mathcal{S}$ *over* P. We might also refer to $\mathcal{B}$ itself the same way, but only when the embedding $\psi : \mathcal{B} \hookrightarrow \mathcal{S}$ is clear, because we need to know which elements $\psi^{-1}(p) \in \mathcal{B}$ correspond to the elements of P in this simulation. Later we will discuss the extent to which $\mathcal{B}_P$ can be said to be uniform in P.

Corollary 7.4 *Two countable structures with correspondence systems over the same uniformly computable cover are isomorphic.*

Proof. Since $\mathcal{S}$ is countable, we simply set $P = \mathcal{S}$ and apply Lemma 7.3, whose proof may now be regarded as a back-and-forth construction of an isomorphism from $\mathcal{B}$ onto $\mathcal{S}$. ■

We are now ready for the main result of this section.

Theorem 7.5 *Every perfectly locally computable structure $\mathcal{S}$ has a computable simulation $\mathcal{A}$, which can be embedded into $\mathcal{S}$ so as to simulate $\mathcal{S}$ over arbitrary countable parameter sets. Specifically, there is a set of elementary embeddings $\psi_p : \mathcal{A} \hookrightarrow \mathcal{S}$, one for each function $p : \omega \to \mathcal{S}$ which enumerates a countable parameter set $Q_p = range(p) \subseteq \mathcal{S}$, such that:*

- *$Q_p \subseteq \psi_p(\mathcal{A})$; and*
- *$\psi_p(\mathcal{A})$ is a simulation of $\mathcal{S}$ over Q_p; and*
- *if p and p' are two such functions and $p\restriction n = p'\restriction n$, then for all $k < n$,*

$$\psi_p^{-1}(p(k)) = \psi_{p'}^{-1}(p'(k)).$$

As a partial converse, every structure which has a computable simulation $\mathcal{A}$ with embeddings ψ_p satisfying these conditions has a uniformly computable cover with a correspondence system.

To make this last claim an actual converse, we would need to show that the correspondence system for $\mathcal{S}$ is perfect. Whether this is true remains open. We also note that it would be equivalent to give the same statement only for finite parameter enumerations p, since the last condition would allow a simulation over a countable parameter set P to be built by taking successive nested finite enumerations $p_m \subseteq p_{m+1}$ with $P = \cup_m \text{range}(p_m)$, and setting $\psi = \lim_m \psi_{p_m}$

Proof. When we assume that $\mathcal{S}$ is perfectly locally computable, the existence of a computable simulation of $\mathcal{S}$ follows from Lemma 7.2, which also ensures that $\mathcal{S}$ and its computable simulation $\mathcal{A}$ both have perfect correspondence systems over the same uniformly computable cover. Therefore Lemma 7.3 shows that $\mathcal{A}$ can be elementarily embedded into $\mathcal{S}$ so as to simulate $\mathcal{S}$ over any parameter set Q_p enumerated by a function $p : \omega \to \mathcal{S}$. Moreover, an examination of the proof of Lemma 7.3 shows that the embedding chooses the $a \in \mathcal{A}$ with $\psi_p(a) = p(k)$ using only the common cover $\mathfrak{A}$, its correspondence systems for $\mathcal{A}$ and $\mathcal{S}$, and the elements $p(0), p(1), \ldots, p(k)$ in $\mathcal{S}$. This proves the claim about parameter enumerations p and p' which agree up to n.

Next we show our partial converse: that the existence of such an A implies that $\mathcal{S}$ has a uniformly computable cover with a correspondence system. $\mathcal{A}$ has a perfect cover $\mathfrak{A}$, by Theorem 6.3. Let M be a perfect correspondence system for $\mathcal{A}$ and $\mathfrak{A}$. The correspondence system N will consist of all maps of the form $\psi_p \circ \alpha$, for all finite $p : n \to \mathcal{S}$ and all $\alpha \in M$ such that range(α) is generated by $\{\psi_p^{-1}(p(i)) : i < n\}$. (Here we think of a finite function $p : n \to \mathcal{S}$ as a function from ω into $\mathcal{S}$ by repeating its image over and over: $p(k + nm) = p(k)$ for all k and m.)

Now each finitely generated $\mathcal{C} \subseteq \mathcal{S}$ with generators enumerated by p lies within the image of ψ_p, and the finitely generated substructure $\psi_p^{-1}(\mathcal{C}) \subseteq \mathcal{A}$ must be the image of some $\mathcal{A}_i \in \mathfrak{A}$ under some $\alpha \in M$, since M is a perfect cover of $\mathcal{A}$. Hence $\mathcal{C} = (\psi_p \circ \alpha)(\mathcal{A}_i)$ is the image of some map in N. Likewise, each $\mathcal{A}_i \in \mathfrak{A}$ is the domain of some $\alpha \in M$, hence also of some map in N. Moreover, each $f \in I_{ij}^{\mathfrak{A}}$, for any i and j, lifts to an inclusion map within $\mathcal{A}$, and then lifts further to an inclusion map within $\mathcal{S}$, via any ψ_p we like. Conversely, any inclusion $\mathcal{C}' \subseteq \mathcal{C}$ of finitely-generated substructures of $\mathcal{S}$ is the lift (via ψ_p, where p enumerates first the generators of $\mathcal{C}'$, and then the generators of $\mathcal{C}$) of an inclusion in $\mathcal{A}$, which in turn is the lift of some f in some $I_{ij}^{\mathfrak{A}}$ via some $\alpha, \beta \in M$. If p' is the restriction of p to the generators of $\mathcal{C}'$, then the inclusion $\mathcal{C}' \subseteq \mathcal{C}$ is the lift of f via $(\psi_{p'} \circ \alpha)$ and $(\psi_p \circ \beta)$, which both lie in N. Thus $\mathfrak{A}$ is a uniformly computable cover of $\mathcal{S}$.

