EFFECTIVIZING LUSIN’S THEOREM

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Abstract. Lusin’s Theorem states that, for every Borel-measurable function \( f \) on \( \mathbb{R} \) and every \( \epsilon > 0 \), there exists a continuous function \( g \) on \( \mathbb{R} \) which is equal to \( f \) except on a set of measure < \( \epsilon \). We give a proof of this result using computability theory, relating it to the near-uniformity of the Turing jump operator, and use this proof to derive several uniform computable versions.

1. Introduction

Lusin’s Theorem, first proven by Nikolay Lusin (or Luzin) in 1912, states informally that every measurable function on the real numbers \( \mathbb{R} \) is “nearly continuous,” in terms of the Lebesgue measure \( \mu \) on \( \mathbb{R} \). The standard formal statement is as follows.

**Theorem 1.1** (Lusin’s Theorem, 1912). For every Borel-measurable function \( f : \mathbb{R} \to \mathbb{R} \) and every \( \epsilon > 0 \), there exists a continuous function \( g : \mathbb{R} \to \mathbb{R} \) such that

\[
\mu(\{ x \in \mathbb{R} : f(x) \neq g(x) \}) < \epsilon.
\]

Alternative versions allow \( \pm \infty \) as values of the functions in question. (Some versions only state that \( f \) restricts to a continuous function on a set of comeasure < \( \epsilon \), but we will consider the stronger version.) A common method of proof of this result involves Egorov’s Theorem, that continuity of a function on a compact set is “nearly” uniform continuity. Indeed, in the standard text [3], Lusin’s Theorem is posed as an exercise, following the exposition of Egorov’s Theorem. On the other hand, [4, Thm. 2.24] proves Lusin’s Theorem from basic principles (and later poses Egorov’s Theorem as an exercise).

Our proof of Lusin’s Theorem demands more background, particularly in computability and descriptive set theory, than the standard ones. The point is not to replace the original proofs, but rather to highlight the connections between Lusin’s Theorem and results in those areas. When given in full, our proof will require substantial attention to detail, but it can be summarized very neatly as the following sequence of steps.

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(1) Borel-measurable functions $f(x)$ are those which can be described (in computable analysis) by the action of a Turing functional whose oracle is the $\alpha$-th jump of the input $x$, for some countable ordinal $\alpha$ that depends on $f$, along with an oracle set $S \subseteq \mathbb{N}$. That is, if $X$ is a representation (normally by a fast-converging Cauchy sequence) of $x$, then

$$\Phi((S \oplus X)(\alpha))$$

computes a Cauchy sequence which converges fast to $f(x)$.

(2) For each fixed ordinal $\alpha < \omega_1$, almost all subsets $X \subseteq \mathbb{N}$ have the property that the $\alpha$-th jump $X^{(\alpha)}$ is Turing-reducible to $S^{(\alpha)} \oplus X$. (For $\alpha = 1$, such an $X$ is said to be generalized low$_1$, or $GL_1$.) More generally, relativizing to a fixed $S \subseteq \mathbb{N}$, almost all $X$ satisfy $S^{(\alpha)} \leq_T X^{(\alpha)}$. (Details appear in [6].)

(3) In the preceding item, “almost all” refers to Lebesgue measure on the space of inputs $X$, and the Turing reducibility, while not uniform, is uniform up to a set of arbitrarily small measure, and moreover is uniform in the bound on that set’s measure. That is, there is a Turing functional $\Psi$ such that, for every $\epsilon > 0$ and every $S$,

$$\mu(\{X : \Psi_{S^{(\alpha)} \oplus X}(\epsilon, \cdot) \neq (S \oplus X)^{(\alpha)}\}) < \epsilon.$$  

We often write $\Psi_\epsilon$ for the unary functional $\Psi(\epsilon, \cdot)$. $\Psi_\epsilon$ is not uniform in $\alpha$.

(4) As shown in [7], the continuous functions $g : \mathbb{R} \to \mathbb{R}$ are precisely those for which there is a Turing functional $\Gamma$ and an oracle $S \subseteq \mathbb{N}$ such that, for all Cauchy sequences $X$ converging fast to any $x$, $\Gamma^{S \oplus X}$ converges fast to $g(x)$.

Taken together, these four items suggest a natural approach for proving Lusin’s Theorem: if $f(x)$ is given by $\Phi((S \oplus X)^{(\alpha)})$, and $S^{(\alpha)} \oplus X$ is given in turn by $\Psi_{\epsilon}^{S^{(\alpha)} \oplus X}$, then putting these together should yield a function

$$\Gamma^{(S^{(\alpha)} \oplus X)} = \Phi^{(\Psi_{\epsilon}^{S^{(\alpha)} \oplus X})}$$

which will be continuous by virtue of item (4) and will equal $f$ up to a set of measure $< \epsilon$ by virtue of item (3).

Of course, many readers will have spotted potential flaws in this argument already, and it will require a good deal of work to address them. In particular, even for those $X$ on which $\Psi^{S^{(\alpha)} \oplus X}$ fails to output $X^{(\alpha)}$, $\Theta$ must still give a coherent output, i.e., a fast-converging Cauchy sequence. Moreover, the sequences computed by $\Theta$ for two distinct $X_0$ and $X_1$ must converge to the same real number whenever $X_0$ and $X_1$ both converge to the same input $x$. Of course, each $x$ is the limit of continuum-many fast-converging Cauchy sequences, many of which may not be handled correctly by $\Psi$, so this appears to be a serious problem. Finally, Lebesgue measure on Cantor space $2^\mathbb{N}$ is used for the statement that almost all $X$ are $GL_1$, whereas we must use Lebesgue measure on $\mathbb{R}$, represented as a quotient of the space of all fast-converging Cauchy sequences, to address Lusin’s Theorem.
Nevertheless, we will overcome these difficulties and give a proof of Lusin’s Theorem essentially following the outline given above. This will require a reasonable background in computable analysis and also in the computability-theoretic notion of the jump operation on subsets of \( \mathbb{N} \) and its iteration through the countable ordinals.

The next two sections are devoted to these two topics, as few readers can be expected to be closely familiar with both of them. For more extensive presentations, we suggest [7] for the computable analysis; [2] for descriptive set theory such as the preceding characterization of the Borel functions; [5, Chapter III] for the basics of the jump; and [1, Chapters 4-5] for iterating the jump through the computable ordinals.

2. Computable functions on \( \mathbb{R} \)

Turing computability normally applies to functions from \( \mathbb{N} \), the set of all nonnegative integers, into itself. By fixing a computable bijection between \( \mathbb{N} \) and \( \mathbb{Q} \), we may equally well consider functions \( \mathbb{Q} \to \mathbb{Q} \), or \( \mathbb{Q} \to \mathbb{N} \), or \( \mathbb{N} \to \mathbb{Q} \).

We write \( a, b, q, r, u, v \) and sometimes \( \epsilon \) for rational numbers, and \( a, b, r, x, y \) for real numbers. Real numbers correspond bijectively to Dedekind cuts under their usual definition: proper nonempty downward-closed subsets of \( \mathbb{Q} \) with no greatest element.

For our purposes, this definition must be adapted slightly.

**Definition 2.1.** The *Dedekind cut* of \( x \in \mathbb{R} \) is the pair \((L, R)\), where \( L = \{ q \in \mathbb{Q} : q < x \} \) and \( R = \{ q \in \mathbb{Q} : q > x \} \). In particular, if \( x \in \mathbb{Q} \), then \( x \notin L \cup R \). We also define *generalized Dedekind cuts* to include the pairs \((\varnothing, \mathbb{Q})\) and \((\mathbb{Q}, \varnothing)\), corresponding to \(-\infty\) and \(+\infty\), in addition to the *proper* Dedekind cuts defined above.

An *enumeration* of a generalized Dedekind cut \((L, R)\) is a set that, when expressed as a join \( A \oplus B \), satisfies \( p_1(A) = L \) and \( p_1(B) = R \), where \( p_1((q, n)) = q \) is projection onto the first coordinate.

It is natural to regard \( A \) and \( B \) as subsets of \( \mathbb{Q} \times \mathbb{N} \), so that an oracle for \( A \oplus B \) allows us to list out the elements in the projections of \( A \) and \( B \), and thus to enumerate both the lower and the upper cuts of the real \( x \).

We can now give a definition of computability for functions on \( \mathbb{R} \) using Dedekind cuts, instead of the usual fast-converging Cauchy sequences, to represent real numbers.

**Definition 2.2.** For each subset \( S \subseteq \mathbb{N} \), a function \( f : \mathbb{R} \to \mathbb{R} \) is *\( S \)-computable* if there exists a Turing functional \( \Phi \) such that, whenever \( X \) is an enumeration of the Dedekind cut of any \( x \in \mathbb{R} \), \( \Phi^{S \oplus x} \) is the characteristic function of an enumeration of the Dedekind cut of \( f(x) \).

The possibility that \( L \cup R \) omits an element of \( \mathbb{Q} \) is the reason for considering enumerations of cuts. If we had simply taken \( L \) as an oracle, rather than an enumeration of \((L, R)\), then for rational \( q \), the characteristic function of the interval \([q, +\infty)\) would have been computable; similarly with \( R \) and the interval \((-\infty, q] \). On the other
hand, if we had required the cut of a rational $y$ to include $y$ itself on one side or the other, then functions such as $f(x) = x^2 - 2$ would not be computable. (For that $f$, on an enumeration of $x = \sqrt{2}$, $\Phi$ would never be able to place 0 on either side with certainty.)

