Computable Differential Fields

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Differential algebra is the study of differential equations from a purely algebraic standpoint. The differential equations studied are polynomials in a variable Y and its derivatives $\delta Y, \delta(\delta Y), \ldots$, with coefficients from a specific field K which admits differentiation on its own elements via the operator δ. Such a field K is known as a *differential field*: it is simply a field with one or more additional unary functions δ on its elements, satisfying the usual properties of derivatives: $\delta(x+y) = (\delta x) + (\delta y)$ and $\delta(x \cdot y) = (x \cdot \delta y) + (y \cdot \delta x)$. It is therefore natural to think of the field elements as functions, and standard examples include the field $\mathbb{Q}(X)$ of rational functions in one variable under differentiation $\frac{d}{dX}$, and the field $\mathbb{Q}(t, \delta t, \delta^2 t, \ldots)$ with a *differential transcendental* t satisfying no differential equation over the ground field $\mathbb{Q}$. Additionally, every field becomes a differential field when the operator $\delta x = 0$ is adjoined; we call such a differential field a *constant field*, since an element whose derivative is 0 is commonly called a *constant*.

Although the natural examples are fields of functions, the treatment of differential fields regards the field elements merely as points. There are strong connections between differential algebra and algebraic geometry, with such notions as the ring $K\{Y\}$ of differential polynomials (namely the algebraic polynomial ring $K[Y, \delta Y, \delta^2 Y, \ldots]$, with each $\delta^i Y$ treated as a separate variable), differential ideal, differential variety, and differential Galois group all being direct adaptations of the corresponding notions from field theory. Characteristically, these concepts behave similarly in both areas, but the differential versions are often a bit more complicated. In terms of model theory, the theories $\mathbf{ACF}_0$ and $\mathbf{DCF}_0$ (of algebraically closed fields and differentially closed fields, respectively, of characteristic 0) are both complete and ω-stable with effective quantifier elimination, but $\mathbf{ACF}_0$ has Morley rank 1, whereas $\mathbf{DCF}_0$ has Morley rank ω.

Just as the algebraic closure $\overline{F}$ of a field F (of characteristic 0) can be defined as the prime model of the theory $\mathbf{ACF}_0 \cup \Delta(F)$ (where $\Delta(F)$ is the atomic diagram of F), the differential closure $\hat{K}$ of a differential field K is normally taken to be the prime model of $\mathbf{DCF}_0 \cup \Delta(K)$. This $\hat{K}$ is unique up to isomorphism over K, but not always minimal: it is possible for $\hat{K}$ to embed into itself over K (i.e. fixing K pointwise) with image a proper subset of itself. This has to do with the fact that some 1-types over K are realized infinitely often in $\hat{K}$, so that the image of the embedding can omit some of those realizations. As a prime model, the differential closure realizes exactly those 1-types which are principal over K, i.e. generated by a single formula with parameters from K. It therefore omits the type of a differential transcendental over K, since this type is not principal, and so every element of $\hat{K}$ satisfies some differential polynomial over K. On the other hand, the type of a *transcendental constant*, i.e. an element x with $\delta x = 0$ but not algebraic over K, is also non-principal and hence is also omitted, even though such an element would be "differentially algebraic" over K.

The goal of the current work in computable differential fields, by the speaker and two co-authors, is to adapt the two fundamental theorems from computable field theory to computable differential fields. These two theorems, each used very frequently in work on computable fields, are the following.

Theorem 1. *(Kronecker's Theorem (1882); see* [5] *or* [2]*)*

(1) *The field $\mathbb{Q}$ has a splitting algorithm. That is, the set of irreducible polynomials in $\mathbb{Q}[X]$, commonly known as the* splitting set *of $\mathbb{Q}$, is decidable.*
(2) *If a computable field F has a splitting algorithm, so does the field $F(x)$, for every element x algebraic over F (within a larger computable field).*
(3) *If a computable field F has a splitting algorithm, then so does the field $F(t)$, for every element t transcendental over F.*

(The algorithms in Parts II and III are different, and no unifying algorithm exists.)

