Definable Incompleteness and Friedberg Splittings

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Abstract

We define a property \( R(A_0, A_1) \) in the partial order \( \mathcal{E} \) of computably enumerable sets under inclusion, and prove that \( R \) implies that \( A_0 \) is noncomputable and incomplete. Moreover, the property is nonvacuous, and the \( A_0 \) and \( A_1 \) which we build satisfying \( R \) form a Friedberg splitting of their union \( A \), with \( A_1 \) prompt and \( A \) promptly simple. We conclude that \( A_0 \) and \( A_1 \) lie in distinct orbits under automorphisms of \( \mathcal{E} \), yielding a strong answer to a question previously explored by Downey, Stob, and Soare about whether halves of Friedberg splittings must lie in the same orbit.

1 Introduction

The computably enumerable sets form an upper semi-lattice under Turing reducibility. Under set inclusion, they form a lattice \( \mathcal{E} \), as first noted by Myhill in [14], and the properties of a c.e. set as an element of \( \mathcal{E} \) often help determine its properties under Turing reducibility. Even before Myhill, Post had suggested that there should be a nonvacuous property of c.e. sets, definable without reference to the Turing degrees, which would imply that the Turing degree of such a set must lie strictly between the computable degree \( 0 \) and the complete c.e. degree \( 0' \).

Post's own attempts to find such a property failed. The properties he defined turned out to be extremely useful in computability theory, but each of them — simplicity, hypersimplicity, and hyperhypersimplicity — actually does hold of some complete set. The existence of a Turing degree between

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0 and 0′ was first proven by completely different means, namely the finite injury constructions of Friedberg and Muchnik ([6], [13]).

The term “Post’s Program” eventually came to denote the search for an \( \mathcal{E} \)-definable property implying incompleteness. Of the properties proposed by Post, all except hypersimplicity turned out to be definable in \( \mathcal{E} \), and other \( \mathcal{E} \)-definable properties, such as maximality, were developed and studied in their own right. Nevertheless, Post’s Program remained unfinished until 1991, when Harrington and Soare ([7]) found a property \( Q(A) \) definable in \( \mathcal{E} \) such that every \( A \) satisfying \( Q \) must be both noncomputable and Turing-incomplete. We give their definition of \( Q(A) \):

\[
Q(A) : (\exists C)_{A \subseteq C} (\forall B \subseteq C) (\exists D \subseteq C) (\forall S \subseteq C) S \subseteq C \implies \left( \exists T \right) \left( C \subseteq T \land A \setminus (S \cap T) = B \setminus (S \setminus T) \right).
\]

Here \( S \sqsubseteq C \) abbreviates \( (\exists \hat{S}) \left[ S \cup \hat{S} = C \land S \cap \hat{S} = \emptyset \right] \). (All variables represent elements of \( \mathcal{E} \), namely c.e. sets.) \( A \sqcup B \) denotes the union of two disjoint sets \( A \) and \( B \). Also, \( A \sqsubseteq m \) \( C \) abbreviates “\( A \) is a major subset of \( C \),” meaning that \( A \sqsubseteq C \) with \( C \setminus A \) infinite such that for every \( W \), if \( C \sqsubseteq W \), then \( \overline{W} \) is finite. Since the property of being finite is \( \mathcal{E} \)-definable, the statement \( A \sqsubseteq m \) \( C \) is \( \mathcal{E} \)-definable as well.

In this paper we generalize the property \( Q(A) \) to an \( \mathcal{E} \)-definable property \( R(A_0, A_1) \) of two c.e. sets. The statement of \( R \) is as follows:

\[
R(A_0, A_1) : A_0 \cap A_1 = \emptyset \land
(\exists C) (\forall B \subseteq C) (\exists D \subseteq C) (\forall S \subseteq C) S \subseteq C \implies
\left[ \left( B \setminus (S - A_0) \right) \cup A_1 = \left( D \setminus (S - A_0) \right) \cup A_1 \right] \land
\left[ C \setminus T \land \left( A_0 \cap S \cap T \right) \cup A_1 = \left( B \cap S \cap T \right) \cup A_1 \right].
\]

This property can be read to say that \( A_0 \) satisfies the \( Q \)-property on \( \overline{A_1} \). Indeed, the statement \( R(A_0, \emptyset) \) is equivalent to \( Q(A_0) \). In Section 2 we prove that just as with the \( Q \)-property, \( R(A_0, A_1) \) implies that \( A_0 \) is not of prompt degree, and hence not Turing complete in \( \Sigma^0_3 \). (A set which is not of prompt degree is said to be \textit{tardy}, and since \( A_0 \) satisfies an \( \mathcal{E} \)-definable property implying tardiness, we say that \( A_0 \) is “definably tardy.” Since all tardy sets are incomplete, we also say that \( A_0 \) is “definably incomplete.”)

Alternatively, we can interpret \( R(A_0, A_1) \) in the lattice \( \mathcal{E} / A \), where \( A \) is the principal ideal in \( \mathcal{E} \) generated by \( A_1 \). (See [15], p. 225.) In this lattice, \( C \sqsubseteq A D \) is defined to mean \( C \subseteq D \cup A_1 \), and \( C \approx A D \) if \( C \sqsubseteq A D \).
and \( D \subseteq A C \). Essentially, \( R(A_0, A_1) \) says that \( Q(A_0) \) holds in \( \mathcal{E}/A \), with containment and equality replaced by \( \subseteq A \) and \( \approx A \). The only differences are that we cannot state the properties \( A_0 \cap A_1 = \emptyset \) or \( A_1 \subseteq C \) in \( \mathcal{E}/A \), and that we have left the quantifier \( (\forall S \subseteq C) \) in \( R(A_0, A_1) \) just as in the original \( Q \)-property, rather than restating it to hold on \( A_1 \). Choosing not to restate it makes the \( R \)-property slightly stronger, but the stronger version can still be satisfied.

In Section 3 we construct c.e. sets \( A_0 \) and \( A_1 \) satisfying \( R \), to show that the \( R \)-property is non-vacuous. \( A_0 \) and \( A_1 \) will also be noncomputable. Thus, the following \( \mathcal{E} \)-definable formula is non-vacuous:

\[
(\exists A_1)[A_0 \supset T \emptyset \& R(A_0, A_1)]
\]

This formula guarantees that \( A_0 \) is noncomputable and incomplete, just as the property \( Q(A) \) does for \( A \). (Recall that computability is equivalent to the property of having a complement in \( \mathcal{E} \).)

We then consider Friedberg splittings. Two disjoint c.e. sets \( B_0 \) and \( B_1 \) form a Friedberg splitting of \( B = B_0 \cup B_1 \) if for every c.e. \( W \):

\[
W - B \text{ is not c.e.} \implies \text{ neither } W - B_0 \text{ nor } W - B_1 \text{ is c.e.}
\]

The sets \( B_0 \) and \( B_1 \) are each said to be \textit{half} of this Friedberg splitting. The sets \( A_0 \) and \( A_1 \) which we construct will have the additional property of forming a Friedberg splitting of their union.

We use the \( R \)-property to show that \( A_0 \) and \( A_1 \) cannot lie in the same orbit under automorphisms of \( \mathcal{E} \). (In the argot of this topic, we say that \( A_0 \) and \( A_1 \) are not \textit{automorphic}. Two sets are automorphic if they lie in the same orbit.) This will follow because the \( A_1 \) we construct will be of prompt degree, hence automorphic to a complete set, by another result of Harrington and Soare in [7].

The orbits of halves of Friedberg splittings have been a subject of interest for some time, at least since the discovery of the hemimaximal sets. A set is hemimaximal if it is half of a nontrivial splitting of a maximal set. This is \( \mathcal{E} \)-definable, and Downey and Stob proved that the hemimaximal sets form an orbit (see [3]).