Classically Definition 2.2 has been expressed using fast-converging Cauchy sequences instead of Dedekind cuts. (A Cauchy sequence $(q_n)_{n \in \mathbb{N}}$ converges fast to $x = \lim_n q_n$ if, for every $n$, $|q_n - x| < 2^{-n}$. ) Readers will understandably be baffled by our choice to use enumerations of Dedekind cuts as inputs and outputs of functions, rather than following tradition. We request forbearance: this will be the simplest way to prove the main theorems we wish to establish here. Were it not for the simplification there, we would gladly use the traditional definition, which has proven appropriate for all work so far in computable analysis.

Definition 2.1 makes Definition 2.2 equivalent to the usual definition of computable functions on $\mathbb{R}$. The next lemma proves this, by showing that we can pass effectively between the different methods of representing real numbers.

**Lemma 2.3.** There exist Turing functionals converting each of the following representations of real numbers $x$ into the other:

- An arbitrary enumeration of a Dedekind cut for $x$ in $\mathbb{Q}$.
- An arbitrary Cauchy sequence that converges fast to $x$.

**Proof.** The lemma states that, for example, there exists a Turing functional $\Upsilon$ such that, whenever $A \oplus B$ is an enumeration of the Dedekind cut in $\mathbb{Q}$ for some real number $x$, $\Upsilon^{AB}$ will be a Cauchy sequence converging fast to the same $x$. Indeed, given an enumeration $A \oplus B$ of the cut of some $x$, on input $n$, $\Upsilon$ searches for $(a, s) \in A$ and $(b, t) \in B$ with $b - a < \frac{1}{2^n}$. For the first such pair of pairs to be found, it outputs $q_n = a + \frac{b}{2}$ as the $n$-th term. Since $x \in (a, b)$, this $q_n$ is strictly within $\frac{1}{2^n}$ of $x$, so $(q_n)_{n \in \mathbb{N}}$ converges fast to $x$. (This is the definition of fast convergence: $|q_n - x| < 2^{-n}$.) Conversely, given any Cauchy sequence $(q_n)_{n \in \mathbb{N}}$ converging fast to $x$, the set

$$\left\{(a, m) \in \mathbb{Q} \times \mathbb{N} : a < q_m - \frac{1}{2^n}\right\} \oplus \left\{(b, n) \in \mathbb{Q} \times \mathbb{N} : b > q_n + \frac{1}{2^n}\right\}$$

enumerates the Dedekind cut of $x$. \hfill $\Box$

With Lemma 2.3 it is clear that the well-known theorem of Weihrauch holds for the functions of Definition 2.2.

**Theorem 2.4** (Weihrauch; see [7]). A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if there exists a set $S \subseteq \mathbb{N}$ such that $f$ is $S$-computable, in the sense of Definition 2.2.

### 3. Approximating the Iterated Jump

In this section we describe known concepts and results that may be unfamiliar to readers outside computability theory. By definition, the *jump*, or *Turing jump*, $A'$ of
a set \( A \subseteq \mathbb{N} \) is the relativization of the Halting Problem to the set \( A \):

\[
A' = \{ e \in \mathbb{N} : \Phi^A_e(e) \text{ halts} \}.
\]

Here \( \Phi_0, \Phi_1, \ldots \) is the standard enumeration of all Turing functionals, that is, of all programs for Turing machines endowed with an arbitrary “oracle set” of natural numbers. One writes \( \Phi^A_e \) for the partial function from \( \mathbb{N} \) into \( \mathbb{N} \) computed by the \( e \)-th such program when using the set \( A \) as its oracle. We refer the reader to [5, Chap. III] for full definitions. The set \( \varnothing' \) is essentially the Halting Problem itself, and just as \( \varnothing' \) is not computable, so likewise \( A \) is always strictly below \( A' \) in Turing reducibility: \( A \preceq_T A' \), by which we mean \( A \preceq_T A' \) but \( A' \npreceq_T A \). (It is not completely obvious that one can compute \( A \) using an \( A' \)-oracle, but the proof is not difficult.)

The jump operator is simply the function \( A \mapsto A' \), sending each subset of \( \mathbb{N} \) to its jump, and thus mapping the power set \( \mathcal{P}(\mathbb{N}) \) into itself. This map is injective but not surjective, and it preserves Turing reductions but can fail to preserve non-reductions: the implication

\[
A \preceq_T B \implies A' \preceq_T B'
\]

always holds, but its converse can fail. Since the jump operator maps \( \mathcal{P}(\mathbb{N}) \) into itself, it is natural to iterate it. We write \( A^{(k+1)} \) for the jump of \( A^{(k)} \), with \( A^{(0)} = A \), thus defining all finite jumps of each set \( A \).

However, this is not the end of the iteration. For \( \omega \), the first infinite ordinal, the \( \omega \)-th jump \( A^{(\omega)} \) of a set \( A \) is a sort of union of all the preceding jumps of \( A \):

\[
A^{(\omega)} = \{(n,k) \in \mathbb{N}^2 : n \in A^{(k)} \}.
\]

One views the \( k \)-th jump of \( A \) (for each \( k \)) as being coded into \( A^{(\omega)} \) as the \( k \)-th column, under the usual computable bijection from \( \mathbb{N}^2 \) onto \( \mathbb{N} \) mapping the ordered pair \( (n,k) \in \mathbb{N}^2 \) to its code number \( \langle n,k \rangle \in \mathbb{N} \). Clearly no finite jump \( A^{(k)} \) can compute \( A^{(\omega)} \), since if it could, it would then have computed the \((k+1)\)-st column \( A^{(k+1)} \), contradicting the fundamental property of the jump operator. We view \( A^{(\omega)} \) as a natural “next” jump, in the sense of ordinals, after all finite jumps have been built. (To be clear: the set \( \omega \) is actually just the set \( \mathbb{N} \), but now viewed as an ordinal.)

Nor yet is this the end of the process. One now continues through successor ordinals as before, with \( A^{(\omega+1)} = \langle A^{(\omega)} \rangle' \) and so on. At subsequent limit ordinals \( \lambda \), it is not always as obvious as for \( \omega \) exactly how to define the \( \lambda \)-th jump, but we can do so if given a computable presentation of the ordinal \( \lambda \) – that is, a computable linear ordering \( < \) of the domain \( \mathbb{N} \) such that \((\mathbb{N},<)\) is isomorphic to \( \lambda \) as a linear order. (Later we will assume that in the order \( < \), the successor and limit relations are computable as well.) Then we can define the \( k \)-th column of \( A^{(\lambda)} \) to represent the \( \alpha \)-th jump \( A^{(\alpha)} \), where \( k \in \mathbb{N} \) is mapped to \( \alpha \in \lambda \) by the isomorphism from \((\mathbb{N},<)\) onto \( \lambda \). This gives a reasonable notion of \( A^{(\lambda)} \), except that it depends on the choice of the presentation \( < \) of \( \lambda \). With proper use of ordinal notations, one can now define \( A^{(\alpha)} \) for all ordinals \( \alpha \) with computable presentations, and the definition will correctly give the Turing degree of each \( A^{(\alpha)} \), although the actual set depends on the notation chosen.
Church and Kleene knew that there must be a countable ordinal with no computable presentation, and the least such ordinal is now known as $\omega^1_{\mathrm{CK}}$. Iterating the jump to $A^{(\omega^1_{\mathrm{CK}})}$ and beyond requires presentations of noncomputable ordinals. In the work in this article, we will frequently be able to use as oracle a (noncomputable) set $S$ capable of giving presentations of noncomputable countable ordinals. Of course, for each $S$ there is a least countable ordinal $\omega^1_S$, which has no $S$-computable presentation (and it then follows that no ordinal $> \omega^1_S$ has any $S$-computable presentation either).

On the other hand, every countable ordinal is isomorphic to some linear order on $\mathbb{N}$, just by the definition of countability, and so, for every countable $\alpha$, there is some oracle set $S$ such that $\alpha < \omega^1_S$, i.e., such that $\alpha$ has an $S$-computable presentation. (Again, we will need a slightly stronger presentation of $\alpha$, with successors and limits known, but with the right oracle this can also be assumed. In fact, Spector showed that $S$ itself can give such a presentation.)

Thus we have (sketchily) described the iterated notion of the jump operator. Having done so, we will now give formal definitions for use in this article.

**Definition 3.1.** A presentation $A$ of a nonzero ordinal $\alpha < \omega_1$ is a linear order $A = (D, \prec)$ isomorphic to $(\alpha, \in)$, whose domain $D$ is a coinfinite subset of $\mathbb{N}$ and whose least element is the number 0 in $\mathbb{N}$. The presentation $A$ is $S$-decidable if $S$ can compute the complete diagram of $A$.

A presentation is normally called $S$-computable if $S$ can compute its atomic diagram. We will need more than just that here, and we could be more precise (as above) about the exact information we require $S$ to compute: the successor function on $A$ and the unary relations on $A$ of having no (immediate) predecessor and of being the greatest element of $A$. Demanding that $S$ compute the complete diagram is overkill, but keeps the definition simple. The requirement that $D$ be coinfinite is unusual in computable structure theory, but helpful for our purposes here: we will need it when $\alpha$ is a successor, in order to have a location in which to code one more jump.

It is important to notice that, for every nonzero $k \in D$, the substructure $B_k$ with domain $\{ j \in D : j < k \}$ is a presentation of an ordinal $\beta_k < \alpha$, and that each $\beta < \alpha$ is isomorphic to $\beta_k$ for some unique $k \in D$. Moreover, $S$ can compute the complete diagram of each $B_k$, uniformly in $k$.