The preceding remarks also proved the first three conditions in Definition 4.1. For Condition (5), fix any $f \in I_{ij}^{\mathfrak{A}}$ for any i and j, along with any $\beta \in N$ with domain $\mathcal{A}_i$. Then $\beta = \psi_p \circ \alpha$ for some $\alpha \in M$ and some $p : n \to \mathcal{S}$ for which $\{\psi_p^{-1}(p(k) : k < n\}$ generates range(α). Since M is a correspondence system, there is a $\gamma \in M$ with domain $\mathcal{A}_j$ such that $\alpha = \gamma \circ f$. But now there is a finite q such that $q{\restriction}n = p$ and $q(n + k) = \psi_p(a_k)$, where $a_0, \ldots, a_m$

generate $\gamma(\mathcal{A}_j)$ within $\mathcal{A}$. So $(\psi_q \circ \gamma) \in N$, and

$$(\psi_q \circ \gamma \circ f) = (\psi_q \circ \alpha) = (\psi_p \circ \alpha),$$

with the last equality following because $p{\restriction}n = q{\restriction}n$ and range(α) is generated by the elements $\psi_p^{-1}(p(k)) = \psi_q^{-1}(q(k))$ for $k < n$. This proves Condition (5).

For Condition (4) of Definition 4.1, fix any $\beta \in N$ with domain $\mathcal{A}_i$ and any finitely generated $\mathcal{C} \subseteq \mathcal{S}$ with $\beta(\mathcal{A}_i) \subseteq \mathcal{C}$. Now $\beta = \psi_p \circ \alpha$ for some $\alpha \in M$ and some finite $p : n \to \mathcal{S}$, with the elements $\psi_p^{-1}(p(k))$ generating range(α). Let $q(k) = p(k)$ for $k < n$, and let $q(n), \ldots, q(n+m-1)$ enumerate the generators of $\mathcal{C}$ in $\mathcal{S}$. By assumption, ψ_q is an elementary embedding of $\mathcal{A}$ into $\mathcal{S}$ whose image contains range(q). Let $\mathcal{D} = \langle \psi_q^{-1}(q(k)) \ : \ k < m \rangle \subseteq \mathcal{A}$. Since $\psi_q^{-1}(q(k)) = \psi_p^{-1}(p(k))$ for all $k < n$, we know that $\alpha(\mathcal{A}_i) \subseteq \mathcal{D}$, and since M is a correspondence system, there is some $\beta \in M$ and some j and $f \in I_{ij}^{\mathfrak{A}}$ with $\mathcal{D} = \text{range}(\beta)$ and $\beta \circ f = \alpha$. But then

$$(\psi_q \circ \beta \circ f) = (\psi_q \circ \alpha) = (\psi_p \circ \alpha),$$

proving Condition (4), since $(\psi_q \circ \beta) \in N$. Thus $\mathcal{A}$ is a uniformly computable cover of $\mathcal{S}$ with correspondence system N. ∎

We can think of $\mathcal{B}_P$ as being built uniformly in the parameter set P if the elements of P are named as elements in different $\mathcal{A}_i$ in the cover $\mathfrak{A}$ of $\mathcal{S}$. That is, suppose that we are given a computable enumeration $\langle (i_k, a_k, f_k) \rangle_{k \in \omega}$ for which there exist maps $\beta_k \in N$ with $a_k \in \mathcal{A}_{i_k} = \text{dom}(\beta_k)$ such that

- each $f_k \in I_{i_k, i_{k+1}}^{\mathfrak{A}}$; and
- $\beta_{k+1} \circ f_k = \beta_k$; and
- $\{\beta_k(a_k) : k \in \omega\} = P$.

Then we could build a computable copy of the simulation $\mathcal{B}_P$ of $\mathcal{S}$ over P, uniformly in the perfect cover of $\mathcal{S}$ and the enumeration $\langle (i_k, a_k, f_k) \rangle_{k \in \omega}$, and enumerate the image of P in $\mathcal{B}_P$. More generally, if the enumeration $\langle (i_k, a_k, f_k) \rangle_{k \in \omega}$ has Turing degree $\boldsymbol{d}$, then with a $\boldsymbol{d}$-oracle we can build a copy of $\mathcal{B}_P$ in which the image of P will be computably enumerable in $\boldsymbol{d}$. It is awkward to think of the set P itself as having Turing degree $\boldsymbol{d}$, because an infinite set P will have distinct enumerations with distinct Turing degrees, but within the cover $\mathfrak{A}$ of $\mathcal{S}$, we can view P as being computably enumerable in $\boldsymbol{d}$, as well as in the degrees of other enumerations. Of course, P itself, viewed as a subset of $\mathcal{S}$, does not admit effective enumeration in any obvious way.

It is immediate from Definition 4.1 that if M is a correspondence system for a cover $\mathfrak{A}$ of $\mathcal{S}$, then likewise M is a correspondence system for the cover $\overline{\mathfrak{A}}$ defined in Section 5. in which we enumerate the identity map on each $\mathcal{A}_i$ into the appropriate set $I^{\mathfrak{A}}_{ii}$ and then close the set $I^{\mathfrak{A}}$ under composition. That is, for every $f \in I^{\mathfrak{A}}_{ij}$ and $g \in \mathcal{I}^{\mathfrak{A}}_{jk}$, we enumerate $(g \circ f)$ into $\mathcal{I}^{\overline{\mathfrak{A}}}_{ik}$. If $\mathfrak{A}$ is uniformly computable, then so is $\overline{\mathfrak{A}}$, since we can build $\overline{\mathfrak{A}}$ from $\mathfrak{A}$ just by applying this rule as we enumerate the sets $I^{\overline{\mathfrak{A}}}_{ij}$. ($\overline{\mathfrak{A}}$ has the same simple cover of $\mathcal{S}$ that $\mathfrak{A}$ has, of course.) It is clear that M is now a correspondence system for the derived cover as well. Moreover, this derived cover $\mathfrak{A}'$ may be viewed as a category, with the elements $\mathcal{A}_i$ of the simple cover as the objects, and with each $I^{\mathfrak{A}'}_{ij}$ as the set of morphisms from $\mathcal{A}_i$ into $\mathcal{A}_j$.

8 Examples

Several examples, mainly involving fields, have been given in the preceding sections. In particular, see Porposition 2.3, Proposition 2.8, Proposition 3.3, Proposition 3.5, and Corollary 4.3. Now we provide an assortment of further examples.