Since the maximal sets themselves form an orbit, and since few orbits are known in \( \mathcal{E} \), this led to the conjecture that if \( \mathcal{O} \) is any orbit in \( \mathcal{E} \), then the collection of “hemi-\( \mathcal{O} \)” sets, i.e. halves of nontrivial splittings of sets in \( \mathcal{O} \), might also be an orbit. Alternatively, it was conjectured that halves of Friedberg splittings of sets in \( \mathcal{O} \) might form an orbit. (For the orbit
of maximal sets, these classes coincide, since any nontrivial splitting of a maximal set is automatically a Friedberg splitting.)

Downey and Stob refuted both conjectures in [5], by producing two Friedberg splittings $B_0 \cup B_1 = C_0 \cup C_1$ of the same set $B$, which were definably different in $\mathcal{E}$. Hence $B_0$ and $C_0$ satisfy different 1-types in the language of inclusion and cannot be automorphic.

The present result goes a step further. Since $A_0$ is definably tardy, every set in its orbit must also be tardy, and hence $A_1$ must lie in a different orbit. This is thus the first example of a single Friedberg splitting with the two halves known to lie in different orbits in $\mathcal{E}$. It is also the first application of Harrington and Soare’s $Q$-property to derive results about Friedman splittings.

Our notation mostly follows that of [16]. The finite sets form an ideal $\mathcal{F} \subset \mathcal{E}$, and we write $\mathcal{E}^*$ for the lattice $\mathcal{E}/\mathcal{F}$. (Computability is definable in $\mathcal{E}$ as the property of possessing a complement, and then finiteness is definable, since a set is finite if and only if all its subsets are computable.) We write $A \subseteq^* B$ if $B - A$ is finite, and $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$.

We use the standard enumeration $\{W_e\}_{e \in \omega}$ of the computably enumerable sets, with finite approximations $\{W_{e,n}\}_{n \in \omega}$ to each. For the c.e. sets which we construct ourselves, we will also give finite approximations, usually writing $A = \bigcup_{e \in \omega} A^e$. If $A$ and $B$ are both enumerated this way, we write $A \setminus B = \{x : (\exists s)[x \in A^e - B^e]\}$, and $A \setminus B = \{x \in A \cap B : (\exists s)[x \in A^e - B^e]\}$. Thus when an element not yet in $B$ enters $A$, we put it into $A \setminus B$, and if it later enters $B$, then we put it into $A \setminus B$ as well.
2 The $R$-Property

In order to guarantee that the set $A_0$ is not automorphic to a complete set, we will force it to satisfy the lattice-definable property $R$ defined in Section 1, and prove that this implies tardiness of $A_0$. Tardiness itself does not guarantee that a set cannot be automorphic to a complete set, of course, but satisfaction of $R$ does, since every other set automorphic to $A_0$ must also satisfy $R$ and therefore must also be tardy, hence incomplete. (A tardy set must be half of a minimal pair under $\leq_T$, as shown in [16], and therefore must be incomplete.) We restate the $R$-property here:

$$R(A_0, A_1) : A_0 \cap A_1 = \emptyset \&$$

$$(\exists C)(\forall B \subseteq C)(\exists D \subseteq C)(\forall S \subseteq C)(\exists T) [A_0 \cup A_1 \subseteq C \&$$

$$[(B \cap (S - A_0)) \cup A_1 = (D \cap (S - A_0)) \cup A_1$$

$$\Rightarrow$$

$$[\overline{C} \subseteq T \& (A_0 \cap S \cap T) \cup A_1 = (B \cap S \cap T) \cup A_1]]$$

**Theorem 2.1** If $A_0$ and $A_1$ are two c.e. sets such that $R(A_0, A_1)$ holds, then $A_0$ is not of prompt degree.

**Proof.** The proof is similar to the corresponding result for the $Q$-property in [7]. Given $A_0$ and $A_1$, we pick a set $C$ as specified in $R(A_0, A_1)$ and fix enumerations $\{A_0^s\}_{s \in \omega}$ of $A_0$ and $\{C^s\}_{s \in \omega}$ of $C$ such that $A_0 \subseteq C \setminus A_0$.

To prove that a given $\varphi_e$ is not a promptness function for $A_0$, we need to find an infinite c.e. set $W_i$ with standard enumeration $\{W_i^s\}_{s \in \omega}$ satisfying the tardiness requirement $T_e$:

$$[(\forall s)\varphi_e(s) \downarrow \geq s] \implies (\forall x)(\forall s)[x \in W_i, s - W_i, s - 1 \implies A_0^s \mid x = A_0^s \varphi_e(s) \mid x]$$

We will prove independently for each $e$ that $T_e$ holds. Having fixed $e$, we will assume for the rest of this section that $\varphi_e(s) \geq s$ for every $s$, since otherwise $T_e$ is automatically fulfilled. We will build a strong array $\{V_{\langle a, k \rangle, n}\}_{k, m \in \omega; a \in \omega \times \omega}$ of c.e. sets with enumerations $\{V^a_{\langle a, k \rangle, n}\}_{s \in \omega}$. The Slowdown Lemma then gives a computable function $f$ such that for each $\langle a, k \rangle$ and each $n, W_{f(\langle a, k \rangle, n)} = V_{\langle a, k \rangle, n}$ and $V_{\langle a, k \rangle, n} \setminus W_{f(\langle a, k \rangle, n)} = V_{\langle a, k \rangle, n},$ so that no element of $V_{\langle a, k \rangle, n}$ enters $W_{f(\langle a, k \rangle, n)}$ until it has already entered $V_{\langle a, k \rangle, n}$. Periodically the strategy for a given $\langle a, k \rangle$ may be injured by a higher-priority strategy. If this happens while we are enumerating $V_{\langle a, k \rangle, n},$ then we give up on $V_{\langle a, k \rangle, n}$ and start enumerating $V_{\langle a, k \rangle, n + 1}$. There will exist an $\langle a, k \rangle$ which is only injured $n$ times (with $n < \omega$), yet receives attention.
at infinitely many stages, and the corresponding \( V_{(\alpha, k), n} \) will be infinite and will be the set which proves satisfaction of \( T_e \).

We define the function \( n((\alpha, k), s) \) to keep track of which \( V_{(\alpha, k), n} \) we are enumerating at stage \( s \). In particular, if the \( (\alpha, k) \)-strategy receives attention at stage \( s + 1 \), then we may add an element to \( V_{(\alpha, k), n((\alpha, k), s + 1)}^{s + 1} \). To avoid notational chaos, however, we will write \( V_{(\alpha, k), n((\alpha, k), s + 1)}^{s + 1} \) in the construction and understand \( V_{(\alpha, k), n((\alpha, k), s + 1)}^{s + 1} \) for it.

To ensure that one of these \( W_{j((\alpha, k), n)} \) will satisfy \( T_e \), we build a c.e. set \( B \) to which to apply the property \( R \). When we want to preserve \( A_0 \upharpoonright x \) from stage \( s \) until stage \( \varphi_e(s) \) so as to satisfy \( T_e \), we do so by restraining all elements \( < x \) from entering \( B \) until stage \( \varphi_e(s) \). The \( R \)-property then prohibits such elements from entering \( A_0 \), since if they did, we would then hold them out of \( B \) forever after, thereby contradicting \( R(A_0, A_1) \).

To apply the \( R \)-property, we need to know which c.e. set \( W_j \) is the \( D \) specified by the property. Of course, we do not have this information, but our strategy is to use \( S \) to cover all the possibilities. Specifically, in the construction we will split \( C \) into the disjoint union of c.e. sets:

$$ C = \bigsqcup_{i \in \omega} S_i; $$

and apply the \( R \)-property to each \( S_i \), with \( S_i \) in the role of \( S \). (Clearly each \( S_i \subseteq C \).) We use each \( S_i \) to handle the possibility that \( D = W_j \).

Of course, the \( R \)-property states that the restraints we place on elements from entering \( B \) only affect \( A_0 \) on \( S \cap T \cap A_1 \). Since \( R(A_0, A_1) \) also states that \( A_0 \cap A_1 \) is empty, we do not need to worry about elements of \( A_1 \), for they can never enter \( A_0 \). We are allowed to choose the \( S_i \), since the matrix of \( R \) applies for all \( S \), and indeed we have already done so above (namely \( S = S_i \), for each \( i \) in turn). However, we can only guess at the set \( T \).