**Definition 3.2.** For a presentation $A$ of a nonzero ordinal $\alpha < \omega_1$, the $A$-jump $C(A)$ is the subset of $\mathbb{N}$ containing those codes $\langle k, n \rangle$ for pairs $\langle k, n \rangle$ such that either:

- $k \in \text{dom}(A)$ and $n \in C(B_k)$; or
- $A$ has a $\prec$-greatest element $j$ and $k$ is the least number $> j$ in $\overline{D}$ and $$(\forall n \in \mathbb{N}) [\langle k, n \rangle \in C(A) \iff \Phi_e^{(C(B_j))}(e) \downarrow].$$

The ordinal $\alpha = 0$ has only one presentation $A_0$, with empty domain, and we define $C(A_0) = C$ for every set $C \subseteq \mathbb{N}$. 

So, for \( k \in D \), the \( k \)-th column of \( C(A) \) is simply the set \( C(B_k) \), meaning that for every \( \beta < \alpha \), the \( \beta \)-th jump of \( C \) (under the presentation \( B_k \)) appears as the \( k \)-th column. If \( \alpha \) is a limit ordinal, all other columns are empty, but for a successor ordinal \( \alpha = \beta + 1 \), with \( j \) the greatest element of \( A \), the \( k \)-th column of \( C(A) \) (for to the least number \( k > j \) in \( D \)) presents the \( \alpha \)-th jump of \( C \) (under the presentation \( A \)). With \( D \) coinfinite, such a column does exist and is found using the diagram of \( A \).

Notice that the 0-th column \( \{ n : (0, n) \in C(A) \} \) of \( C(A) \) is just the set \( C \) itself, for every presentation \( A \) of any nonzero ordinal, because 0 is always the least element of the presentation. This gives us a uniform way to recover \( C \) from \( C(A) \), independent of the presentation.

When \( \alpha = m \) is finite, \( C(A) \) is not literally the same set as the jump \( C(m) \) discussed above, but for our purposes in computability they are equivalent: many columns of \( C(A) \) actually are the set \( C(m) \), all others are computable from \( C(m) \), and we have a 1-reduction from the \( k \)-th column to \( C(m) \) uniformly in \( m \). More generally, for a presentation \( A \) of a successor \( \alpha = \beta + 1 \), one can view \( C(A) \) as the jump of \( C(B_j) \), where \( j \) is the greatest element in \( A \): this essentially says that the \( \alpha \)-th jump is the jump of the \( \beta \)-th jump. (\( C(B_j) \) itself also appears inside \( C(A) \), just as the \( \beta \)-th jump is 1-reducible to the \( \alpha \)-th jump.) The uniformity of Definition 3.2 across finite and infinite ordinals will simplify our arguments below.

The key fact about these iterations, to be exploited in our subsequent discussion, is that they are fairly close to being uniform. In its strictest interpretation, uniformity of the jump operator would require that some Turing functional \( \Phi \) have the property that \( \Phi^A = A' \) for all \( A \), and this is clearly false; indeed it would be impossible for it to hold of any single set \( A \), let alone all sets. A more relaxed definition allows a fixed oracle set \( S \), requiring that, for all \( A \), \( \Phi^{S \oplus A} = A' \). Again, it is impossible for this to hold for all \( A \) (and in particular for \( A = S \)), but with \( S = \emptyset' \) it comes close, in the sense of the usual Lebesgue measure \( \mu \) on \( 2^\mathbb{N} \). Theorem 2 of [3] proves that a Turing reduction \( A' \leq_T \emptyset' \oplus A \) exists for measure-1-many sets \( A \). It is not uniform on a measure-1 set. (That is, no single functional \( \Phi \) suffices, even up to a set of measure 0.) Moreover, the measure used there is Lebesgue measure on Cantor space \( 2^\mathbb{N} \), the space of all subsets \( A \) of \( \mathbb{N} \). We need a more specific theorem, using only (strict) Dedekind cuts \( L \oplus R \) of real numbers as our oracles. Since the set of all such cuts has measure 0 under Lebesgue measure on Cantor space (viewed here as \( 2^{(\mathbb{Q} \oplus \mathbb{Q})} \)), we must re-prove the result of [3] for our own measure, namely Lebesgue measure on \( \mathbb{R} \).

The first lemma is a warm-up for the main theorem, demonstrating the basic technique. Fix a computable enumeration \( q_0, q_1, \ldots \) of all rationals in the interval \((0, 1)\). We write \( L_x \) for the strict left Dedekind cut of a real number \( x \), and \( R_x \) for its strict right cut. For each \( a = q_j \) and \( b = q_k \) in \( \mathbb{Q} \) such that \( 0 \leq a < b \leq 1 \) and every \( q_i \in (a, b) \) has \( i > \max\{j, k\} \), we define the binary strings \( \lambda_{a,b} \) and \( \rho_{a,b} \) each to have length \( l \),
where \( l \) is least with \( q_l \in (a, b) \), and set
\[
\lambda_{a,b}(i) = \begin{cases} 
1, & \text{if } q_i \leq a \\
0, & \text{if } q_i \geq b
\end{cases}
\quad \text{and} \quad
\rho_{a,b}(i) = 1 - \lambda_{a,b}(i) = \begin{cases} 
1, & \text{if } q_i \geq b \\
0, & \text{if } q_i \leq a.
\end{cases}
\]

Thus the real numbers \( x \) in the open interval \((a, b)\) are precisely those \( x \) such that
\( \lambda_{a,b} \) is an initial segment of \( L_\sigma \) (viewed as an infinite binary string) and \( \rho_{a,b} \) is an initial segment of \( R_\sigma \). (These strings are examined further at the end of the proof of Theorem 3.4.) For intervals \((a, b)\) where some \( q_i \in (a, b) \) has \( i < j \) or \( i < k \), it is possible to divide the interval into finitely many subintervals with the property required above, effectively, and to consider a string \( \lambda \oplus \rho \) for each subinterval.

**Lemma 3.3.** For every rational \( \epsilon > 0 \), there exists a Turing functional \( \Psi_\epsilon \) such that, for every \( S \subseteq \mathbb{N} \),
\[
\mu(\{ x \in (0, 1) : \Psi_\epsilon^{S \oplus L_\sigma \oplus R_\sigma} = (S \oplus L_\sigma \oplus R_\sigma)' \}) > 1 - \epsilon.
\]
Moreover, there is a computable function \( h \) mapping each \( \epsilon > 0 \) to an index \( h(\epsilon) \in \mathbb{N} \) such that \( \Phi_{h(\epsilon)} = \Psi_\epsilon \). So this process is uniform in \( \epsilon \), although when \( \epsilon = 0 \), no \( \Psi \) suffices.

**Proof.** Given an \( e \in \mathbb{N} \), \( \Psi_\epsilon^{S \oplus L_\sigma \oplus R_\sigma} \) searches for a rational number \( r \in [0, 1] \), a finite initial segment \( \sigma \subseteq S \), and a finite collection \((a_0, b_0), \ldots, (a_m, b_m)\) of disjoint open rational subintervals of \((0, 1)\) and a stage \( s \) such that:

- \((\forall i \leq m) \Phi_{e,s}^{\sigma \oplus \lambda_{a_i,b_i} \oplus \rho_{a_i,b_i}}(e) \downarrow \); and
- \( \sum_{i \leq m} (b_i - a_i) > r \); and
- there do not exist a number \( t \), a \( \tau \subseteq S \), and finitely many disjoint rational intervals \((c_0, d_0), \ldots, (c_n, d_n)\) within \((0, 1)\) such that \( \sum_{i \leq n} (d_i - c_i) \geq r + \frac{\epsilon}{2^{2^t t}} \) and \((\forall i \leq n) \Phi_{e,t}^{\tau \oplus \lambda_{a_i,b_i} \oplus \rho_{a_i,b_i}}(e) \downarrow \).

The \( S' \)-oracle allows \( \Psi_\epsilon \) to recognize the truth or falseness of the final statement for any specific \( r \), while the first two statements are decidable. For the \( r \) that is found, we have \( r < \mu(\{ y \in (0, 1) : \Phi_{e}^{S \oplus L_\sigma \oplus R_\sigma}(y) \downarrow \}) \leq r + \frac{\epsilon}{2^{2^t t}} \). (This also makes it clear why such an \( r \) must exist: finitely many initial segments of oracles \( L_\sigma \oplus R_\sigma \) cover all but \( \frac{\epsilon}{2^{2^t t}} \)-much of the total measure of this set.) Now \( \Psi_\epsilon \) examines the \( L_\sigma \oplus R_\sigma \)-portion of its oracle. If, for some \( i \leq m \), \( \lambda_{a_i,b_i} \oplus \rho_{a_i,b_i} \subseteq L_\sigma \oplus R_\sigma \), then it outputs 1, since such an \( x \) will lie in one of the intervals \((a_i, b_i)\). For all other \( x \), it outputs 0, meaning that it thinks that \( \Phi_{e}^{S \oplus L_\sigma \oplus R_\sigma}(e) \uparrow \). This output 0 may be wrong for certain values \( x \), but only for \( \frac{\epsilon}{2^{2^t t}} \)-many. Since this holds for every \( e \), incorrect outputs can only occur for at most \( \epsilon \)-much of the interval \([0, 1]\). Moreover, it is clear that this procedure is uniform in \( \epsilon > 0 \).