As an example of perfect local computability, we propose the linear order $\mathcal{L}$ on Cantor space. The domain of $\mathcal{L}$ is the uncountable set 2^ω, with the relation $<$ being simply the lexicographic order. This structure is well known in mathematics, and fairly straightforward to describe via the "middle thirds" construction, but we know (from Proposition 3.5, for instance) that such characteristics do not always ensure even 1-extensional local computability.

Of course, a subset $S \subseteq 2^\omega$ generates only the substructure $(S, <)$, and so the finitely generated substructures of $\mathcal{L}$ are just the finite linear orders. (This would hold with any infinite linear order in place of $\mathcal{L}$.) So it is trivial to show that $\mathcal{L}$ is locally computable. Building a perfect cover $\mathfrak{A}$, on the other hand, will require some description.

Consider the larger signature containing the relation $<$, one other binary relation G, and four unary relations L, R, G_L, and G_R. The intuition is that we use L and R to designate the left and right end points of the order, and L_G and R_G to name left and right end points of the "gaps" in $\mathcal{L}$, i.e. the end points of the missing middle thirds. Gxy holds iff x and y are the left and right end points of the same gap. One can extend $\mathcal{L}$ itself to be a structure in this signature, indeed with the new relations all definable from $<$: $G^{\mathcal{L}}$ is just the adjacency relation in $\mathcal{L}$, while $L_G^{\mathcal{L}}x$ holds iff $(\exists y)G^{\mathcal{L}}xy$ and $R_G^{\mathcal{L}}x$ iff $(\exists y)G^{\mathcal{L}}yx$. $L^{\mathcal{L}}x$ and $R^{\mathcal{L}}y$ hold only of the left and right end points x and y of the entire order, respectively. We use this new signature to build our cover.

First, for any finite structure $\mathcal{B}$ in the new signature, it is decidable whether $\mathcal{B}$ satisfies the following axioms.

1. $(\mathcal{B}, <)$ is a linear order.
2. For each $x \in \mathcal{B}$, at most one of Lx, Rx, L_Gx, or R_Gx holds.
3. $(\forall x)[(Lx \rightarrow \forall y\ x \leq y)\ \&\ (Rx \rightarrow \forall y\ y \leq x)]$.
4. $(\forall x \forall y)[Gxy \rightarrow (x < y\ \&\ L_Gx\ \&\ R_Gy\ \&\ \forall z \neg(x < z < y))]$.

The idea is that all finite substructures of $\mathcal{L}$ should satisfy these axioms. We do not require $\mathcal{B}$ to have any x with $L^{\mathcal{B}}x$, nor any y with $R^{\mathcal{B}}y$, because an arbitrary finite subset of $\mathcal{L}$ will not necessarily contain the end points of $\mathcal{L}$. Similarly, it is allowed for $L_G^{\mathcal{B}}x$ to hold even if there is no $y \in \mathcal{B}$ with $G^{\mathcal{B}}xy$, and likewise for $R_G^{\mathcal{B}}$.

Now we may compute, uniformly, a list $\mathcal{B}_0, \mathcal{B}_1, \ldots$ (with no repetitions) of all models $\mathcal{B}$ of these axioms such that the domain of $\mathcal{B}$ is an initial segment of ω and $<^{\mathcal{B}}$ is just the standard relation $<$ on that domain. The simple cover $\mathfrak{A}$ consists of all $\mathcal{A}_i$ with $i \in \omega$, where $\mathcal{A}_i$ is the reduct of $\mathcal{B}_i$ to the signature with just $<$. (So in fact $\mathcal{A}_i$ is just the linear order $0 < 1 < \cdots < |\mathcal{A}_i| - 1$.) Of course, the generating set of $\mathcal{A}_i$ is its entire domain, and clearly this does form a uniformly computable simple cover of $\mathcal{L}$.

The maps in $I_{ij}^{\mathfrak{A}}$ will be defined as embeddings of $\mathcal{B}_i$ into $\mathcal{B}_j$, since such an embedding is clearly also an embedding $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$. We enumerate into $I_{ij}^{\mathfrak{A}}$ all strong homomorphisms $f : \mathcal{B}_i \rightarrow \mathcal{B}_j$ in the (larger) signature of these structures. Such an f must be injective, since it is a homomorphism of strict linear orders, and must also satisfy $\forall x(L^{\mathcal{B}_i}x \leftrightarrow L^{\mathcal{B}_j}f(x))$ and $(\forall x \forall y)[G^{\mathcal{B}_i}xy \rightarrow G^{\mathcal{B}_j}f(x)f(y)]$, and likewise for the other symbols in the larger signature.. (However, it is allowed for $G^{\mathcal{B}_j}$ to hold of the pair $\langle f(x), z\rangle$, even if there was no $y \in \mathcal{B}_i$ such that $G^{\mathcal{B}_i}$ held of $\langle x, y\rangle$. On the other hand, if such a y did exist, then $G^{\mathcal{B}_j}$ holds of $\langle f(x), f(y)\rangle$, by our rules above, and the axioms for G and $<$ then ensure that $z = f(y)$.)

The perfect correspondence system M will contain all maps $\alpha : \mathcal{A}_i \to \mathcal{L}$ which are homomorphisms when viewed as maps from $\mathcal{B}_i$ into $\mathcal{L}$ in the larger signature (to which $\mathcal{L}$ was extended above). That is, they must satisfy:

- $\alpha(x) = 0000\cdots$ iff $L^{\mathcal{B}_i}(x)$.
- $\alpha(x) = 1111\cdots$ iff $R^{\mathcal{B}_i}(x)$.
- $\alpha(x)$ has tail $0000\cdots$ (that is, $\alpha(x)$ contains only finitely many 1's) iff $R_G^{\mathcal{B}_i}(x)$ or $L^{\mathcal{B}_i}(x)$.
- $\alpha(x)$ has tail $1111\cdots$ iff $L_G^{\mathcal{B}_i}(x)$ or $R^{\mathcal{B}_i}(x)$.
- $G^{\mathcal{B}_i}(x, y)$ iff there is some $\sigma \in 2^{<\omega}$ with $\alpha(x) = \sigma\hat{\ }01111\cdots$ and $\alpha(y) = \sigma\hat{\ }10000\cdots$.