To determine the index \( j \) such that \( T = W_j \) corresponds to the set \( S \) which we choose, we use a \( \Pi^0_2 \) guessing procedure, since the conclusion in the matrix of \( R \) is a \( \Pi^0_2 \) property. The \( j \) for which \( T = W_j \) will be the least \( j \) which receives infinitely many guesses under this procedure. (We ensure that the hypothesis of the matrix holds, by periodically putting all elements of \( D^g \cap (S^g - A_0^g) \) into \( B^g \).) Moreover, in the construction, we will subdivide each \( S_i \) into the disjoint union of c.e. sets \( S_{i,j} \):

$$ S_i = \bigsqcup_{j \in \omega} S_{i,j}. $$

\( S_{i,j} \) is used to handle the possibility that \( T = W_j \), so we pay attention to \( S_{i,j} \).
each time $j$ is named by the guessing procedure. Thus the $S_{i,j}$ corresponding to the correct $T$ will receive attention infinitely often.

To simplify the notation, we let the variable $\alpha = (i, j)$ range over $\omega \times \omega$, and define:

$$
D_\alpha = W_i \\
S_\alpha = S_{i,j} \\
T_\alpha = W_j.
$$

We order the elements $\alpha$ of $\omega \times \omega$ by pulling back the usual order $<$ on $\omega$ to $\omega \times \omega$ via a standard pairing function. Thus each $\alpha$ has only finitely many predecessors under $<$.

For each $\alpha$, let $F(\alpha)$ be the conjunction of the hypothesis and conclusion in the matrix of the $R$-property:

$$
F(\alpha): \quad (B \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1 & \quad (1) \\
\neg C \subseteq T_\alpha \& (A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 = (B \cap S_\alpha \cap T_\alpha) \cup A_1] \quad (2)
$$

Then $F(\alpha)$ is a $\Pi^0_2$ condition, uniformly in $\alpha$, so there is a computable total function $g$ such that $F(\alpha)$ holds just if $g^{-1}(\alpha)$ is infinite. We enumerate the c.e. set $Z_\alpha = g^3(\alpha)$ by setting $Z_\alpha^* = \{t : g(t) = \alpha\}$.

Now we narrow down each $T_\alpha$ to a c.e. subset $U_\alpha$, enumerated by:

$$
U_\alpha^* = U_\alpha^{\alpha - 1} \cup \{x \in T_\alpha^* \setminus C^* : x < \lfloor Z_\alpha^* \rfloor\}
$$

Thus, if $T_\alpha$ actually is the $T$ corresponding to $S_i$, then $U_\alpha$ will contain all of $T_\alpha$ except certain elements of $C$. Hence $F(\alpha)$ will hold with $U_\alpha$ in place of $T_\alpha$. On the other hand, if $F(\alpha)$ fails, then $Z_\alpha$ and $U_\alpha$ are both finite.

If $F(\alpha)$ holds, then $C \subseteq U_\alpha$, so $T_\alpha \subseteq U_\alpha \cup A_1$, because $A_0 \cup A_1 \subseteq C$. For the least $\alpha$ such that $F(\alpha)$ holds, our construction of $S_\alpha^{\alpha+1}$ will yield $C \subseteq A_0 \subseteq S_\alpha \cup A_1$, with $S_\beta$ finite for all $\beta < \alpha$. Hence there will exist a $k$ such that

$$
C = A_0 \subseteq S_\alpha \cup A_1 \cup \{0, 1, \ldots, k - 1\}
$$

Line (3) is a $\Pi^0_1$ statement, uniformly in $k$ and $\alpha$, since our definition of $S_\alpha$ will be uniform in $\alpha$. Therefore, there exists a total function $h_\alpha$ such that (3) holds if and only if $h_\alpha^{-1}(k)$ is infinite. We define:

$$
h(s) = h_{g(s)}(n), \text{ where } n = \{| t < s : g(t) = g(s) |\}.
$$

We will enumerate sets $V_{(\alpha, k), n}$ for each $\alpha$, $k$ and $n$. For the least $\alpha$ with $Z_\alpha$ infinite and the least $k$ with $h_\alpha^{-1}(k)$ infinite, the set $V_{(\alpha, k), n}$ (for some
n) will be the $W_i$ required by $T_e$. Elements of each $V_{(a,k),n}$ (the “witness elements” for the requirement $T_e$) will be denoted $v_{(a,k)}^s$. Each $v_{(a,k)}^s$ will enter $V_{(a,k),n}$ for at most one $n$.

The Slowdown Lemma (see [16], p. 284) then yields a computable function $f$ such that, for every $\langle \alpha, k \rangle$ and every $n$, $V_{(a,k),n} = W_{f(\langle \alpha, k \rangle,n)}$, and at every stage $s$,

$$(V_{(\alpha,k),n}^s - V_{(\alpha,k),n}^{s-1}) \cap W_{f(\langle \alpha, k \rangle,n),s} = \emptyset.$$ 

When a witness element $v_{(\alpha,k)}^s$ enters $V_{(\alpha,k),n}$, we will find the stage $t_{(\alpha,k)}^s > s$ at which $v_{(\alpha,k)}^s$ enters $W_{f(\langle \alpha, k \rangle,n)}$ and restrain (with priority $\langle \alpha, k \rangle$) elements $\leq v_{(\alpha,k)}^s$ from entering $A_0$ until stage $\varphi_e(t_{(\alpha,k)}^s)$. (Recall that $T_e$ assumes $\varphi_e$ to be total.) Thus we will have $A_0^{t_{(\alpha,k)}^s} \uparrow v_{(\alpha,k)}^s = A_0^{\varphi_e(t_{(\alpha,k)}^s)} \uparrow v_{(\alpha,k)}^s$. If we can achieve this for all $v_{(\alpha,k)}^s$ in the (infinite) set $V_{(\alpha,k),n}$ for some $n$, then the set $W_{f(\langle \alpha, k \rangle,n)}$ will be the set required by $T_e$ to prove that $\varphi_e$ is not a promptness function for $A_0$.

At stage 0, for all $\langle \alpha, k \rangle$, we set $n(\langle \alpha, k \rangle, 0) = 0$ and $V_{(\alpha,k),0}^0 = \emptyset$, with $v_{(\alpha,k)}^0 \uparrow$ and $t_{(\alpha,k)}^0 \uparrow$. Also, let every $S_{\alpha}^0 = \emptyset$ and let $B^0 = \emptyset$.

At stage $s + 1$, we first define each $S_{\alpha}^{s+1}$. For each $x \in C^{s+1} - C^s$, find the least $\alpha$ such that $x \in U_{\alpha}^s$ and put $x$ into $S_{\alpha}^{s+1}$. If there is no such $\alpha$, put $x$ into $S_{\alpha}^{s+1}$. (The c.e. set $S_\alpha$ simply collects elements which enter $C$ without entering any $S_{\alpha}$. Thus $C = \bigsqcup_{\alpha \leq \omega} S_{\alpha}$.)

Set $\alpha = g(s)$, and define:

$$B_{s+1} = B_s \cup \{x : \begin{array}{l} x \in C^s - A_0^s \& (\exists \beta \leq \alpha) [x \in D_{\beta}^{s+1} \& S_{\beta}^{s+1} \& (\forall \delta \leq \beta)(\forall k < s)[S_{\delta}^{s+1}\downarrow \Rightarrow x \geq v_{(\delta,k)}^s]] \end{array} \}$$

For each strategy which is injured at stage $s + 1$, we begin enumerating a new witness set. To this end, set $n(\langle \gamma, k \rangle, s + 1) = n(\langle \gamma, k \rangle, s) + 1$ and $v_{(\gamma,k)}^{s+1} \uparrow$ and $t_{(\gamma,k)}^{s+1} \uparrow$ for each $\langle \gamma, k \rangle$ satisfying any of the following conditions:

- $\gamma > \alpha$.
- $\gamma = \alpha$ and $k > h(s)$.
- There exists $x < k$ with $x \in A_0^{s+1} - A_0^s$.
- There exists $\beta < \gamma$ with $S_{\beta}^{s+1} \neq S_{\beta}^s$.
- There exists $\beta < \gamma$ such that $U_{\beta}^{s+1}$ contains an element $\geq m$, where $m = \min(B_{s+1}^{s+1} - B^s)$.
For all other $\langle \gamma, k \rangle$, set $r((\gamma, k), s + 1) = r((\gamma, k), s)$.