Finally, we will need to know that Lemma 3.3 holds not only for the jump operator, but for all iterations and relativizations of it. Here is the full result.
Theorem 3.4. There exists a Turing functional $\Psi_\varepsilon$, uniform in the rational $\varepsilon > 0$, such that for every set $S \subseteq \mathbb{N}$ and every fixed $S$-decidable presentation $A$ of any ordinal $\alpha < \omega_1^S$ (with $\alpha$-jumps $C(A)$ defined using this presentation), the “error set”

$$U_{\varepsilon,S,A} = \{ x \in [0,1] : \Psi_\varepsilon^{S(A)} \oplus L_x \oplus R_x \neq (S \oplus L_x \oplus R_x)^{(A)} \}$$

has measure $< \varepsilon$ and is an $S(A)$-effective union of rational open intervals, uniformly in $\varepsilon$, $S$, $\alpha$, and $A$. Moreover, for all $x$, $\Psi_\varepsilon^{S(A)} \oplus L_x \oplus R_x$ is total.

Proof. We give the procedure of $\Psi_\varepsilon$ on an input $(k,e)$, using the presentation $A$ of $\alpha$, whose complete diagram $\Psi_\varepsilon$ can compute using its $S$-oracle.

For elements $k \in D = \text{dom}(A)$, the program runs in a highly recursive manner, computing the $k$-th column of its output using (finitely much information from) those columns whose numbers $i$ satisfy $i < k$ (according to the diagram of $A$). Since $A$ is well-ordered by $<$, this procedure is well-founded and will eventually halt. On input $(k,e)$ with $k \in D$, the program checks whether $k$ is 0 or a limit point in $A$. For $k = 0$, it uses its oracle to decide whether $e \in S \oplus L_x \oplus R_x$ and outputs the answer. For a limit point $k$, it decodes $e = (k',e')$ and runs itself on this pair, since $(k',e'),k) \in (S \oplus L_x \oplus R_x)^{(A)}$ just if $(k',e') \in (S \oplus L_x \oplus R_x)^{(A)}$. If $k$ is a successor, then the program finds the immediate predecessor $i$ of $k$ under $<$, using the diagram of $A$, and attempts to determine whether $\Phi_i(e) \downarrow$, where $I$ is the $i$-th column of the program’s own output. This requires running the program itself many times, recursively, but only on finitely many inputs (and only on pairs $(i',e')$ with $i' \leq i$). The procedure is the same as in Lemma 3.3; the program runs until it has found a finite set of some measure $r$ of initial segments $\lambda_{a,b} \oplus \rho_{a,b}$ that will cause $\Phi_e(e)$ with oracle $\Psi_\varepsilon^{S(B_i)} \oplus \lambda_{a,b} \oplus \rho_{a,b}$ to halt, and has been told by the oracle $S^{(A)}$ that the oracles of this form that cause $\Phi_e(e)$ to halt have total measure at most $r + \frac{\varepsilon}{2 \varepsilon + r^2}$. Here we are using $\Psi_\varepsilon^{S(B_i)} \oplus \lambda_{a,b} \oplus \rho_{a,b}$ as an approximation to $(S \oplus L_x \oplus R_x)^{(B_i)}$, which is the actual content of the $i$-th column $I$. The approximation is not always correct; below we will consider the measure of the set on which it is incorrect, but for now the important point is that it does always give an output, instead of diverging. Finally, $\Psi_e$ determines whether $L_x \otimes R_x$ begins with any of the finitely many strings $\lambda_{a,b} \oplus \rho_{a,b}$ that were found to make $\Phi_e^{\Psi_e^{S(B_i)} \oplus \lambda_{a,b} \oplus \rho_{a,b}}(e)$ halt. If so, it outputs 1, while if not, it outputs 0.

For elements $k \notin D$, the program checks whether any $j < k$ is the greatest element of $A$, and if so, whether $k$ is the $<$-least number in $D$ greater than that $j$. If not, then it immediately outputs 0. If so, then it runs in a similar manner to the program in Lemma 3.3 recursively using the column $J = \{ (j,n) : n \in \mathbb{N} \}$ of its own output. For definiteness we specify that on input $(k,e)$ (with $k \notin D$) it uses a tolerance of $\frac{\varepsilon}{2(k+\varepsilon)^2}$ to approximate $J'$.

Since $<$ well-orders $A$, it is readily seen (by induction on columns whose numbers lie in $D$, ordered according to $<$) that this program halts on every input $(k,n)$ with $k \in D$. If $\alpha$ is a successor, the same proof then applies to the $k \notin D$ determined above,
and for all other $k \notin D$ it halted immediately. So the program $\Psi_e$ with arbitrary oracle $S(A) \oplus L_x \oplus R_x$ always computes a total function. Next we consider the set $U_{e,S,A}$ of those $L_x$ on which it fails to compute $(S \oplus L_x \oplus R_x)^{(A)}$. This can happen in many ways.

For the very first jump, when $k_1$ is the second-to-left point of $A$, the computation on input $(k, e)$ will be incorrect on a set of $x$ of measure $< \frac{\epsilon}{2^1 + k^2}$, and so the set of those $x$ for which there is an error anywhere in this column has measure $< \frac{\epsilon}{2^1 + k^2}$. For the second jump, in column number $k_2$, there are two reasons the computation could be incorrect: either $x$ lies in the set of measure $< \frac{\epsilon}{2^1 + k^2}$ on which the approximation goes wrong, or else the approximation was using an incorrect version of $S(L_x \oplus R_x)^{(B_k)}$ from column $k_1$. However, we already counted those $x$ for which the $k_1$ column was incorrect, so the reals $x$ added to the set $U_{e,S,A}$ on account of column $k_2$ have total measure $< \frac{\epsilon}{2^1 + k_2}$. Similarly, for every $x$ in $U_{e,S,A}$, either there is some $<\text{least } k \in D$ such that the computation for $x$ goes wrong in column number $k$, or else $\alpha$ is a successor and the computation went wrong in column $k = \min(D)$. Therefore, the total measure of $U_{e,S,A}$ is at most

$$\left(\sum_{k \in D} \frac{\epsilon}{2^1 + k}\right) + \frac{\epsilon}{2^{\min(D)}} \leq \sum_{k \in \mathbb{N}} \frac{\epsilon}{2^1 + k} = \epsilon.$$  

The foregoing paragraph already essentially explained how we can uniformly enumerate the open set $U_{e,S,A}$ from an $S(A)$-oracle. Those $x$ in $U_{e,S,A}$ for which the first column $k_1$ was incorrect all have $\Psi_e^{S(A) \oplus L_x \oplus R_x}(e) = 0$ for some $e$ such that eventually $\Phi_e^{S \oplus L_x \oplus R_x}(e)$ halted. With the $S(A)$-oracle we can run both of these computations with arbitrary strings of the form $\lambda_{a,b} \oplus \rho_{a,b}$ in place of $L_x$. When we find any $\lambda_{a,b} \oplus \rho_{a,b}$ and $e$ for which $\Psi_e(e)$ gave 0 but $\Phi_e^{S \oplus \lambda_{a,b} \oplus \rho_{a,b}}(e) \downarrow$, we enumerate the open interval $(a, b)$ of $\mathbb{R}$ into $U_{e,S,A}$ (since all $x$ there have $\lambda_{a,b} \oplus \rho_{a,b} \subseteq L_x \oplus R_x$). For column $k_2$, we do the same, using $\Psi_e^{S \oplus L_x}$ to compute the oracle for the computation $\Phi_e^{S \oplus L_x \oplus R_x}(e)$; if it does so incorrectly, then this $x$ was already enumerated at the previous step, while if it does so correctly, then we will enumerate $x$ into $U_{e,S,A}$ just if $\Phi_e^{S \oplus L_x \oplus R_x}(e)$ halts and $\Psi_e^{S(A) \oplus L_x \oplus R_x}(e) = 0$. Likewise, every $x \in U_{e,S,A}$ will eventually be enumerated by this process, because some finite initial segment $\lambda \oplus \rho \subseteq L_x \oplus R_x$ must have been adequate to cause both of these events to occur, and we will eventually find that segment (which must be of the form $\lambda_{a,b} \oplus \rho_{a,b}$, with $a = \max(\lambda^{-1}(1) \cup \{0\})$ and $b = \min(\lambda^{-1}(0) \cup \{1\})$) and enumerate $x$ into $U_{e,S,A}$.

Finally we discuss the situation of isolated points in $U_{e,S,A}$. The strings $\lambda_{a,b} \oplus \rho_{a,b}$ (and their substrings) are not the only possible initial segments of oracles $L_x \oplus R_x$: the other possibility occurs when $x$ itself is equal to the rational number $q_j$, in which case $q_j \notin L_x$ and $q_j \notin R_x$. Initial segments $\lambda \oplus \rho$ of such strings still satisfy the property $\min(\lambda^{-1}(0)) \leq \max(\rho^{-1}(0))$, but they are allowed to have at most one $j < |\lambda|$ for which $\lambda(q_j) = \rho(q_j) = 0$ (with $\rho(i) = 1 - \lambda(i)$ for all other $i$).
Intuitively (and by definition), all rational numbers \( q_j \) should lie in the error set \( U_{e,S,A} \), because the particular functional \( \Phi_e^{S\oplus L_x \oplus R_x} \) that halts just if \( q_j \notin L_x \cup R_x \) will always cause \( \Psi_e \) to make an error when \( x = q_j \). However, \( \Phi_e \) makes this error only on the isolated point \( q_j \), not on an open interval. To show that \( U_{e,S,A} \) is indeed open, notice that for each \( q_j \), we can effectively find the indices \( c \) and \( d \) of two other relevant functionals:

- \( \Phi_c^{S(A)\oplus L_x \oplus R_x} \) halts just if \( L_x \) contains some rational \( > q_j \) (that is, just if \( x > q_j \));
- \( \Phi_d^{S(A)\oplus L_x \oplus R_x} \) halts just if \( R_x \) contains some rational \( < q_j \) (that is, just if \( x < q_j \)).