Now for any finite substructure $\mathcal{C}$ of $\mathcal{L}$, there is a unique i such that an $\alpha \in M$ maps $\mathcal{A}_i$ onto $\mathcal{C}$. The elements of $\mathcal{B}_i$ must be $0, 1, \ldots, |\mathcal{C}| - 1$, under the usual $<$ relation, and the other relations on $\mathcal{B}_i$ are determined by these conditions. Our enumeration $\mathcal{B}_0, \mathcal{B}_1, \ldots$ clearly must include some such $\mathcal{B}_i$, and conversely, it is uniquely determined by the choice of $\mathcal{C}$. On the other hand, given any i, the conditions above show how to pick out a substructure $\mathcal{C} \subseteq \mathcal{L}$ and a bijection $\alpha \in M$ from $\mathcal{B}_i$ onto $\mathcal{C}$. There will almost always be more than one such $\mathcal{C}$, of course: $\mathcal{C}$ is uniquely determined only if every $x \in \mathcal{B}_i$ satisfies either $L^{\mathcal{B}_i}$ or $R^{\mathcal{B}_i}$. This shows that our M satsifies the first three conditions of Definition 4.1, and also satisfies the Uniqueness Condition described subsequently.

Now suppose that $\mathcal{C} \subset \mathcal{D}$ are finite substructures of $\mathcal{L}$, and that $\alpha \in M$ with image $\mathcal{C}$. As argued above, there is a unique $\beta \in M$ with image $\mathcal{D}$, and $(\beta^{-1} \circ \alpha)$ defines a homomorphism (in the larger signature) from $\mathcal{B}_i = \mathrm{dom}(\alpha)$ onto $\mathcal{B}_j = \mathrm{dom}(\beta)$. This homomorphism must lie in $I_{ij}^{\mathfrak{A}}$, so Condition 4 is fulfilled. Conversely, if we have $\alpha \in M$, with domain $\mathcal{A}_i$ and image $\mathcal{C}$, and are given j and $f \in I_{ij}^{\mathfrak{A}}$, then we define $\beta : \mathcal{A}_j \to \mathcal{L}$ as follows, starting with $\beta \restriction f(\mathcal{A}_i) = \alpha \circ f^{-1}$. For each element $x \in \mathcal{A}_j - f(\mathcal{A}_i)$, starting with the smallest, we ask first whether there is a $y \in \mathcal{A}_j$ such that $G^{\mathcal{B}_j}$ holds of $\langle x, y\rangle$ or of $\langle y, x\rangle$. If so, then either $\beta(x)$ is determined by $\alpha(f^{-1}(y))$ (if $y \in f(\mathcal{A}_i)$), or else we choose $\beta(x)$ and $\beta(y)$ to be the end points of an appropriate gap in $\mathcal{L}$. If there is no such y, then we simply choose $\beta(x)$ in the appropriate interval in $\mathcal{L}$ satisfying either L, R, L_G, R_G, or none of the above there, according to the relation satisfied by x in $\mathcal{B}_j$, but with $\neg G^{\mathcal{L}}xz$ and $\neg G^{\mathcal{L}}zx$ for each z

already in the image of β. This β, and its image $\beta(\mathcal{A}_j) \subseteq \mathcal{L}$, satisfy Condition 5 in Definition 4.1, so M is indeed a correspondence system. Moreover, we have already noted that M satisfies the Uniqueness Condition, hence must be perfect.

The main point of this example is that building a cover of a structure $\mathcal{S}$, especially a perfect cover (or at least a highly extensional cover), usually requires us to sort out exactly what the important attributes of the various elements of $\mathcal{S}$ may be. In this case, those attributes mainly involved the gaps in $\mathcal{L}$: being the left or right end point of a gap, first of all, and recognizing the corresponding right or left end point, if this was the case. If we had used the same unary relations but omitted G from the signature, then we could have built a 1-extensional cover, but not a 2-extensional cover, since recognizing the corresponding right or left end point requires two quantifiers. Cantor space is sufficiently homogeneous that we can do all of this effectively. Of course, it could be far more difficult to build a 1- or 2-extensional cover, let alone a perfect cover, for a suborder of $\mathcal{L}$ in which certain end points of gaps were removed, while others remained.

Likewise, in the case of algebraically closed fields, building a perfect cover requires recognizing the essential attributes of elements of the ACF: their algebraic relationships to each other and to the ground field $\mathbb{Q}$. We invite the reader to build extensional covers of ordinals: by doing so, he or she will find that the salient attribute of elements there is being a limit ordinal, and in particular the specific limit level, as described by the Cantor-Bendixson rank (e.g. ω^2 is a limit of limits, ω^3 a limit of limits of limits, etc.). The extent to which one must worry about the Cantor-Bendixson rank depends on the level of extensionality one demands of the cover. Again, the process of finding a highly extensional cover leads one to an understanding of the most relevant characteristics of elements of the structure.

The odd fact is that for countable structures, this is not so much the case. As shown in Theorem 6.3, every (globally) computable structure has a perfect cover, yet there this cover is built not by recognizing specific aspects or attributes to be described. The construction of the cover there comes directly from the computable presentability of the original stucture, wherein all those attributes are wrapped up.

By Theorem 7.5, there is a computable simulation of the linear order $\mathcal{L}$ of Cantor space. We omit the details here, but this simulation may be envisioned by taking the Cantor middle-thirds set C within the real unit interval $[0, 1]$ and intersecting C with $\mathbb{Q}$. This linear order is soon seen to be computably presentable: one way is to present $(\mathbb{Q}, <)$ computably and then take the suborder of those rationals with ternary expressions using only the digits 0 and 2; while another presentation can be given by a computable set of pairs $\langle \sigma, \tau \rangle \in (2^{<\omega} \times 2^{<\omega})$, with $\langle \sigma, \tau \rangle$ representing the string $\sigma\hat{}\tau\hat{}\tau\hat{}\tau\hat{}\cdots$ in 2^{ω}. We leave the reader to show that both of these methods build computable simulations of $\mathcal{L}$.

We conclude this section with some examples of countable structures that distinguish various of our notions. For example, the following is our first proof that having a computable cover does not imply local computability.