We now define the witness sets at stage $s + 1$. For each $\langle \beta, k \rangle \leq \langle \alpha, h(s) \rangle$ (in the lexicographic order) which was not injured at stage $s + 1$:

1. If $v^s_{\langle \beta, k \rangle} \uparrow$ and $\langle \beta, k \rangle \neq \langle \alpha, h(s) \rangle$, let $v^{s+1}_{\langle \beta, k \rangle}$ and $t^{s+1}_{\langle \beta, k \rangle}$ diverge also, with $V_{\langle \beta, k \rangle, n} = V^s_{\langle \beta, k \rangle, n}$.

2. If $v^s_{\langle \alpha, h(s) \rangle} \uparrow$, let $v^{s+1}_{\langle \alpha, h(s) \rangle} = s + 1$, with $V_{\langle \alpha, h(s) \rangle, n} = V^s_{\langle \alpha, h(s) \rangle, n}$ and $t^{s+1}_{\langle \alpha, h(s) \rangle} \uparrow$.

3. If $v^s_{\langle \beta, k \rangle} \downarrow$ but $t^s_{\langle \beta, k \rangle} \uparrow$, let $v^{s+1}_{\langle \beta, k \rangle} = v^s_{\langle \beta, k \rangle}$, and ask whether the following holds:

$$\forall y \mid y \leq v^{s+1}_{\langle \beta, k \rangle} \left[ y \in A^{s+1}_0 \lor y \in A^{s+1}_1 \lor y \in (U^{s+1}_{\beta} - C^{s+1}) \lor y \in (C^{s+1} - B^{s+1}) \cap S^{s+1}_{\beta} \cap U^{s+1}_{\beta} \right]$$

(4)

If (4) holds, let $V^{s+1}_{\langle \beta, k \rangle, n} = V_{\langle \beta, k \rangle, n} \cup \{v^{s+1}_{\langle \beta, k \rangle}\}$ and $t^{s+1}_{\langle \beta, k \rangle} = \mu t \mid v^{s+1}_{\langle \beta, k \rangle} \in W_f(\langle \beta, k, n \rangle, t)$.

(Such a $t$ must exist, since $W_f(\langle \beta, k, n \rangle) = V_{\langle \beta, k, n \rangle}$. If (4) fails, then let $V_{\langle \beta, k \rangle, n} = V^s_{\langle \beta, k \rangle, n}$ and $t^{s+1}_{\langle \beta, k \rangle} \uparrow$.

4. If $v^s_{\langle \beta, k \rangle} \downarrow$ and $t^s_{\langle \beta, k \rangle} \downarrow$ and $\varphi_{r,s}(t^s_{\langle \beta, k \rangle}) < s$, then let $v^{s+1}_{\langle \beta, k \rangle} \uparrow$ and $t^{s+1}_{\langle \beta, k \rangle} \uparrow$, with $V_{\langle \beta, k \rangle, n} = V^s_{\langle \beta, k \rangle, n}$.

5. If $v^s_{\langle \beta, k \rangle} \downarrow$ and $t^s_{\langle \beta, k \rangle} \downarrow$ but either $\varphi_{r,s}(t^s_{\langle \beta, k \rangle}) \geq s$ or $\varphi_{r,s}(t^s_{\langle \beta, k \rangle})$ diverges, then let $V^{s+1}_{\langle \beta, k \rangle, n} = V_{\langle \beta, k \rangle, n}$, $v^{s+1}_{\langle \beta, k \rangle} = v^s_{\langle \beta, k \rangle}$, and $t^{s+1}_{\langle \beta, k \rangle} = t^s_{\langle \beta, k \rangle}$.

This completes the construction.

We now use the sets $B$ and $S_\alpha$ to prove that requirement $T_{\alpha}$ is satisfied.

**Lemma 2.2** If $Z_\beta$ is finite, then there exists a stage $s_1$ such that $t^s_{\langle \beta, k \rangle} \uparrow$ for all $s \geq s_1$ and all $k$.

**Proof.** Pick a stage $s_0$ such that no $s \geq s_0$ satisfies $g(s) = \beta$, and let $k' = \max \{ h(s) : g(s) = \beta \}$. Then for all $k > k'$, $v^s_{\langle \beta, k \rangle} \uparrow$ for all $s$, and hence $t^s_{\langle \beta, k \rangle} \uparrow$ for all $s$. (The construction makes it clear that for any $k$ and $s$, $t^s_{\langle \beta, k \rangle}$ can converge only if $v^s_{\langle \beta, k \rangle}$ converges.)

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Now suppose \( k \leq k' \) and \( v^s_{(\beta,k)} \downarrow \) for all \( s \geq s_0 \). This means that we never execute Step (4) in the construction after stage \( s_0 \), and that the \( (\beta,k) \) strategy is never injured after stage \( s_0 \). But if \( t^s_{(\beta,k)} \) ever converges after stage \( s_0 \), then eventually we must reach Step (4), since we assumed \( \varphi_e \) to be total. Hence \( t^s_{(\beta,k)} \) must diverge for all \( s \geq s_0 \).

Finally, suppose \( k \leq k' \) and \( v^s_{(\beta,k)} \uparrow \) for some \( s_1,k \geq s_0 \). Then \( v^s_{(\beta,k)} \) will diverge for all subsequent \( s \), since it can only be newly defined at a stage \( s \) with \( g(s) = \beta \). Thus \( t^s_{(\beta,k)} \) will diverge for all subsequent \( s \) as well. Letting \( s_1 = \max_{k \leq k'} s_1,k \) completes the proof.

\[ \square \]

**Lemma 2.3** \( F(\alpha) \) holds for some \( \alpha \), and for the least such \( \alpha \), there exists a \( k \) such that \( h_0^{-1}(k) \) is infinite.

**Proof.** First we claim that some \( Z_\alpha \) must be infinite. Suppose not, so \( Z_\alpha \) is finite for all \( \alpha \), and \( F(\alpha) \) fails for all \( \alpha \). However, the \( R \)-property holds, so there must be some \( \alpha \) for which line (1) fails. Choose the least such \( \alpha \). Then

\[
(B \cap (S_\alpha - A_0)) \cup A_1 \neq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.
\]

Suppose \( x \in B \cap (S_\alpha - A_0) \). Pick \( s \) such that \( x \in B^{s+1} - B^s \). Now to go into \( B^{s+1} \), \( x \) must have been in \( A^{s+1}_\beta \cap S^{s+1}_\beta \) for some \( \beta \). Since \( x \in S_\alpha \), we know \( x \notin S_\beta \) for all \( \beta \neq \alpha \). Hence \( x \in D_\alpha \), and so

\[
(B \cap (S_\alpha - A_0)) \cup A_1 \subseteq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.
\]

Therefore, there must be some element \( x \in \overline{A_1} \cap \overline{B} \cap D_\alpha \cap (S_\alpha - A_0) \). Assume \( x \) is the least such element. Now for every \( \beta < \alpha \), line (1) must hold and line (2) must fail, since we chose \( \alpha \) to be minimal satisfying the \( R \)-property. Hence for all \( \beta < \alpha \),

\[
(B \cap (S_\beta - A_0)) \cup A_1 = (D_\beta \cap (S_\beta - A_0)) \cup A_1.
\]

Now since every \( Z_\beta \) with \( \beta < \alpha \) is finite, there is a stage \( s_0 \) such that for all \( s \geq s_0 \), \( g(s) > \alpha \), and we may also assume that \( s_0 \) is so large that \( x \in S_0^\alpha \cap D_0^\alpha \cap C^\alpha \). (Notice that \( x \in S_\alpha \) forces \( x \in C_\).)