\( \Phi_e \) will contribute an error interval to \( U_{e,S,A} \) of the form \((q_j, q)\) for some rational \( q > q_j \), and \( \Phi_d \) will contribute one of the form \((r, q_j)\). Therefore, including \( q_j \) itself in \( U_{e,S,A} \) keeps it open, as now the entire interval \((r, q)\) is contained in \( U_{e,S,A} \). So, along with the error intervals previously enumerated into \( U_{e,S,A} \), we also enumerate the interval \((r, q)\) defined here, for this \( q_j \); and for every other rational in [0, 1] as well, noting that \( r \) and \( q \) were computed effectively from \( j \). Thus \( U_{e,S,A} \) is still \( S(A) \)-effectively open, and the countably many new points do not change its measure. \( \square \)

In Theorem 3.4, it would have been difficult to work with all of \( \mathbb{R} \) at once, but any interval \([a, b]\) of finite measure (with \( S \)-computable end points) could have been used in place of [0, 1], and the argument would be uniform in those end points. However, now that the theorem is proven, it is an easy matter to do it uniformly simultaneously for all \([n, n + 1]\) with \( n \in \mathbb{Z} \), and with \( \epsilon_n = \frac{4}{3} \cdot \frac{\epsilon}{2^{n+1}} \), to do the same for all of \( \mathbb{R} \).

**Corollary 3.5.** There exists a Turing functional \( \Psi_e \), uniform in the rational \( \epsilon > 0 \), such that for every set \( S \subseteq \mathbb{N} \) and every fixed \( S \)-decidable presentation \( A \) of any ordinal \( \alpha < \omega_1^\text{CK} \) (with \( \alpha \)-jumps \( C(A) \) defined using this presentation), the “error set”

\[
U_{e,S,A} = \{ x \in \mathbb{R} : \Psi_e^{S(A)\oplus L_x \oplus R_x} \neq (S \oplus L_x \oplus R_x)^{(A)} \}
\]

has measure \(< \epsilon \) and is an \( S(A) \)-effective union of rational open intervals, uniformly in \( \epsilon, S, \alpha, \) and \( A \). Moreover, for all \( x \), \( \Psi_e^{S(A)\oplus L_x \oplus R_x} \) is total. \( \square \)

The buffer sets are tied to the set \( \mathbb{Q} \), which is special because it is the set in which we took our Dedekind cuts. This has nothing to do with \( \mathbb{Q} \) being the prime subfield of \( \mathbb{R} \); \( \mathbb{Q} \) was simply a convenient dense set of uniformly computable real numbers for us to use. One could rewrite this entire paper using Dedekind cuts in a different dense set of uniformly computable reals, such as \( \{ \pi + \frac{1}{2^n} : n \in \mathbb{N} \} \), and all the results would still hold, but the buffer sets would cover this set rather than covering \( \mathbb{Q} \). On the other hand, \( U_{e,S,A} \) would still contain all of \( \mathbb{Q} \); in fact, it is readily seen to contain all computable real numbers.
4. COMPUTING DISCONTINUOUS FUNCTIONS

Theorem 2.4 shows that it is impossible to compute a discontinuous function $f(x)$ just from enumerations of Dedekind cuts for the input $x$. However, if we allow ourselves more information about $x$, then it becomes possible.

**Definition 4.1.** A function $f : \mathbb{R} \to \mathbb{R}$ is *jump-computable* if there exists a Turing functional $\Phi$ such that, for every enumeration $X$ of the Dedekind cut of each $x \in \mathbb{R}$, the function

$$\Phi^X : \mathbb{Q} \to \{0, 1\}$$

computes an enumeration of the Dedekind cut of $f(x)$.

More generally, for an ordinal $\alpha < \omega_1$ and an oracle set $S \subseteq \mathbb{N}$, $f$ is $\alpha$-jump $S$-computable if there exists $\Phi$ such that, for every enumeration $X$ of the Dedekind cut of each $x \in \mathbb{R}$,

$$\Phi^{(S \oplus X)(\alpha)} : \mathbb{Q} \to \{0, 1\}$$

computes an enumeration of the Dedekind cut of $f(x)$.

If $\alpha$ is a countable noncomputable ordinal, then the $\alpha$-th jump is not well-defined in general, as we noted in Section 3. However, since we are allowed an oracle $S$, we can choose $S$ complex enough to compute the complete diagram of a presentation $A$ of $\alpha$. So it is possible to discuss the situation $\alpha \geq \omega^CK_1$ uniformly. The principal theorem relevant here can be found in [2]. By Lemma 2.3, this theorem holds with $x$ and $f(x)$ represented either by fast-converging Cauchy sequences or by enumerations of Dedekind cuts.

**Theorem 4.2.** For every Borel function $f : \mathbb{R} \to \mathbb{R}$, there is a Turing functional $\Phi$, an oracle $S \subseteq \mathbb{N}$, and an $S$-decidable presentation $A$ of some countable ordinal such that, for every enumeration $A \oplus B$ of the Dedekind cut of any $x \in \mathbb{R}$, $\Phi^{(S \oplus X)(A)}$ enumerates the Dedekind cut of $f(x)$.

Computable analysts have commonly approached jump-computability the opposite way, by taking the limit of a computable function:

$$f(x) = \lim_{s \to \infty} \Phi^X(\cdot, s).$$

That is, for each $s \in \mathbb{N}$, one computes an approximation to $f(x)$, and the actual value $f(x)$ is the limit of these approximations. As an example, the derivative of a differentiable function $h(x)$ could be given by letting $\Phi^X$ be the difference quotient $s \cdot (h(x + \frac{1}{s}) - h(x))$. The Turing functional $\Phi$ is readily defined from the functional and oracle computing $h$. (The derivative of a computable differentiable function is
not in general computable, so this is often the best that can be done.) The connection
between this method and ours is given by the Transparency Lemma.

**Lemma 4.3** (Transparency Lemma (folklore)). A function is jump-computable if and
only if it is the limit of a computable function (in the sense immediately above).

So one could attack Lusin’s Theorem by iterating the limit operation instead of
the jump operation. Our choice to use the jump is dictated mainly because it allows
us to apply the known results of Section 3 on near-uniform continuity of the jump,
especially Theorem 3.4.

5. Lusin’s Theorem

Now we may approach Lusin’s Theorem.

**Theorem 5.1** (Lusin, 1912). Every Borel-measurable function \( f : \mathbb{R} \to \mathbb{R} \) is nearly
continuous.

**Proof.** By Theorem 4.2, \( f \) is given by

\[
f(x) = \Phi^{(S \oplus X)^{(A)}}(A),
\]

for some oracle \( S \) and some \( S \)-decidable presentation \( A \) of an ordinal \( \alpha < \omega_1^S \). By
this we mean that, whenever \( X = A \oplus B \) enumerates a Dedekind cut representing \( x \),
the right-hand side is the characteristic function of an enumeration of the Dedekind
cut for the real number \( y = f(x) \). We wish to give a procedure \( \Gamma \) which will accept
an arbitrary enumeration \( q_0, q_1, \ldots \) of all rational numbers. Also fix an enu-
meration of the error set \( U_{\epsilon,S,A} \) given by Corollary 3.5 for this \( \epsilon, S, \) and \( A \). Recall
from the discussion there that we have a computable function mapping each \( j \) to a
“buffer set” for \( q_j \), namely, a rational interval \( (b_j, b'_j) \) within \( U_{\epsilon,S,A} \) containing \( q_j \).

Having fixed \( \Phi \) and \( A \) as above, we now use them and an \( S^{(A)} \)-oracle to enumerate
three sets: upper computations, lower computations, and error intervals. For these
we define a functional \( \Theta \) that attempts to compute \( f \) by running \( \Psi_{\epsilon} \) on an oracle and
applying \( f \) to the output:

\[
\Theta^D = \Phi^{(\Psi_{\epsilon}^D)}.
\]

If the oracle \( D \) is of the form \( (S^{(A)} \oplus L_x \oplus R_x) \), then the \( \Psi_{\epsilon} \) computation will output
\( (S \oplus (L_x \oplus R_x)^{(A)}) \) (except on a set of measure \( < \epsilon \)), and in turn \( \Phi \) will use this
oracle to enumerate the upper and lower cuts of \( f(x) \). The next definition specifies
our ability to recognize when \( \Theta \) will enumerate various rational numbers into the
upper and lower cuts of \( f(x) \). We use the strings \( \lambda_{a,b} \) and \( \rho_{a,b} \) defined in the proof of
Theorem 3.4.
Definition 5.2. An upper computation (on an interval \((c, c')\)) is a triple \((c, c', u) \in \mathbb{Q}^3\) with \(c < c'\) such that, for some \(n \in \mathbb{N}\),

\[
\Theta(S^{(A)} \oplus_{\lambda, e, \rho, c, c'} ((u, 2n + 1))) \downarrow = 1,
\]

meaning that \(u\) is being enumerated into the upper cut of the output. A lower computation (on an interval \((d, d')\)) is a triple \((d, d', v) \in \mathbb{Q}^3\) with \(d < d'\) such that, for some \(n \in \mathbb{N}\),

\[
\Theta(S^{(A)} \oplus_{\lambda, d, d', \rho, d, d'}) ((v, 2n)) \downarrow = 1,
\]

enumerating \(v\) into the lower cut.

The set of all such computations (for all \(c, c'\) and all \(d, d'\)) is therefore \(S^{(A)}\)-computably enumerable.