Proposition 8.1 *Let $\mathcal{R}_c$ be the ordered field containing all computable real numbers. Then $\mathcal{R}_c$ has a computable cover, but no uniformly computable cover.*

A real number is computable if the lower cut which it defines in $\mathbb{Q}$ is computable; equivalently, if its binary expansion is $\sum_{n\in\omega} f(n) \cdot 2^{-n}$ for some computable function f such that $f(n) \leq 1$ for all $n > 0$. That is, f computes all bits in the binary expansion of the real. The computable reals form a countable real closed subfield of the reals, of infinite transcendence degree over $\mathbb{Q}$. (The field $\mathcal{R}_c[i]$ of computable complex numbers, on the other hand, is algebraically closed, hence cannot be extended to an ordered field, and by Corollary 4.3, it is perfectly locally computable.)

Proof. For any finite set of generators of a subfield $\mathcal{B} \subseteq \mathcal{R}_c$, we already have a computable presentation of $\mathcal{B}$ as a field, given in Proposition 2.3. To compute the order $<$ on $\mathcal{B}$, we need only know the upper and lower cuts of each generator. Since these generators are all computable real numbers, there are algorithms for computing these cuts, and ordinary algebra then allows us to extend our computation of $<$ to all rational functions of these generators, hence to all of $\mathcal{B}$. Thus the (countable) set of all finitely generated substructures of $\mathcal{R}_c$ forms a computable cover of $\mathcal{R}_c$, with the inclusion maps as the embeddings for this cover.

However, suppose $\mathfrak{A}$ were a uniformly computable cover for $\mathcal{R}_c$. Then for every element x of every $\mathcal{A}_i \in \mathfrak{A}$, we could compute the binary expansion of x, since the upper and lower cuts of rationals above and below x can easily be determined by the computable relation $<$ in $\mathcal{A}_i$. Since this could be done uniformly in i and x, we could give a single algorithm which would list out all these binary expansions. However, for every computable real $r \in \mathcal{R}_c$, the ordered subfield $\mathbb{Q}(r)$ would be isomorphic to some $\mathcal{A}_i$, and since the isomorphism would preserve the order, the binary expansion of r would appear (infinitely often, in fact) on the list above. This is impossible: it is well known (via an easy diagonalization) that there is no universal computable set. ■

Notice, however, that $\mathcal{R}_c$ is not too far from being locally computable. In particular, we could say that $\mathcal{R}_c$ is *locally* $\emptyset''$*-computable*, in the sense that there is a $\emptyset''$-computable enumeration of a computable cover of $\mathcal{R}_c$. In fact, by relativizing Theorem 6.3 using a $\emptyset''$-oracle, we could show that the ordered field $\mathcal{R}_c$ is perfectly locally $\emptyset''$-computable, since it has a presentation computable in $\emptyset''$.

We also give a simple example to show that even for countable structures, local computability is not equivalent to extensional local computability (and hence, by Theorem 6.3, not equivalent to computable presentability either). For any nonempty set $S \subseteq \omega$, let T_S be the countable tree, in the language of strict partial orders $\prec$ with a constant r for the root, built as follows. The root of T is $r = 0$, and we put all odd numbers at level 1 in T. Then, writing $S = \{n_0 < n_1 < \cdots\}$, for each $k \in \omega$, we add a chain of n_k nodes above each of the nodes $2\langle k, i\rangle + 1$ at level 1. (If $|S| = j < \omega$, then we partition the nodes at level 1 into j countable classes instead, so that every node at level 1 has a chain above it. The chain could have length 0 if $0 \in S$.) Thus T_S branches only at the root, with countably many branches starting at level 1, and for each $n \in \omega$, n lies in S iff T_S has some branch (hence infinitely many) containing exactly $n + 1$ nodes. (Here we do not regard the root as a node on any branch.) Clearly this structure T_S can be built to have the same Turing degree as S.

Proposition 8.2 *The following are equivalent.*

1. *T_S is perfectly locally computable.*
2. *T_S is extensionally locally computable.*
3. *S is the range of a limitwise monotonic function.*

Proof. Recall that a (total) function f is limitwise monotonic if there exists a total computable binary function h which is monotonic, i.e. $h(z, s) \leq h(z, s+1)$ for all z and s, and such that $f(z) = \lim_s h(z, s)$. If such a function exists, then it is a simple matter to use it to build a computable tree isomorphic to T_S, and then Theorem 6.3 shows that T_S is perfectly locally computable. Thus $(3 \implies 1)$.

$(1 \implies 2)$ is immediate. To see that $(2 \implies 3)$, suppose that $\mathfrak{A}$ is an extensional cover of T_S. We define a total computable function $h(\langle x, i\rangle, s)$. First fix a single element $n_0 \in S$, and let $h(\langle x, i\rangle, s) = n_0$ whenever x is not a node in the structure $\mathcal{A}_i$ of $\mathfrak{A}$. (Since the language has no function symbols, $\mathcal{A}_i$ contains only its generators and r, so the domain of $\mathcal{A}_i$ is computable.) Also, $h(\langle r, i\rangle, s) = n_0$ for all i and s. If x does lie in $\mathcal{A}_i$ and $x \neq r$, then for each of the finitely many embeddings f which have appeared in any $I_{ij}^{\mathfrak{A}}$ by stage s, we ask how many other nodes lie on the branch containing $f(x)$:

$$h(\langle x, i\rangle, s) = \max\ |\{y \in \mathcal{A}_j : r \prec y \prec f(x) \text{ or } f(x) \prec y\}|,$$

taking the maximum over all $j \in \omega$ and all $f \in I_{ij,s}^{\mathfrak{A}}$. Then h is computable, increasing in s, and total (since we put $h(\langle x, i\rangle, s) = 0$ if no embeddings f have appeared yet).