Now use Lemma 2.2 to find a stage \( s_1 \geq s_0 \) such that:

\[
(\forall s \geq s_1)(\forall \beta \leq \alpha)(\forall k)(v^s_{(\beta,k)} \uparrow).
\]

Since \( \varphi_e \) is total, there must be a stage \( s \geq s_1 \) such that \( t^s_{(\alpha,k)} \uparrow \), and once we reach this stage \( s \), \( x \) must go into \( B^{s+1} \), contradicting our assumption that \( x \notin B \).
Thus, there must be some $\alpha$ such that $Z_\alpha$ is infinite. Let $\alpha$ be the least such. Then every $U_\beta$ with $\beta < \alpha$ is finite. Since $F(\alpha)$ holds, we have $\overline{C} \subseteq T_\alpha$, so by our construction, $\overline{C} \subseteq U_\alpha$, and by the major subset property, $\mathcal{A}_0 \subseteq^* U_\alpha \cup A_1$.

For this $\alpha$, we claim that $C - A_0 \subseteq^* S_\alpha \cup A_1$. Suppose $x \in C - A_0$. All but finitely many such $x$ lie in $U_\alpha \cup A_1$, as noted above. If $x \in A_1$, we are done. For each sufficiently large $x \in C - A_0 - A_1$, there exists $s$ such that $x \in U_\alpha - U_\alpha^{s-1}$. By definition of $U_\alpha$, we must have $x \notin C^s$. But $x \in C$, so $x \in C^{t+1} - C^t$ for some $t \geq s$. Hence $x \in S_\alpha^{t+1}$ by definition of $S_\alpha^{t+1}$, unless there exists $\beta < \alpha$ with $x \in U_\beta$. But all $U_\beta$ with $\beta < \alpha$ are finite, by our choice of $\alpha$, so all but finitely many of these $x$ lie in $S_\alpha$. Therefore, line (3) holds for some $k$, and $h_\alpha^{-1}(k)$ is infinite.

Use Lemma 2.3 to take the lexicographically least $\langle \alpha, k \rangle$ such that $F(\alpha)$ holds and $h_\alpha^{-1}(k)$ is infinite. Then there are infinitely many stages $s$ for which $g(s) = \alpha$ and $h(s) = k$, but only finitely many for which $\langle g(s), h(s) \rangle$ precedes $\langle \alpha, k \rangle$ in the lexicographic ordering. Let $s_0$ be the least stage with $\langle g(s_0), h(s_0) \rangle = \langle \alpha, k \rangle$ such that:

- $A_0^{s_0} \upharpoonright k = A_0 \upharpoonright k$, and
- $B^{s_0} \upharpoonright m = B \upharpoonright m$, where $m = \max \cup_{\beta < \alpha} U_\beta$, and
- for all $s \geq s_0$, $\langle g(s), h(s) \rangle \geq \langle \alpha, k \rangle$ lexicographically, and
- $S_\beta^{s_0} = S_\beta$ for all $\beta < \alpha$.

The final condition is possible since each $S_\beta \subseteq U_\beta$, which is finite for every $\beta < \alpha$. We also let $s_0 < s_1 < s_2 < \cdots$ be all the stages $s \geq s_0$ with $\langle g(s), h(s) \rangle = \langle \alpha, k \rangle$.

Now the $\langle \alpha, k \rangle$-strategy is never injured after stage $s_0$, so for every $s \geq s_0$, $n(\langle \alpha, k, s_0 \rangle) = n(\langle \alpha, k, s \rangle)$, and we write $n = n(\langle \alpha, k, s_0 \rangle)$. (Thus $n$ is the number of times the $\langle \alpha, k \rangle$-strategy was injured during the construction.) Moreover, minimality of $s_0$ implies that this strategy was injured at some stage $s \leq s_0$ such that there is no $s_{-1}$ with $s \leq s_{-1} < s_0$ and $\langle g(s_{-1}), h(s_{-1}) \rangle = \langle \alpha, k \rangle$. Therefore, $V^s_{\langle \alpha, k \rangle, n} = V^{s_0}_{\langle \alpha, k \rangle, n}$ is empty.

We claim that the subset $V_{\langle \alpha, k \rangle, n}$ satisfies requirement $T_\epsilon$. For this we need:

**Lemma 2.4** For this $\langle \alpha, k \rangle$, and for each $y \geq k$, there exists an $s$ such that the matrix of line (4) holds of $y$, $\langle \alpha, k \rangle$, and $s$. 

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Proof. Let $y \geq k$. If $y \in A_0 \cup A_1$, we are done. If $y \in \overline{C}$, then $y \in T_0$ since $F(\alpha)$ holds. But $Z_0$ is infinite, so $T_0 - C \subseteq U_0$, and $y$ is in $U_0 - C$, hence in some $U^{s+1} - C^{s+1}$.

So suppose $y \in C - A_0 - A_1$. Now since $h^{-1}_\gamma(k)$ is infinite and $y \geq k$, we know by line (3) that $y \in S_\alpha$. But $S_\alpha \subseteq U_\alpha \subseteq T_\alpha$ by definition of $S^{s+1}_\alpha$. Since $y \notin (B \cap S_\alpha \cap T_\alpha) \cup A_1$ by line (2), we know $y \notin B$. Thus there is an $s$ with $y \in (C^{s+1} - B^{s+1}) \cap S^{s+1}_\alpha \cap U^{s+1}_\alpha$. This proves the Lemma.

Now $V_{(a,k),n} = W_{j((a,k),n)}$, and if $s'$ is the stage at which $v^{s'}_{(a,k)}$ enters $V_{(a,k),n}$, then $t^{s'}_{(a,k)} > s'$ by our choice of $f$ from the Slowdown Lemma. Let $s'' = \varphi_{s'}(t^{s'}_{(a,k)})$. Then $s' < s''$, since we assumed $\varphi_{s'}$ to be increasing.

**Lemma 2.5** $V_{(a,k),n}$ is infinite. Moreover, for any element $v^{s'}_{(a,k)}$ of $V_{(a,k),n}$, with $s'$ and $s''$ as above, we have:

$$B^{s''} \upharpoonright v^{s'}_{(a,k)} = B^{s''} \upharpoonright (a,k)$$ and $$A^{s'}_{0} \upharpoonright v^{s'}_{(a,k)} = A^{s''}_{0} \upharpoonright v^{s'}_{(a,k)}.$$

**Proof.** For each $v^{s'}_{(a,k)}$ with $s \geq s_0$, Lemma 2.4 guarantees that there will be a stage at which Step (3) of the construction applies. The first such stage will be $s'$, since at that stage $v^{s'}_{(a,k)} = v^{s'}_{(a,k)}$ will enter $V_{(a,k),n}$ and $t^{s'}_{(a,k)}$ will be defined. But since $\varphi_{s'}$ is total, we will eventually reach the stage $s'' > s'$ at which Step (4) applies, leaving $v^{s''+1}_{(a,k)}$ undefined. Then at the next $s_m > s''$, we will define $v^{s_{m}+1}_{(a,k)} = s_m + 1$, which is not yet in $V_{(a,k),n}$. Thus, $V_{(a,k),n}$ must be infinite.

Now pick $v^{s'}_{(a,k)} \in V_{(a,k),n}$, with $s'$ and $s''$ as above. Since $V^{s_0}_{(a,k),n}$ is empty, we know that $s' > s_0$. If $s$ is any stage with $s' \leq s < s''$, then we see from the definition of $B^{s+1}$ that an element $y$ can only enter $B^{s+1}$ on behalf of some $\gamma$ such that $y \in S^{s+1}_\gamma$. But then $y \in U^{s+1}_\gamma$. Since we chose $s_0$ to let $B^{s_0} \upharpoonright m = B \upharpoonright m$, we must have $\gamma \geq \alpha$. But $t^{s'}_{(a,k)} \downarrow$, so $y \geq v^{s'}_{(a,k)} = v^{s'}_{(a,k)}$ by definition of $B^{s+1}$.