Of course, \(\Theta\) is not always correct, due to errors that \(\Psi\) may make in attempting to compute \((S \oplus L_x \oplus R_x)^{(A)}\). Corollary 3.5 explained how to enumerate the errors.

Definition 5.3. An error interval is an interval \((e, e')\) with rational end points \(e < e'\) that is enumerated into \(U_{c,S,A}\) as described in Corollary 3.5. We write \((e_s, e'_s)\) for the \(s\)-th such interval in the fixed \(S^{(A)}\)-computable enumeration given there.

Now we explain the process by which our functional \(\Gamma\), which computes \(g(x)\) from the oracle \(S^{(A)} \oplus A \oplus B\), uses the arbitrary enumeration \(A \oplus B\) of the Dedekind cuts of an arbitrary real number \(x\). Recall that \(p_1(A) = L_x\) and \(p_1(B) = R_x\). Fix \(t \in \mathbb{N}\), and for each \(u \in \mathbb{N}\), let \(L_{A,u} = \{m \in \mathbb{N} : (\exists n \leq u) \ (q_m, n) \in A\}\) and \(R_{B,u} = \{m \in \mathbb{N} : (\exists n \leq u) \ (q_m, n) \in B\}\). Find the least \(u\) such that the difference \((\{0, 1, \ldots, t-1\} - (L_{A,u} \cup R_{B,u}))\) contains at most one element. Then we define \(\lambda_t \oplus \rho_t\), our approximation of length \(t\) to \(x\), by:

\[
\lambda_t(i) = \begin{cases} 
1, & \text{if } i < t \land (\exists j < u) \ [q_i \leq q_j \land j \in L_{A,u}]; \\
0, & \text{otherwise (for } i < t). 
\end{cases}
\]

\[
\rho_t(i) = \begin{cases} 
1, & \text{if } i < t \land (\exists j < u) \ [q_i \geq q_j \land j \in R_{A,u}]; \\
0, & \text{otherwise (for } i < t). 
\end{cases}
\]

Notice that, if \(i \notin L_{A,u} \cup R_{B,u}\) but some \(j \in L_{A,u}\) has \(q_j > q_i\), this defines \(\lambda_t(i) = 1\), because we know that \(i\) will eventually enter \(L_x\); similarly if \(j \in R_{B,u}\) and \(q_j < q_i\). This \(\lambda_t \oplus \rho_t\) has the property discussed in the proof of Theorem 3.3, there is at most one \(i < t\) (possibly none at all) with \(\lambda_t(i) = 0 = \rho_t(i)\), and for all other \(i < t\), \(\lambda_t(i) = 1 - \rho_t(i)\).

Because \(\lambda_t \oplus \rho_t\) may not accurately reflect \(L_x\) and \(R_x\), we also include two other versions. Define \(\lambda_t^+ = \lambda_t\) and \(\rho_t^- = \rho_t\), and then define \(\lambda_t^+(i) = 1 - \rho_t^-(i)\) and \(\rho_t^+(i) = 1 - \lambda_t^-(i)\). Thus \(\lambda_t^+ \oplus \rho_t^-\) and \(\lambda_t^- \oplus \rho_t^+\) are the two other possibilities: that the \(i\) with \(\lambda_t(i) = 0 = \rho_t(i)\) could eventually appear in either cut. (If there is no such \(i\), then all three versions are equal.)

Lemma 5.4. Fix any \(x \in \mathbb{R}\), and let \(\lambda_{\infty}, \rho_{\infty} \in 2^\omega\) be the strict Dedekind cuts \(L_x\) and \(R_x\), viewed as binary sequences. Then for each enumeration \(A \oplus B\) of the Dedekind
cuts of \( \mathbf{x} \), and for every \( t \), \((\lambda_\infty \uparrow t) \oplus (\rho_\infty \uparrow t)\) is equal to one of the strings \( \lambda_t \oplus \rho_t \), \( \lambda_t^+ \oplus \rho_t^+ \), or \( \lambda_t^- \oplus \rho_t^- \) defined above from \( A \oplus B \). Moreover, \( \lambda_\infty \oplus \rho_\infty = \lim_t (\lambda_t \oplus \rho_t) \), and when \( \mathbf{x} \notin \mathbb{Q} \), this limit also equals \( \lim_t (\lambda_t^+ \oplus \rho_t^+) \) and \( \lim_t (\lambda_t^- \oplus \rho_t^-) \).

**Main Step for stage** \( s + 1 \). The six strings of each length \( t \) computed from \( A \oplus B \), define closed rational intervals of positive length: \([a_t, b_t] = [\max \lambda_t^{-1}(1) \cdot \min \rho_t^{-1}(1)]\), \([a_t^+, b_t^+] = [\max(\lambda_t^+)^{-1}(1) \cdot \min(\rho_t^+)^{-1}(1)]\), and \([a_t^-, b_t^-] = [\max(\lambda_t^-)^{-1}(1) \cdot \min(\rho_t^-)^{-1}(1)]\), each with \( \mathbf{x} \) in its interior. For each of these three intervals, the intersection at \( t \rightarrow \infty \) is the singleton \( \{\mathbf{x}\} \), so it is clear that, for sufficiently large \( t \), at least one of the following holds. (Item (3) covers the cases \( \mathbf{x} = e_j \) and \( \mathbf{x} = e'_j \), while either (1) or (2) must apply to all other \( \mathbf{x} \).)

1. Each of \([a_t, b_t] \), \([a_t^+, b_t^+] \), and \([a_t^-, b_t^-] \) is disjoint from \((\bigcup_{i=0}^{s}(e_i, e'_i))\).
2. For some \( j \leq s \), each of \([a_t, b_t] \), \([a_t^+, b_t^+] \), and \([a_t^-, b_t^-] \) is a subset of \((e_j, e'_j)\).
3. For some \( j \leq s \), each of \([a_t, b_t] \), \([a_t^+, b_t^+] \), and \([a_t^-, b_t^-] \) lies within the buffer set around \( e_j \) or around \( e'_j \) (as defined at the end of the proof of Theorem 3.4).

We find the least \( t \) for which any of these holds. If item (3) holds and (2) fails, we do nothing at this stage. (For some \( k > s \), the buffer set around this \( e_j \) or \( e'_j \) will be defined as \((e_k, e'_k)\), and then item (2) will apply.) Otherwise we ignore (3) and act according to which of (1) or (2) holds.

If item (1) holds, then \( \mathbf{x} \) does not lie in any error interval as yet, so we turn to the first \( s \) computations enumerated by our \( S(A) \)-oracle (as in Definition 5.2). Going through these in the order they are enumerated, when we come to an upper computation of the form \((c, c', u)\) for which all of \([a_t, b_t] \), \([a_t^+, b_t^+] \), and \([a_t^-, b_t^-] \) are subsets of \((c, c')\), we enumerate \( u \) into the upper cut of the output, provided that all numbers \( v \) already enumerated into the lower cut (at this stage or previous stages) satisfy \( v < u \). Likewise we enumerate new lower bounds \( v \) whenever they do not contradict any previously enumerated upper bounds \( u \). (If such a contradiction occurs, then an error interval will ultimately arise here, as the function \( f \) could not have computed this without some essential use of one or more jumps of \( X \).)

To describe the action when item (2) holds, we fix the least \( j \) for which \((e_j, e'_j)\) contains all three of the closed intervals. The index \( j \) is also the number of the stage at which we began to consider this error interval, and at that stage we gave a recipe for defining upper and lower bounds and outputs, in such a way as to make the function \( g \) piecewise-linear on \((e_j, e'_j)\). (This recipe appears directly below, as the “Secondary Step for stage \( s + 1 \).”) With our three closed rational intervals approximating \( \mathbf{x} \), and knowing that \( \mathbf{x} \in (e_j, e'_j) \), we simply follow this recipe exactly as described back at stage \( j \). This completes the Main Step for stage \( s + 1 \).

The **Secondary Step for stage** \( s + 1 \) deals with the new error interval \((e_s, e'_s)\).

There may have been finitely many upper and lower bounds already enumerated for rational subintervals of \((e_s, e'_s)\) before this stage: for example, if an upper computation \((c, c', u)\) appeared before stage \( s + 1 \) (and no earlier error interval intervened), then \( u \)
may have been enumerated as an upper bound on the output for various inputs already known to lie within \((c,c')\). Exactly which inputs \(y\) depends on how soon the closed rational intervals around \(y\) were recognized to lie within \((c,c')\), but we know from the previous stages which (finitely many) upper and lower bounds have been enumerated on which rational intervals within \((e_s,e_s')\). Here at stage \(s+1\), the computation must respect these existing bounds on these subintervals, of course. The procedure at stage \(s+1\) is the following. Let \((q_0,q_1),\ldots,(q_{2m},q_{2m+1})\) be the rational subintervals of \((e_s,e_s')\) in question. For each end point \(q_i\) except for \(e_s\) and \(e_s'\) themselves, let \(u_i \leq +\infty\) be the least of the upper bounds assigned to any of these subintervals whose closure contains \(q_i\), and let \(v_i \geq -\infty\) be the least of the lower bounds assigned to them. \((\pm\infty\) indicates that no upper/lower bounds had yet been defined here.) If no such rationals \(q_i\) lie within \((e_s,e_s')\), then we artificially create \(q_{2m+2} = \frac{1}{2} \cdot (e_s + e_s')\), so as to have one. Now within the error set, there is no need to worry about keeping \(g\) equal to \(f\), so we simply declare that the following will hold.