Now for any $\mathcal{A}_i$ and any $x \neq r$ in $\mathcal{A}_i$, we have an extensional match β mapping $\mathcal{A}_i$ onto some finite substructure $\mathcal{B} \subseteq T_S$. Consequently, any $f \in I_{ij}^{\mathfrak{A}}$ corresponds to an extension of $\mathcal{B}$ in T_S, and so $\lim_s h(\langle x, i\rangle, s)$ is the number of other nodes on the branch in T_S containing $\beta(x)$. Thus $\lim_s h(\langle x, i\rangle, s)$ exists and lies in S. Conversely, every $n \in S$ corresponds to a branch of length $n + 1$ in T_S, and that branch has an extensional match with some $\mathcal{A}_i \in \mathfrak{A}$, so every $n \in S$ lies in the range of $\lim_s h(\ \cdot\ , s)$. ∎

Corollary 8.3 *There exists a countable, locally computable tree T (in the language of partial orders) which is not extensionally locally computable.*

Proof. Fix a set S which is not the range of any limitwise monotonic function. The finitely generated substructures of the corresponding tree T_S are just the finite substructures, and we can list these out easily, since they contain precisely those finite trees which do not branch above the root. Since these are finite objects, it is easy to enumerate all possible embeddings of one into another, and every such embedding corresponds to an extension of one finite substructure of T_S to another one. Thus we have a uniformly computable cover of T_S, but no extensional cover, by Proposition 8.2. ∎

Now consider T_ω, the tree we build by taking $S = \omega$. That is, T_ω branches infinitely often at its root, branches nowhere else, and for each $n \in \omega$ has countably many branches of length $n+1$. Then T_ω has exactly the same uniformly computable cover as any other tree T_S (for any infinite S). However, T_ω is computable, and moreover T_ω is not elementarily equivalent to any T_S with $S \neq \omega$. (In particular, for any $n \notin S$, the Σ_2-sentence saying that there exists a branch containing exactly $n+1$ nodes holds in T_ω but not in T_S.) This establishes the following.

Corollary 8.4 *There exist countable structures with the same uniformly computable cover, such that one structure is computable (and hence perfectly locally computable), but the other is not computably presentable, indeed not even extensionally locally computable.* ∎

Our next corollary, in concert with Corollary 3.9, helps distinguish local computability from perfect local computability.

Corollary 8.5 *There exist 2^ω-many countable, pairwise elementarily non-equivalent structures with the same uniformly computable cover. Indeed, these structures all have distinct Σ_2-theories.*

Proof. Just consider T_S for every nonempty $S \subseteq \omega$. ∎

Finally, we use these results to show that the converse of each statement in Proposition 3.8 and Theorem 3.10 is false.

Theorem 8.6 *There exists a tree T which is not extensionally locally computable, yet such that for every $\theta < \omega_1^{CK}$, the Σ_θ-theory of T is itself Σ^0_θ.*

Proof. By a result of Hirschfeldt, Miller, and Podzorov in [6, Lemma 3.1], there exists a set S which is not the range of any limitwise monotonic function, yet which is low, i.e. its jump S' is Turing equivalent to $\emptyset'$. The corresponding tree T_S is not extensionally locally computable, by Proposition 8.2. However, the atomic diagram of T_S has the same Turing degree as S, and more generally, the Σ_θ theory of T_S has the same Turing degree as the θ jump $S^{(\theta)}$, which for $\theta > 0$ is just the degree of $\emptyset^{(\theta)}$.

Now the Σ_1-theory of T_S simply describes all finite subtrees of T_S. But these subtrees are precisely those finite trees which branch only at the root, so the Σ_1-theory is computable. Now let $m > 0$ and let $\varphi(\vec{x})$ be any Π_m formula. The Σ_{m+1} sentence $\exists\vec{x}\varphi(\vec{x})$ holds iff there exist elements $\vec{a}$ in T_S such that $\varphi(\vec{a})$ holds. But a $\emptyset^{(m)}$ oracle will decide whether $\varphi(\vec{a})$ holds in T_S, uniformly for any fixed $\vec{a}$, so the Σ_{m+1}-theory of T_S is enumerable using a $\emptyset^{(m)}$ oracle, hence is Σ_{m+1}, as required. A similar argument covers hyperarithmetical formulas. ∎

The following analogous result shows that the hierarchy of extensionalities does not completely collapse.

Theorem 8.7 *There exist structures which are* 1*-extensionally locally computable but not* 5*-extensionally locally computable. Moreover, the* Σ_6^0*-theory of such a structure can be of arbitrary non-*Σ_6^0 *Turing degree.*

Proof. Fix any set $U \subset \omega$ which is not Π_6^0. The structure $\mathcal{L}_U$ will be the computable well-order of order type ω^4, given by the lexicographic order $\prec$ on the domain ω^4, with a constant symbol 0 for the least element and an additional unary function symbol S such that for $k \in S$, the $(k+1)$-st iterate $S^{k+1}(0)$ is the least element $\succ S^k(0)$ of the form $\langle j, 0, 0, 0\rangle$; and for $k \notin U$, $S^{k+1}(0)$ is the least element $\succ S^k(0)$ of the form $\langle i, j, 0, 0\rangle$. For all $x \notin \{f^k(0) : k \in \omega\}$, we define $S(x) = x$.

Notice that $\mathcal{L}_U$ has no computable presentation, for the complement of U is definable by a Σ_6-formula in the language of $\mathcal{L}_U$, yet is not Σ_6^0. By similar reasoning, $\mathcal{L}_U$ cannot be 5-extensionally locally computable: Proposition 3.8 shows that if it were, then the Σ_6-theory of $\mathcal{L}_U$ would be Σ_6^0.

We now construct a uniformly computable cover of $\mathcal{L}_U$ and show that our cover is 1-extensional. Finitely generated substructures of $\mathcal{L}_U$ consist of all $f^k(0)$ and finitely many other points. In our cover $\mathfrak{A}$, however, we include more information. A structure $\mathcal{A}_i$ in $\mathfrak{A}$ consists of such a set of points, along with a function q_i such that

- $q_i(a, b) \in \omega \cup \{\infty\}$ for all $a, b \in \mathcal{A}_i$ with $a \prec b$; and
- $q_i(a, b) + q_i(b, c) = 1 + q_i(a, c)$ for all $a, b, c \in \mathcal{A}_i$ with $a \prec b \prec c$; and
- $(\forall k > 0)(\forall a) q_i(a, f^k(0)) = \infty$.