Hence $B^{s''} \upharpoonright v^{s'}_{(a,k)} = B^{s''} \upharpoonright v^{s'}_{(a,k)}$.

Having seen that no $y < v^{s'}_{(a,k)}$ can enter $B$ between stages $s'$ and $s''$, we prove that no such $y$ can enter $A_0$ at those stages either. First, we know that $A^{s'}_{\alpha} \upharpoonright k = A^{s'}_{\alpha} \upharpoonright k$ by choice of $s_0$. So suppose $k \leq y < v^{s'}_{(a,k)}$. Now since $v^{s'}_{(a,k)}$ entered $V_{(a,k),n}$ at stage $s'$, we know by line (4) that

$$y \in A^{s'}_{0} \cup y \in A^{s'}_{1} \cup y \in (U^{s'}_{\alpha} - C^{s'}) \cup y \in (C^{s'} - B^{s''}) \cap S^{s'}_{\alpha} \cap U^{s'}_{\alpha}.$$

If $y \in A^{s'}_{0}$, then $A^{s'}_{0}(y) = A^{s''}_{0}(y)$, and if $y \in A_1$, then $y \notin A_0$ at all. Therefore, we will assume that $y \notin A^{s'}_{0} \cup A_1$ and prove that $y \notin A^{s''}_{0}$.
If the final clause holds, then \( y \in (C^{s'\prime} - B^{s'}) \cap S^{s'}_\alpha \cap U^{s'}_\alpha \). Hence \( y \not\in B^{s''} \), by the first half of the lemma. If \( y \in A^{s''}_0 \), then \( y \not\in B \), since no element that has entered \( A_0 \) can later enter \( B \). But then

\[
(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1
\]

since \( y \) is on the left side and not on the right side. (Notice that \( y \in U_\alpha \) implies \( y \in T_\alpha \).) This contradicts line (2), which we know holds because \( F(\alpha) \) holds. Therefore \( y \not\in A^{s''}_0 \).

So suppose the third clause holds, i.e., \( y \in (U^{s'\prime}_\alpha - C^{s'\prime}) \). Then \( y \not\in B^{s''} \) since \( B^{s'} \subseteq C^{s'} \), and so \( y \not\in B^{s''} \). If \( y \in A^{s''}_0 \), then we must have \( y \in C^{s''-1} \); since we chose enumerations such that \( A_0 \subseteq C \setminus A_0 \). Pick \( s \) such that \( y \in C^{s'} - C^{s'\prime} \); then \( s' < s < s'' \) and \( y \not\in A_0 \). Now \( y \in U^{s'\prime}_\alpha \subseteq T^{s'}_\alpha \), and by definition of \( S^{s'}_\alpha \) we will have \( y \in S^{s'}_\alpha \). (Recall that \( s_0 \) was chosen so large that \( S^{\beta}_{\beta} = S_\beta \) for all \( \beta < \alpha \).) But now \( y \not\in A^{s''}_0 \), since otherwise

\[
(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1
\]

just as in the preceding paragraph. \( \blacksquare \)

Hence \( V_{(\alpha,k),n} = W_{f_{(\alpha,k),n}} \) is an infinite c.e. set which satisfies the tardiness requirement \( T_\varepsilon \). This completes the proof of Theorem 2.1. \( \blacksquare \)
3 Satisfaction of $R$

We now prove that the $R$-property defined in Section 2 is nontrivial. The theorem establishes several other properties of the sets $A_0$ and $A_1$ as well, in order to yield the corollaries.

**Theorem 3.1** There exists a c.e. set $A$ with Friedberg splitting $A = A_0 \sqcup A_1$ such that all of the following hold:

1. $A$ is promptly simple of high degree.
2. $A_1$ has prompt degree.
3. $R(A_0, A_1)$.

**Corollary 3.2** The formula in one free variable $A_0$:

$$(\exists A_1)[A_0 > T \emptyset \& R(A_0, A_1)]$$

is definable in $\mathcal{E}$ and non-vacuous, and implies that $A_0$ is a noncomputable incomplete set.

**Proof of Corollary.** The statement $A_0 > T \emptyset$ is equivalent to the statement that $A_0$ does not have a complement in $\mathcal{E}$, hence is $\mathcal{E}$-definable. The $A_0$ and $A_1$ constructed in Theorem 3.1 satisfy the matrix, since halves of a Friedberg splitting must be noncomputable. Finally, Theorem 2.1 shows that $A_0$ is tardy, hence incomplete.

**Corollary 3.3** There exists a Friedberg splitting $A = A_0 \sqcup A_1$ such that $A_0$ and $A_1$ are not automorphic in the lattice of c.e. sets.

**Proof of Corollary.** Take the splitting given by Theorem 3.1. If an automorphism $\Phi$ of $\mathcal{E}$ satisfied $\Phi(A_0) = A_1$, then $R(A_1, \Phi(A_1))$ would have to hold. By Theorem 2.1, then, $A_1$ would be tardy, contradicting the promptness of $A_1$.

**Proof of Theorem.** Let $C$ be any promptly simple set, with computable enumeration $C = \{C^s\}_{s \in \omega}$. Then $C$ is also of prompt degree, so let $v$ and $w$ be the prompt-simplicity and promptness functions for this enumeration of $C$, satisfying for every $i$:

$$W_i \text{ infinite } \implies (\exists^\infty s)(\exists x \in W_i,s - W_i,s-1)[x \in C^{v(x)}]$$

$$W_i \text{ infinite } \implies (\exists^\infty s)(\exists x \in W_i,s - W_i,s-1)[C^{w(x)} [x \neq C^s | x]]$$
We construct disjoint sets $A_0$ and $A_1$ and auxiliary sets $D_i$ and $T_{i,j}$, and set $A = A_0 \cup A_1$. The approximations to $A$, $A_0$, and $A_1$ at stage $s$ will be written $A^s$, $A^s_0$, and $A^s_1$, and will be defined so that $A^s = A^s_0 \cup A^s_1 \subseteq C^s$ for all $s$. The construction will satisfy the following requirements for all $i$ and $j$:

$$\mathcal{N}_{(i,j)} \quad (\text{matrix of R-property}) :$$

$$[W_i \subseteq C \& W_j \subseteq C \& C - W_j \text{ c.e. } \&$$

$$(W_i \cap (W_j - A_0)) \cup A_1 = (D_i \cap (W_j - A_0)) \cup A_1] \implies (\exists T) \overline{T} \subseteq T \& (A_0 \cap W_j \cap T) \cup A_1 =^* (W_i \cap W_j \cap T) \cup A_1]$$

$$\mathcal{M}_i \quad (\text{major subset requirement}) :$$

$$\overline{T} \subseteq W_i \implies \mathcal{A} \subseteq^* W_i$$

$$\mathcal{P}_i \quad (\text{prompt simplicity of } A_i) :$$

$$W_i \text{ infinite } \implies (\exists s) (\exists x \in W_{i,s} - W_{i,s-1}) [x \in A^s_i]$$

$$\mathcal{Q}_i \quad (\text{promptness of } A_i) :$$

$$W_i \text{ infinite } \implies (\exists s) (\exists x \in W_{i,s} - W_{i,s-1}) [A^s_i \neq A^s_1 \mid x]$$