- \(g(q_i) = u_i\) if \(u_i < +\infty\) as defined above;
- \(g(q_i) = v_i\) if \(v_i > -\infty\) but \(u_i = +\infty\);
- \(g(q_i) = 0\) otherwise.

In the interval from one \(q_i\) to the next-leftmost \(q_k\), \(g\) will be a linear function, from the point \((q_i,g(q_i))\) defined above to the point \((q_k,g(q_k))\).

It remains to define \(g\) on the first and last subintervals of \((e_s,e_s')\), since the cases \(q_i = e_s\) and \(q_i = e_s'\) were specifically omitted just now. The leftmost subinterval is \((e_s,q_i)\) for some \(i\), and we have declared the value \(g(q_i)\) already. We do not specify here any value for \(g(e_s)\), because this value cannot yet be determined. However, our intention is that \(g\) will be linear between \(e_j\) and this \(q_i\) as well. So, for whatever the smallest upper bound \(u\) currently enumerated for any interval containing \(e_j\) may be, we define the upper bounds on \((e_j,q_i)\) linearly from the point \((e_j,u)\) to the point \((q_i,g(q_i))\); similarly for the lower bounds. Notice that, by our choice of the rationals \(q_0,\ldots,q_{2m+1}\), all upper and lower bounds on \(e_j\) also apply on the entire subinterval \((e_j,q_i)\), and in particular \(g(q_i)\) was chosen in accordance with these bounds. If no upper bound, or no lower bound, is yet defined at \(e_j\), then the corresponding bounds on \((e_s,q_i)\) also remain undefined. Eventually a value \(g(e_j)\) will be defined, once we reach the first error interval containing \(e_j\) (its buffer set, if not sooner). Once that happens, the Secondary Step will define \(g\) linearly on this subinterval, and in the meantime item (3) of the Main Step will stop any other bounds from being defined on it.

We then execute the analogous procedure on the rightmost subinterval \((q_j,e_s')\), using the fixed value \(g(q_j)\) and any already-defined bounds for \(g(e_s')\). This completes the Secondary Step, and also completes stage \(s+1\). To finish the proof of Theorem 5.1 now requires two lemmas, one saying that the functional \(\Gamma\) constructed here does indeed define a function \(g : \mathbb{R} \to \mathbb{R}\), and a second one saying that \(g = f\) up to a set of measure < \(\epsilon\).
Lemma 5.5. For each fixed \( x \in \mathbb{R} \), the functional \( \Gamma^{S(A)@X} \) outputs an enumeration of the same Dedekind cuts, defining a single real number which we therefore call \( g(x) \), independent of the choice of enumeration \( X = A@B \) of the Dedekind cuts of \( x \).

Proof. Write \( \lambda_\infty \oplus \rho_\infty \) for the Dedekind cuts of \( x \) as binary sequences. Then, for every enumeration \( A@B \) of those Dedekind cuts, Lemma 5.4 noted that either \( \lambda_t \oplus \rho_t \) or \( \lambda_t \oplus \rho_t^+ \) is equal to \( (\lambda_\infty@t) \oplus (\rho_\infty@t) \). Since the construction never does anything unless all three closed rational intervals sanction it, every step it takes must be the step that it would take under the direction of \( (\lambda_\infty@t) \oplus (\rho_\infty@t) \), which is the same regardless of the choice of enumeration.

Now if \( x \notin \mathbb{Q} \), then for every \( t \in \mathbb{N} \), there is some \( s_0 \) such that all \( s \geq s_0 \) have \( \lambda_s@t = \lambda_s^+@t = \lambda_s^-@t = \lambda_\infty@t \). Therefore, for any action that \( (\lambda_\infty@t) \oplus (\rho_s@t) \) would like to take, at sufficiently large stages all three closed rational intervals will permit it to take that action. So, for irrational \( x \), the use of three distinct approximations never stops the computation by \( \Gamma \); at worst it slows it down.

If \( x \in \mathbb{Q} \), say \( x = q_m \), then for \( t > m \) we will always have \( \lambda_t(m) = \rho_t(m) = 0 \) but \( \lambda_t^+(m) = 1 = \rho_t^+(m) \), so the unification described above for irrational \( x \) no longer occurs. However, with \( x = q_m \), we will eventually reach a stage \( s_0 \) at which the buffer set \( (e_{s_0}, e_{s_0}') \) for \( q_m \) is discovered, and then a stage \( s_1 > s_0 \) by which all three closed rational intervals are seen to lie within this \( (e_{s_0}, e_{s_0}') \). Thereafter the instructions for that error interval will be applied, and for those instructions the three distinct intervals clearly do not create any problems, since in all three intervals, both endpoints ultimately converge to \( x \) as \( t \to \infty \). □

Of course, being computed by \( \Gamma \), this \( g \) must be continuous, by Theorem 2.4. Only one claim in Theorem 5.1 remains to be proven.

Lemma 5.6. The function \( g \) defined above has the property that
\[
\mu(\{x \in \mathbb{R} : g(x) \neq f(x)\}) < \epsilon.
\]

Proof. The error set \( \mathcal{U}_{\epsilon,S,A} \) has measure \( < \epsilon \), and we claim that all \( x \) outside it satisfy \( f(x) = g(x) \). Indeed, if \( x \notin \mathcal{U}_{\epsilon,S,A} \), then \( \Psi^{S(A)@L_x@R_x}_\epsilon \) must equal \( (S@L_x@R_x)^{(A)} \), and so all computations by
\[
\Theta \Psi^{S(A)@L_x@R_x}_\epsilon = \Theta (S@L_x@R_x)^{(A)}
\]
are enumerated in the list of upper and lower computations. Now for every rational \( u > f(x) \), there is such an upper computation \( (c, c', u) \) with \( x \in (c, c') \). Suppose this computation appears at stage \( s \) in the enumeration of computations. Since \( x \) is not in the error set, items (2) and (3) in the Main Step will never be applied to \( x \), and as the three closed rational intervals \([a_t, b_t]\), etc. close in on \( x \) at subsequent stages, they must eventually sit inside \((c, c')\). Once that happens, the computation \((c, c', u)\) will be applied, enumerating \( u \) into the upper cut of \( g(x) \) (since clearly no other computation contradicts this one). Similarly, every \( v < f(x) \) will eventually be enumerated into the lower cut for \( g(x) \), and so \( f(x) = g(x) \) as required. □
Formally this completes the proof, and a few last words about it may aid comprehension. We note that the error set \( \mathcal{U}_{\epsilon,S,A} \) is open and contains \( \mathbb{Q} \), hence is dense in \( \mathbb{R} \). So, for an \( x \) as in Lemma 5.6, error intervals do approach \( x \) from both sides. However, no relevant \([a_t,b_t]\) is ever contained in an error interval, nor in the buffer set around an end point, and so our procedure finds shorter and shorter intervals \([a_t,b_t]\) as the error intervals close in from both sides. For every computation \((c,c',u)\) with \( x \in (c,c') \), all three closed rational intervals will eventually be short enough to fit inside \((c,c')\), and so the computation will in fact be applied. Meanwhile, a connected component of \( \mathcal{U}_{\epsilon,S,A} \) is itself an open interval, necessarily with irrational end points and necessarily composed of infinitely many error intervals \((e,e')\), since these each have rational end points. For some of these end points, the values \( g(e) \) or \( g(e') \) may be left undefined when \((e,e')\) appears, but they will be fixed later on, as the next overlapping error interval in the connected component appears. As these end points \( e \) approach the end point \( x \) of the connected component, each of them will have been subjected to more of the computation intervals around \( x \), so will have had certain upper and lower bounds placed on \( g(e) \) before \((e,e')\) was recognized as an error interval. These bounds are what force the values \( g(e) \) to approach \( g(x) \) as the rationals \( e \) themselves approach \( x \), so that the function \( g \) is indeed continuous – as it must be, being computed by the functional \( \Gamma \).

\[ \square \]

6. Uniformity

It was not the original purpose of this article to prove anything new. The intention was to present a new proof of Lusin’s Theorem in real analysis, using known facts from computability theory and descriptive set theory, and thus to illustrate and illuminate a connection between the principles used in standard proofs of Lusin’s Theorem and the principles from computability which make our proof here work. Nevertheless, certain uniformities and computability results became apparent during the creation of the proof in Section 5, and in the end we have an effective version of Lusin’s Theorem. Sometimes new ideas entail new results, even when not intended to do so.

The substantial uniformity in the creation of \( g \) from \( f \) in our proof of Theorem 5.1 yields the function \( h \) that we describe here.

**Theorem 6.1.** There is a computable total function \( h : \mathbb{Q} \times \mathbb{N} \to \mathbb{N} \) such that, for each fixed \( \alpha \) and \( S \) and each \( S \)-decidable presentation \( A \) of \( \alpha \), whenever an \( \alpha \)-jump-computable function \( f \) is given by the oracle computation

\[
\Phi_e^{(S \oplus A \oplus B)^{(A)}}
\]

for all enumerations \( A \oplus B \) of each \( x \in \mathbb{R} \), a function \( g(x) \) realizing Lusin’s Theorem 5.1 for this \( f \) and an arbitrary rational \( \epsilon > 0 \) is given by

\[
\Phi_{h(e,\epsilon)}^{E(A) \oplus S^{(A)} \oplus A \oplus B}
\]

where \( E(A) \) is the elementary diagram of \( A \), given as a subset of \( \mathbb{N} \) by a Gödel coding.
That is, the Turing functional for computing \( g \) can be determined uniformly from that for \( f \), uniformly in \( \epsilon \) and independently of the level of \( f \) in the Borel hierarchy. (Of course, the oracle \( S^\langle A \rangle \) used to compute \( g \) does depend on the Borel level of \( f \), as well as on \( S \) itself.)