The last condition makes it clear that q_i constitutes finitely much information. The second condition shows that q_i is determined by its values on pairs of consecutive points in $\mathcal{A}_i$. The intuition is that $q_i(a, b)$ tells how many points are allowed to go in between a and b in extensions of $\mathcal{A}_i$. The last condition ensures that every $f^k(0)$ is a limit point. We use the symbol ∞ rather than ω to emphasize that we are not naming order types of intervals, but only cardinalities.

The set $I_{ij}^{\mathfrak{A}}$ consists of those maps $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ which respect the symbols 0, S, and $\prec$ and satisfy, for every $a \prec b$ in $\mathcal{A}_i$, that $\mathcal{A}_j$ contains at most

$q_i(a,b)$ elements between $f(a)$ and $f(b)$. Due to the final condition, there are only finitely many intervals to be checked, so in fact each $I_{ij}^{\mathfrak{A}}$ is computable, uniformly in i and j.

The 1-extensional match for an $\mathcal{A}_i$ will be any substructure $\mathcal{B} \subset \mathcal{L}_U$, with a bijection $\beta : \mathcal{A}_i \hookrightarrow \mathcal{B}$, satisfying the condition that for each $a \prec b$ in $\mathcal{A}_i$, the open interval $(\beta(a), \beta(b))$ in $\mathcal{L}_U$ contains exactly $q_i(a,b)$ points. This β will be the 1-extensional match. Since the interval between $S^k(0)$ and $S^{k+1}(0)$ contains an interval of type ω^2, this is possible no matter how many points $f^k(0) \prec a_1 \prec \cdots \prec a_m \prec f^{k+1}(0)$ lie in $\mathcal{A}_i$, even if all $q_i(a_j, a_{j+1}) = \infty$.

Conversely, for any finitely generated $\mathcal{B} \subset \mathcal{L}_U$, there is a function $q_{\mathcal{B}}$ telling the number of points of $\mathcal{L}_U$ in each open interval between points from $\mathcal{B}$, and the 1-extensional match for $\mathcal{B}$ will be that $\mathcal{A}_i$ with exactly the same function q_i attached to it, with the obvious β as the 1-extensional match.

It is quickly seen that in each of these cases, the map β we have described really is 1-extensional. Our use of the functions q_i and our definition of the sets $I_{ij}^{\mathfrak{A}}$ were designed to ensure this. Each finitely generated $\mathcal{C} \subset \mathcal{S}$ gives an isomorphic $\mathcal{A}_j \in \mathfrak{A}$ with an embedding from $I_{ij}^{\mathfrak{A}}$ to match the inclusion of $\mathcal{B}$ in $\mathcal{C}$, and vice versa. ■

We conjecture that similar results using ω^{m+4} can be proven which show for all m that $(2m+1)$-extensional local computability need not imply $(2m+5)$-extensional local computability. Also, it is likely that by allowing the functions q to be subcomputable, we could show that $(2m+1)$-extensional local computability need not imply $(2m+3)$-extensional local computability. We leave these ideas for a different paper.

9 Questions

Many questions arise from the notions we have introduced in this paper, and here we list some of the most compelling ones.

1. We can consider the field $\mathcal{R}$ (and $\mathcal{C}$, the complex numbers) with additional function symbols. To what extent are these structures locally computable? For example, do $\mathcal{R}$ and $\mathcal{C}$ stay locally computable when the exponential function e^x is added to the language? Similar questions apply to the trigonometric and other standard functions on $\mathcal{R}$ and $\mathcal{C}$. We have formulas for $e^{(a+b)}$ and $e^{(a \cdot b)}$, of course, but it is possible for e^x to be algebraic for transcendental x, and similarly for the trigonometric

functions, so a good deal of work needs to be done to build a uniformly computable cover $\mathfrak{A}$, including the enumeration of the embeddings in $I^{\mathfrak{A}}$. Indeed, these structures might turn out not to be locally computable. In fact, if a function $f(x)$ such as e^x or $\cos x$ were included in the language, then the existence of even a non-uniform computable cover would imply that for every $x \in \mathcal{R}$, the set

$$\{n \in \omega : f^n(x) \text{ is algebraic}\}$$

is c.e. The author is not aware of any known results along these lines.

2. Lemma 7.2 can be viewed as a sort of downwards Löwenheim-Skølem Theorem for perfectly locally computable structures. Is there an upwards version? The answer is not always positive. For example, the natural cover of the computable structure $\mathcal{S} = (\omega, 0, S)$ cannot be a perfect cover for any uncountable structure: such a structure would have to be elementarily equivalent to $\mathcal{S}$, but an uncountable model of this theory must have a nonstandard element, which would generate a substructure not isomorphic to any in the natural cover of $\mathcal{S}$. In order for an upwards version to hold, it appears necessary that the structure contain infinitely many realizations of at least one 1-type, and this should remain true even if we construe types as sets of computable infinitary formulas. Is this sufficient?

3. We have inclusions of the following classes of structures, illustrated by

the diagram:

$$\begin{array}{l}
\{\text{structures with computable covers}\} \\
\quad \supseteq \{\text{locally computable structures}\} \\
\quad\quad \supseteq \{\text{extensionally locally computable structures}\} \\
\quad\quad\quad \supseteq \{\text{2-extensionally locally computable structures}\} \\
\qquad\qquad\qquad \vdots \\
\quad\quad\quad\quad \supseteq \{\theta\text{-extensionally locally computable structures}\} \\
\quad\quad\quad\quad\quad \supseteq \{(\theta+1)\text{-extensionally locally computable structures}\} \\
\qquad\qquad\qquad \vdots \\
\quad\quad\quad\quad\quad\quad \supseteq \{\infty\text{-extensionally locally computable structures}\} \\
\quad\quad\quad\quad\quad\quad\quad \supseteq \{\infty\text{-extensionally locally comp. structures with AP}\} \\
\quad\quad\quad\quad\quad\quad\quad\quad \supseteq \{\text{perfectly locally computable structures}\}.
\end{array}$$

Certain of these inclusions are known to be strict. For instance, the ordered field $\mathcal{R}_c$ of computable real numbers and the field $\mathcal{R}$ of real numbers show that the first two inclusions do not reverse. Likewise, in Section 8 we saw countable structures which had some extensionality but were not computably presentable, hence not perfectly locally computable (by Theorem 6.3). Theorem 8.7 built a structure which showed that not all inclusions between 1-extensional and 5-extensional can reverse, and suggested similar results at other levels. We believe that the related structure $(\omega^{(\theta+1)}, <, P)$ can be useful here: the linear order on the ordinal $\omega^{(\theta+1)}$, for $\theta < \omega_1^{CK}$, with a unary relation P which holds exactly of those elements of the form $\omega^\theta \cdot n$ with $n \in \emptyset^{(\theta)}$. The ordinal ω_1^{CK} itself, viewed as a linear order with no further relations on it, should also distinguish some two levels of this hierarchy. In general we conjecture that there is no collapse within the diagram shown above, but this conjecture remains open. The case of θ-extensional and $(\theta+1)$-extensional structures with $\theta \geq \omega_1^{CK}$ seems especially mysterious.