$$\mathcal{F}_i \quad (\text{Friedberg requirement for } A_0) :$$

$$W_i \setminus A \text{ infinite } \implies W_i \cap A_0 \neq \emptyset$$

$$\mathcal{G}_i \quad (\text{Friedberg requirement for } A_1) :$$

$$W_i \setminus A \text{ infinite } \implies W_i \cap A_1 \neq \emptyset$$

In the requirement $\mathcal{N}_{(i,j)}$, of course, $W_i$ plays the role of $B$ and $W_j$ the role of $S$ in the matrix of the $R$-property. We will construct c.e. sets $T_{i,j}$ for each $i$ and $j$, and then refine them to form the $T$ demanded by each $\mathcal{N}_{(i,j)}$. Once again we order $\omega \times \omega$ in order type $\omega$ and write $\alpha = \langle i, j \rangle$, this time with:

$$B_\alpha = W_i$$

$$D_\alpha = D_i$$

$$S_\alpha = W_j$$

$$\hat{S}_\alpha = W_j^n$$

$$T_\alpha = T_{i,j}$$

$$\mathcal{N}_\alpha = \mathcal{N}_{(i,j)}$$

Thus $\mathcal{N}_\alpha$ says:

$$[B_\alpha \subseteq C \& S_\alpha \cup \hat{S}_\alpha = C \&$$

$$(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1] \implies (\exists T) \overline{T} \subseteq T \& (A_0 \cap S_\alpha \cap T) \cup A_1 =^* (B_\alpha \cap S_\alpha \cap T) \cup A_1].$$

$\mathcal{N}_\alpha$ is a negative requirement, trying to keep elements from entering $A_0$ until they can do so without harming the $R$-property (if ever). All the other requirements are positive ones, trying to put elements into $A_0$ or $A_1$. There
are no negative restraints on elements of $C$ entering $A_1$, except that they cannot already be in $A_0$.

Each element which we try to put into $A_0$ to satisfy some $F_e$ or $M_e$ must receive permission to enter $A_0$ from each $N_e$ with $\alpha \leq e$. The 
/restraint function $g(x,s)$ will give the greatest $\alpha \leq e$ which has not yet given this permission as of stage $s$. The 
/priority function $p(x,s)$ keeps track of which requirement $F_e$ or $M_e$, wanted $x$ to enter $A_0$. This can change from stage 
/to stage, for several reasons. If a higher-priority requirement decides at stage $s+1$ that it needs $x$ to enter $A_0$, then $p(x,s+1) < p(x,s)$. Alternatively, an 
$F_e$ could find itself satisfied by another $x' \in A_0^{s+1}$ and no longer need to put 
$x$ into $A_0$, although in this case we leave $p(x,s+1) = p(x,s)$ so as not to 
disrupt the flow of elements into $A_0$. Finally, a higher-priority requirement could make $x$ enter $A_0^{s+1}$, in which case we define $p(x,s+1) \uparrow$, removing $x$ from the flow of elements into $A_0$ since we need $A_0 \cap A_1 = \emptyset$.

We use the Recursion Theorem on our construction of $A_0$, $C$, and $D_0$ to 
define the following $\Pi^0_0$ statement $F(\alpha)$ for each $\alpha$:

$$(B_0 \cap (S_0 - A_0)) \cup A_1 = (D_0 \cap (S_0 - A_0)) \cup A_1 \cup B_0 \subseteq C \& S_0 \cup \hat{S}_0 = C.$$

Since $F(\alpha)$ is $\Pi^0_0$, there is a computable function $g : \omega \to \omega \times \omega$ such that $F(\alpha)$ holds if and only if the set $Z_0 = g^{-1}(\alpha)$ is infinite. We let 
$Z_0^c = g^{-1}(\alpha) \cap \{0, 1, \ldots, s - 1\}$. Monitoring $|Z_0^c|$ will help us determine 
for which $\alpha$ the hypothesis in the matrix of the R-property is satisfied. 
For those $\alpha$ for which the hypothesis fails, $|Z_0^c|$ is finite, and $N_0$ will only 
restrain finitely many elements from entering $A_0$, since we need not satisfy the 
conclusion of the R-property for such an $\alpha$.

At stage $s = 0$, we set $A_0^0 = A_0^0 = \emptyset$. Also, let all $p(x,0)$ and $q(x,0)$ 
diverge.

At stage $s + 1$, we first define $T_0^{s+1}$ for each $\alpha$:

$$T_0^{s+1} = T_0^s \cup \{x \in C_0^{s+1} : x < |Z_0^{s+1}|\}.$$ 

Next we determine which elements of $C^{s+1}$ to add to $A_0^s$ to create $A_0^{s+1}$. For 
this, we need movable markers for elements currently in $C - A$. Write 

$$C^{s+1} - A^s = \{d_0^{s+1}, d_1^{s+1}, \ldots, d_{m+1}^{s+1} \}$$

preserving the order of the markers from the preceding stage. (That is, if 
$d_i = d_i^{s+1}$ and $d_j = d_j^{s+1}$, then $i < j$ iff $i' < j'$; and if $d_i^{s+1} \in C^s$ and 
$d_j^{s+1} \notin C^s$, then $i < j$.)
For the sake of $\mathcal{M}_c$, we define

$$V_{c}^{s+1} = V_{c}^{s} \cup \{x \in W_{c,s+1} - C^{s+1} : (\forall y \leq x)(y \in W_{c,s+1} \cup C^{s+1})\}.$$ 

(For each $c$, the sets $V_{c}^{s}$ enumerate a c.e. set $V_{c}$. If $C \not\subseteq W_{c}$, then $V_{c}$ will be finite, but if $C \subseteq W_{c}$, then $V_{c} \subseteq V_{c} \subseteq W_{c}$.)

For each $c \leq s$, define the $c$-state of each $d_{k}^{s+1}$ at stage $s + 1$ to be:

$$\sigma(c, d_{k}^{s+1}, s + 1) = \{i < c : d_{k}^{s+1} \in V_{i}^{s+1}\}.$$ 

We order the different possible $c$-states by viewing them as binary strings.

Find the least $i \leq s$ such that there exist $c$ and $j$ with $c < i < j \leq s$ and $\sigma(c, d_{i}^{s+1}, s + 1) = \sigma(c, d_{j}^{s+1}, s + 1)$, and $d_{i}^{s+1} \not\in V_{c}^{s+1}$ and $d_{j}^{s+1} \in V_{c}^{s+1}$. For the least such $c$ and the least corresponding $j$, we say that $\mathcal{M}_c \text{ wants to put into } A_0$ all the elements $d_{i}^{s+1}, d_{i+1}^{s+1}, \ldots d_{j-1}^{s+1}$, so as to give the marker $d_{j}$ a higher $(c + 1)$-state at subsequent stages.

Now we consider the requirements $\mathcal{F}_c$. For each $c \leq s$ with $W_{c,s} \cap A_0^{s} = \emptyset$ and for each $x$ such that

$$x \in (W_{c,s} \cap C^{s+1}) - A_0^{s} - \{d_{0}^{s+1}, d_{1}^{s+1}, \ldots d_{i}^{s+1}\},$$

we say that $\mathcal{F}_c \text{ wants to put } x \text{ into } A_0$.

We set $p(x, s + 1) \uparrow$ for all $x \not\in C - A_0^{s}$. Otherwise $x = d_{k}^{s+1}$ for some $k$, and $p(x, s + 1)$ is the least $\leq k$ (if any) such that either $p(x, s) \downarrow = c$ or $\mathcal{M}_c$ or $\mathcal{F}_c \text{ wants to put } x \text{ into } A_0$. Thus, the function $p(x, s + 1)$ gives the priority currently assigned to putting $x$ into $A_0$. If there is no such $c$, let $p(x, s + 1) \uparrow$.

We now follow the following steps for each $x \leq s$:

1. If $p(x, s + 1) \uparrow$, then $q(x, s + 1) \uparrow$ also.

2. If $p(x, s + 1) \downarrow$ but $q(x, s) \uparrow$, we ask if every $\alpha \leq p(x, s + 1)$ satisfies either $x \in S_{\alpha}^{s+1} \cup \hat{S}_{\alpha}^{s+1}$ or $x \not\in T_{\alpha}^{s+1}$. If so, set $q(x, s + 1) = p(x, s + 1) + 1$. If not, then $q(x, s + 1) \uparrow$.