By the Padding Lemma (see e.g. [8, Lemma I.3.2]), \( h \) may also be assumed injective.

This can be taken further. It is not necessary to know all of the elementary diagram \( E(A) \); it suffices to know which elements of \( (A, \prec) \) are limit points from the left, which is the zero element, which pairs \((m, n)\) are adjacencies (with \( m < n \) and no elements between them), and whether \( A \) itself is a limit ordinal, a successor, or zero. So \( E(A) \) could be replaced by an \( S \)-computable atomic diagram of a presentation of \( \alpha \) in a signature with that much information (adjacency relation, limit points, etc.). Notice that for uniformity, even the finite information, such as knowing which element is the left end point of \( A \), must be given. On the other hand, it is not necessary to be given this diagram itself: since an \( S \)-oracle is given, one only needs to know an index for computing the diagram from \( S \). In summary, for a fixed \( S \), our procedure is uniform across all \( S \)-computable presentations of ordinals \( < \omega_1^S \).

Nowhere in the proof of Theorem 5.1 did we use the fact that the values \( f(x) \) were finite real numbers (as opposed to \( \pm \infty \)). Indeed, the same proof would work even if enumerations of improper Dedekind cuts were allowed as outputs. Moreover, the construction in Theorem 5.1 ensures that the points \( x \) where \( g(x) \neq f(x) \) all have \( g(x) \) finite, as they all lie in buffer sets or error intervals, and \( g \) is finite on all those intervals (thanks to the assurance that at least one rational \( x \)-coordinate \( z \) was fixed inside the interval and given a finite \( y \)-value). Therefore we have the following result.

**Theorem 6.2.** For every Borel-measurable function \( f: \mathbb{R} \to \mathbb{R} \cup \{ \pm \infty \} \) and every \( \epsilon > 0 \), there exists a continuous function \( g: \mathbb{R} \to \mathbb{R} \cup \{ \pm \infty \} \) such that \( \mu(\{ x \in \mathbb{R} : f(x) = g(x) \}) < \epsilon \). Moreover, the value \( g(x) \) is infinite only when \( f(x) \) is, and all uniformities described in Theorem 6.1 still hold here. \( \square \)

7. The Causes of Discontinuity

Our proof of Theorem 5.1 emphasizes a remarkable fact about Lusin’s Theorem. Once the parameters \( \epsilon, S, \) and \( A \) are fixed, the proof uses the exact same error set \( U_{\epsilon, S, A} \) for every function \( f \) it is given. So one may legitimately argue that the non-continuity of functions \( f \) at this level of the Borel hierarchy is the “fault” of the real numbers \( x \) in

\[
U_{S, A} = \bigcap_{\epsilon > 0} U_{\epsilon, S, A},
\]

namely, the set of those \( x \in \mathbb{R} \) that are not generalized-\( \alpha \)-low relative to \( S \) (using the presentation \( A \) of \( \alpha \)). In this sense, the functions themselves are not the obstacle: their non-continuity was caused by our inability to approximate \((S \oplus L_x \oplus R_x)^\langle A \rangle \) for those \( x \) in the error intervals.
Proposition 7.1. Fix $\alpha$, $S$, an $S$-decidable presentation of $\alpha$, and a rational $\epsilon > 0$. Then for every $\alpha$-jump $S$-computable function $f : \mathbb{R} \to \mathbb{R}$, the procedure in Theorem 6.1 produces a continuous $g$ such that the set $\{ x \in \mathbb{R} : f(x) \neq g(x) \}$ is always contained within the same open set $U_{\epsilon,S,A} \subseteq \mathbb{R}$ of measure $< \epsilon$, independent of the choice of $f$. Indeed, for $\epsilon_0 \leq \epsilon_1$, we have $U_{\epsilon_0,S,A} \subseteq U_{\epsilon_1,S,A}$.

The immediate objection to this proposition is that, just by translating $f$ by a certain fixed parameter $c$, one could define a function $f_c(x) = f(x - c)$ for which most of the discontinuities of $f$ move out of $U_{\epsilon,S,A}$. This is true, but it requires $c$ to be noncomputable, indeed not $S$-computable, and so $f_c$ does not belong to the class of functions considered in Proposition 7.1. In fact, $U_{\epsilon,S,A}$ is closed under translation by $S(A)$-computable parameters, and under other similar gambits one might concoct.

The more informed objection to the proposition is that it is obvious: there are only countably many $\alpha$-jump $S$-computable functions, so by applying Lusin’s Theorem to the $n$-th such function with tolerance $\epsilon_{2n+1}$, we immediately prove the proposition. This is correct, but the spirit of the proposition is that it was not necessary to slice up the $\epsilon$-amount of measure this way: our proof of Theorem 5.1 defined $U_{\epsilon,S,A}$ using basic computability theory, and then uniformly constructed some continuous $g$ for each $f$ such that they differed only within $U_{\epsilon,S,A}$. Probably the best way to express this is to note that the restriction of every such $f$ to the complement of each $U_{\epsilon,S,A}$ is itself $S(A)$-computable and hence continuous on this domain, and that each such domain is simply a $\Pi_1^{S(A)}$ set of real numbers.

In contrast, however, the restriction of such an $f$ to $U_{S,A}$ (defined just above) need not be continuous. Analogously, while only measure-0-many real numbers fail to be generalized-$\alpha$-low relative to $S$, no single Turing functional can compute $(S \oplus X)^{(\alpha)}$ from $S^{(\alpha)} \oplus X$ for all but measure-0-many $X$. For a counterexample, let $f$ be the characteristic function of $(0, +\infty)$: for any continuous $g$ assuming the values $0$ and $1$, the $g$-preimage of $(0,1)$ will be a nonempty open set (hence of measure $> 0$) where $g \neq f$. Thus it is fruitless to try to use these ideas to build a continuous $g$ equal to the given $f$ outside a set of measure $0$: that extension of Theorem 5.1 is simply false.

8. Computing Continuous Functions

When Lusin’s Theorem is applied to a function $f$ that is already continuous, it holds trivially: just take $g = f$. One might ask whether the procedure given in Section 5 reflects this. The immediate answer is that it does not: if $f$ is continuous but is presented to us as an $\alpha$-jump-computable function, applying the procedure there will often produce a $g$ that, while satisfying the requirements of Lusin’s Theorem, is not in fact equal to $f$, not even up to a set of measure 0. For future investigation, we conjecture that this is inherent: no uniform procedure (as in Theorem 6.1) instantiating Lusin’s Theorem can also succeed in making $g = f$ when $f$ is itself continuous.

However, if we ask the same question restricted entirely to continuous functions $f : \mathbb{R} \to \mathbb{R}$, then it is possible to produce a procedure for computing the function
from a procedure for \(\alpha\)-jump-computing it. (In general a stronger oracle is required, though.) This situation could plausibly arise: for example, perhaps we can only determine a jump-computation for a solution \(f\) to some differential equation under certain initial conditions, although such an \(f\), being differentiable, must be continuous.

**Theorem 8.1.** Let \(\alpha\) be a countable ordinal and \(A\) an \(S\)-decidable presentation of \(\alpha\). Then there exists a computable total injective function \(h : \mathbb{N} \to \mathbb{N}\) such that, whenever

\[
f = \Phi_{\varepsilon}((S \oplus A \oplus B)^{(A)}) : \mathbb{R} \to \mathbb{R}
\]

is an \(\alpha\)-jump \(S\)-computation of a continuous \(f\), we have a 0-jump \(S^{(A+1)}\)-computation

\[
\Phi_{h(e)}((S^{(A+1)} \oplus A \oplus B)) = f.
\]

Here \(A + 1\) is the presentation of the ordinal \(\alpha + 1\) with \(\text{dom}(A + 1) = \text{dom}(A) \cup \{k\}\), where the number \(k = \min(\mathbb{N} - \text{dom}(A))\) is adjoined to \(A\) as a new greatest element.

**Proof.** Where in Theorem 5.1, the rational numbers were a hindrance to be handled by buffer sets, here instead they serve as our guide. For every \(q \in \mathbb{Q}\), the left and right cuts \(L_q\) and \(R_q\) are computable uniformly in \(Q\), so \((S \oplus L_q \oplus R_q)^{(A)}\) is \(S^{(A)}\)-computable, uniformly in \(q\), and an \((S \oplus L_q \oplus R_q)^{(A+1)}\) oracle can decide the set

\[
D = \{(a,b,u,v) \in \mathbb{Q}^4 : (\forall q \in [a,b]) \; v < f(q) < u\}.
\]

The elements of \(D\) are “boxes” \((a,b) \times (v,u)\) in \(\mathbb{R}^2\) within which the graph of \(f\) (restricted to \((a,b)\)) must lie. Now for any \(x \in \mathbb{R}\) and any enumeration \(A \oplus B\) of the cut of \(x\), we get an \(S^{(A+1)}\)-computable enumeration of

\[
E_x = \{(u,v) \in \mathbb{Q}^2 : (\exists a \in p_1(A))(\exists b \in p_1(B)) \; (a,b,u,v) \in D\}.
\]

By continuity there are boxes in \(D\) with \(u - v\) arbitrarily small, and so the projections \(p_3\) and \(p_4\) of \(E_x\) are the right and left cuts \(R_{f(x)}\) and \(L_{f(x)}\). Thus we have a computation of \(f\) below an \(S^{(A+1)}\)-oracle, whose program is uniform in the index \(e\). \(\square\)

**References**


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