4. In Section 3, we saw that for a θ-extensionally locally computable structure $\mathcal{S}$, the Σ_ζ-theory of $\mathcal{S}$ is itself Σ^0_ζ whenever $\zeta \leq \theta + 1$. However, Theorem 8.6 showed that the converse of Proposition 3.8 can fail: even if the Σ_ζ-theory of $\mathcal{S}$ is Σ^0_ζ for every $\zeta < \omega_1^{CK}$, $\mathcal{S}$ can still fail to be

even 1-extensionally locally computable. What additional conditions on the structure $\mathcal{S}$ might yield converses to Proposition 3.8? That is, we wish to say that, if the Σ_ζ theory of $\mathcal{S}$ is Σ^0_ζ for all $\zeta \leq \theta + 1$ and $\mathcal{S}$ satisfies some further condition, then $\mathcal{S}$ must be θ-extensionally locally computable; and similarly that if the Σ_ζ theory of $\mathcal{S}$ is Σ^0_ζ for all ζ and $\mathcal{S}$ satisfies some (different?) further condition, then $\mathcal{S}$ must be ∞-extensionally locally computable. Related versions of these questions can be posed about the Σ_ζ-theory of $(\mathcal{S}, \vec{p})$ over all finite tuples $\vec{p}$ of parameters from $\mathcal{S}$, of course.

5. For locally (i.e. 0-extensionally) computable structures, it was proven that for any $n > 1$, the Σ_n-theory can fail to be Σ_n, since the tree T constructed in Proposition 8.2 has a Σ_2 theory which computes the set S from which T was built, and we can take S to have arbitrarily high degree. Are there analogous examples of 1-extensionally locally computable structures with arbitrarily complex Σ_3-theory? And can this be extended to θ-extensionally locally computable structures and the $\Sigma_{\theta+2}$-theory?

6. The notion of θ-extensional local computability could be viewed as a measure of how far a countable structure is from being computably presentable. Traditionally, the spectrum of a structure has provided another measure of this distance, so one might ask how the spectrum is related to the strength of the local computability. (The spectrum of $\mathcal{S}$ is the set of Turing degrees of all structures isomorphic to $\mathcal{S}$ with domain ω.) Proposition 8.2 is a small step in this direction, showing that countable locally computable structures can have arbitrarily high Turing degree. (More exactly, the spectrum of a countable locally computable structure can have an arbitrarily high lower bound.) On the other hand, we can relativize the notion of a uniformly computable cover to any Turing degree $\mathbf{d}$. For example, the structure $\mathcal{R}_c$ of Proposition 8.1, while not locally computable, was shown to be $\emptyset''$-computable in this sense. What relations (if any) exist between the spectrum of a countable structure and the degrees in which it is locally computable, or extensionally locally computable, or perfectly locally computable?

7. What can be said about homomorphisms or isomorphisms between locally computable structures? Do they induce embeddings or other actions on the uniformly computable covers? Or vice versa? Is there

any case in which one can recover the automorphism group of a structure from a perfect cover of the structure (if one exists)? Or from a perfect cover and a correspondence system?

The analogy to category theory suggests natural transformations as the most reasonable definition of interest when one considers "nice" maps from one cover to another. One would like these maps between covers to correspond as closely as possible to homomorphisms or isomorphisms between the structures covered. Of course, it is possible for two structures of distinct cardinality to have the same perfect cover, and so even an isomorphism from one cover onto another (under any reasonable definition) need not yield an embedding for the structures they cover. Possibly this can be rectified by including cardinality considerations and/or the correspondence system in the definition.

8. Since one of the basic results of local computability is that adding the $<$ relation to the field of real numbers destroys all computability, it is much more reasonable to extend our studies of local computability into algebraic topics than into analytic topics. An obvious next step would be the consideration of algebraic groups, over $\mathbb{R}$ or $\mathbb{C}$ or other locally computable fields, since those are defined by polynomial maps, with no use of $<$. Differential algebra over these base fields could also be a fruitful topic of study.

9. A model theorist might make use of Definition 2.9 without the restriction to the countable case. In that situation, the least possible cardinality of an ω-extensional cover of a structure $\mathcal{S}$ would likely correspond to the size of the type space, with some appropriate adjustment for other levels of extensionality. Proposition 4.4 and Corollary 3.9 both can be adapted to settings where the covers need not be computable or even countable (but must still have correspondence systems). It would also be possible for a pure model theorist to drop the countability restrictions and to consider either covers by uncountably many finitely generated structures, or covers (of a structure of power λ, say) by structures with generating sets of size $< \kappa$, for some fixed $\kappa \leq \lambda$.

10. In the analogy to category theory, the structure $\mathcal{S}$ itself appears as a sort of inverse limit of its perfect cover. ($\mathcal{S}$ actually is the inverse limit of the category $\mathbf{FGSub}(\mathcal{S})$, and its countable simulation is the inverse

limit of its perfect cover.) What about direct limits? For example, there are possible ways to view the automorphism group of a countable structure $\mathcal{C}$ as a direct limit of the set of partial automorphisms of $\mathcal{C}$, especially in the case of an algebraic field, and when $\mathcal{C}$ is computable, this may lead to effectiveness notions on the (quite possibly uncountable) automorphism group. Are these dual in some way to local computability?

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