Theorem 2. *(Rabin's Theorem (1960); see* [7]*)*

(1) *Every computable field F has a* Rabin embedding, *i.e. a computable field embedding $g : F \to E$ such that E is a computable, algebraically closed field which is algebraic over the image $g(F)$.*
(2) *For every Rabin embedding g of F, the image $g(F)$ is Turing-equivalent to the splitting set S_F of F.*

For differential fields, the analogue of the first part of Rabin's Theorem was proven in 1974 by Harrington, who showed that for every computable differential field K, there is a computable embedding g of K into a computable, differentially closed field L such that L is a differential closure of the image $g(K)$. Harrington's proof used a different method from that of Rabin, and therefore did not address the question of the Turing degree of the image. Indeed, the first question to address, in attempting to adapt either of these theorems for differential fields, is the choice of an appropriate analogue for the splitting set S_F in the differential context.

Kronecker saw the question of reducibility of a polynomial in $F[X]$ simply as a natural question to ask. With twentieth century model theory, we understand better the reasons why it is important. Specifically, every irreducible polynomial $p(X) \in F[X]$ generates a principal type over the theory $\mathbf{ACF}_0 \cup \Delta(F)$, and every principal type is generated by a unique monic irreducible polynomial. (More exactly, the formula $p(X) = 0$ generates such a type.) On the other hand, no reducible polynomial generates such a type (with the exception of powers $p(X)^n$ of irreducible polynomials, in which case $p(X)$ generates the same type). So the splitting set S_F gives us a list of generators of principal types, and every element of $\overline{F}$ satisfies exactly one polynomial on the list. Moreover, since these generating formulas are quantifier-free, we can readily decide whether a given element satisfies a given formula from the list or not. Thus, a decidable splitting set allows us to identify elements of $\overline{F}$ very precisely, up to their orbit over F.

From model theory, we find that the set $\overline{T_K}$ of *constrained pairs* over a differential field K plays the same role for the differential closure. A pair (p, q) of differential polynomials from $K\{Y\}$ is *constrained* if p is monic and irreducible and of greater order than q (i.e. for some r, $p(Y)$ involves $\delta^r Y$ nontrivially while

$q(Y) \in K[Y, \delta Y, \ldots, \delta^{r-1}Y]$) and, for every $x, y \in \hat{K}$, if $p(x) = p(y) = 0$ and $q(x) \neq 0 \neq q(y)$, then there exists $h \in K\{Y\}$ such that either $h(x) = 0 \neq h(y)$ or $h(y) = 0 \neq h(x)$. This says that, if x and y both *satisfy* the pair (p, q), then the differential fields $K\langle x\rangle$ and $K\langle y\rangle$ which they generate within $\hat{K}$ must be isomorphic, via an isomorphism fixing K pointwise and mapping x to y. This is sufficient to ensure that the formula $p(Y) = 0 \neq q(Y)$ generates a principal type over $\mathbf{DCF}_0 \cup \Delta K$, and conversely, every principal type is generated by such a formula with (p, q) a constrained pair. With this background, we may state our results, first addressing Rabin's Theorem and then Kronecker's.

Theorem 3. *For every embedding g of a computable differential field as described by Harrington in* [3]*, the image $g(K)$ is Turing-computable from the set $\overline{T_K}$. So too is algebraic independence of finite tuples from $\hat{K}$, and also the function mapping each $x \in \hat{K}$ to its minimal differential polynomial over K. However, there do exist such embeddings g for which $\overline{T_K}$ has no Turing-reduction to $g(K)$.*

Theorem 4. *Let K be a computable nonconstant differential field, with $z \in \hat{K}$. Then $\overline{T_{K\langle z\rangle}}$ is Turing-computable from $\overline{T_K}$.*

So the middle part of Kronecker's Theorem holds. We believe that we also have a proof for constant fields, and for the third part, but this remains to be checked.

Conjecture 5. *Let K be a computable differential field, and z an element differentially transcendental over K within some larger comptuable differential field. Then $\overline{T_{K\langle z\rangle}}$ is Turing-computable from $\overline{T_K}$.*

It remains to determine whether the set $\overline{T_{\mathbb{Q}}}$ of constrained pairs over the constant differential field $\mathbb{Q}$ is decidable; we regard this as the most important question currently open in this area of study. A positive answer would likely give us a much better intuition about the structure of various simple differentially closed fields, well beyond any current understanding. It would also be desirable to make the failure of the second part of Rabin's Theorem more precise, by finding sets which are always equivalent to the Rabin image $g(F)$, and by finding sets which are always equivalent to $\overline{T_K}$.

References

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