3. If $p(x, s + 1) \downarrow$ and $q(x, s) \downarrow = p(x, s + 1)$, then set $q(x, s + 1)$ to be the greatest $\alpha \leq p(x, s + 1)$ satisfying all four of the following conditions:

   (a) $S_{\alpha}^{s+1} \cap \hat{S}_{\alpha}^{s+1} = \emptyset$.

   (b) $x \not\in \hat{S}_{\alpha}^{s+1}$.

   (c) $x \in T_{\alpha}^{s+1}$.

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(d) \( \forall \beta < \alpha \), either \( \beta \) fails one of the three conditions (a)-(c), or 
\( \beta = (i', j') \) and \( \alpha = (i, j) \) with \( i \neq i' \).

Also, enumerate \( x \) in \( D^{s+1}_{q(x, s + 1)} \). (For future reference, notice that if \( \alpha \) satisfies (a)-(c), then some \( \beta \leq \alpha \) with the same first coordinate as \( \alpha \) must satisfy (a)-(d).)

If there is no such \( \alpha \), set \( q(x, s + 1) = -1 \).

4. If \( p(x, s + 1) \downarrow \) and \( q(x, s) \downarrow \) with \( 0 \leq q(x, s) \leq p(x, s + 1) \), we ask whether \( x \in B^{s+1}_q(x, s) \). If so, or if \( q(x, s) \) no longer satisfies the conditions (a)-(d), set \( q(x, s + 1) \) to be the greatest \( \alpha < q(x, s) \) satisfying the conditions (a)-(d) above, and let \( x \in D^{s+1}_q(x, s + 1) \). (If there is no such \( \alpha \), let \( q(x, s + 1) = -1 \).) Otherwise, let \( q(x, s + 1) = q(x, s) \).

5. If \( p(x, s + 1) \downarrow \) and \( q(x, s) \downarrow = -1 \), enumerate \( x \in A^{s+1}_0 \), and let \( q(x, s + 1) \uparrow \).

This completes our enumeration of \( A^{s+1}_0 \). Next we determine which elements to add to \( A^{s+1}_1 \):

1. Find the least \( \epsilon \leq s \) (if any) such that \( Q_\epsilon \) is not yet satisfied and there is an element \( x \in W_{\epsilon, t} - W_{\epsilon, t-1} \) for some \( t \leq s \) such that \( w(t) > s \), and there exists \( y < x \) such that \( y \in C^{s+1} - A^{s+1}_0 \) and \( y \not\in A^{s+1}_1 \cup \{a_0^{s+1}, \ldots, a_\epsilon^{s+1}\} \) and no \( F_i \) with \( i < \epsilon \) wants to put \( y \) into \( A_0 \). Put the greatest such \( y \) into \( A^{s+1}_1 \). This forces \( A^{s+1}_1 \mid x \neq A^{s+1}_1 \mid x \), satisfying \( Q_\epsilon \) permanently. (If there is no such \( \epsilon \), do nothing.)

2. Find the least \( \epsilon \leq s \) (if any) such that \( P_\epsilon \) is not yet satisfied and there is an element \( x \in C^{s+1} \cap (W_{\epsilon, t} - W_{\epsilon, t-1}) \) for some \( t \leq s \) with \( v(t) > s \), such that \( x \not\in \{d_0^{s+1}, \ldots, d_\epsilon^{s+1}\} \) and no \( F_i \) with \( i < \epsilon \) wants to put \( x \) into \( A_0 \). If no such \( x \) lies in \( A^s \cup A^{s+1}_0 \), then put the least such \( x \) into \( A^{s+1}_1 \). This forces \( x \in A^{s+1}_1 \), satisfying \( P_\epsilon \) permanently.

3. Find the least \( \epsilon \leq s \) (if any) such that \( G_\epsilon \) is not yet satisfied and there is an element \( x \in (W_{\epsilon, t+1} \cap C^{s+1}) - A^{s+1}_0 \) with \( x \not\in \{d_0^{s+1}, \ldots, d_\epsilon^{s+1}\} \), such that no \( F_i \) with \( i < \epsilon \) wants to put \( x \) into \( A_0 \). Put this \( x \) into \( A^{s+1}_1 \). This satisfies \( G_\epsilon \) forever.

Let \( A^{s+1} = A^{s+1}_0 \cup A^{s+1}_1 \). This completes the construction.

\textbf{Lemma 3.4} \( C - A \) \textit{is infinite}.
Proof. We prove by induction on $\epsilon$ that $d_\epsilon = \lim_s d_\epsilon^s$ exists. Assume that this holds for all markers $d_i$ with $i < \epsilon$, and let $s_0 \geq \epsilon$ be a stage such that $d_i^{s_0} = d_i$ for all $i < \epsilon$. Now each $F_j$, $G_j$, $P_j$, and $Q_j$ with $j > \epsilon$ cannot put any of the elements $d_i^{s_0}, \ldots d_s^s$ into $A_1$ at stage $s + 1$, so none of these requirements ever moves the marker $d_\epsilon^s$. Also, each $G_i$, $P_i$, and $Q_i$ with $i \leq \epsilon$ puts at most one element into $A_0$, hence moves the markers at most once. Let $s_1 \geq s_0$ be a stage so large that no $G_i$, $P_i$, or $Q_i$ with $i \leq \epsilon$ moves any markers at any stage $s \geq s_1$.

By the construction, $d_\epsilon^s$ can only be moved at stage $s \geq s_1$ by a requirement $M_i$ or $F_i$ with $i \leq \epsilon$. Furthermore, when $F_i$ ($i \leq \epsilon$) moves a marker, it puts an element into $A_0$, so it is satisfied at that point. Before then it may have tried to put finitely many other elements into $A_0$ as well, and any of them may go into $A_0$ or $A_1$ at a later stage, moving markers in the process. However, since there are only finitely many such elements, $d_\epsilon$ is moved only finitely many times on behalf of $F_i$.

Now $M_0$ moves $d_\epsilon$ at most $2^{s+1}$ times after stage $s_1$: once to put $d_0$ into $V_0$, possibly twice to put $d_1$ into $V_0$, and so on. Once $M_0$ has finished moving $d_\epsilon$, $M_1$ moves it at most $2^s$ more times, to put markers into $V_1$. Similarly, once each $M_i$ has moved $d_\epsilon$ for the last time, $M_i+1$ may move it at most $2^{s-i}$ more times. Hence we eventually reach a stage $s_2$ after which $d_\epsilon$ never moves again. Possibly $d_\epsilon^s \uparrow$, but since $C$ is infinite and every $d_i$ with $i < \epsilon$ has already converged to its limit, we know that $d_\epsilon^s$ will be defined at some stage $t > s_2$. Since it never moves again, this yields $d_\epsilon = \lim_s d_\epsilon^s$. ■

Lemma 3.5 For each $\epsilon$, the requirements $N_\epsilon$, $P_\epsilon$, $Q_\epsilon$, $F_\epsilon$, and $G_\epsilon$ are all satisfied.

Proof. We proceed by induction on $\epsilon$. Assume the lemma holds for all $i < \epsilon$. We write $\alpha$ for the pair coded by $\epsilon$, and prove first that $N_\alpha$ is satisfied. Suppose $(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1$ and $B_\alpha \subseteq C$ and $S_\alpha \subseteq \hat{S}_\alpha = C$. Then $F(\alpha)$ holds and $Z_\alpha$ is infinite. The construction of $T_\alpha$ then guarantees that $C \subseteq T_\alpha$. Let $G_\alpha$ be the intersection of all those $V_\alpha$ with $i < \alpha$ such that $V_\alpha$ is infinite, and let $\tilde{T}_\alpha = T_\alpha \cap G_\alpha$. Thus $C \subseteq \tilde{T}_\alpha$, since $C \subseteq V_\alpha$ whenever $V_\alpha$ is infinite.

Sublemma 3.6 For each $\alpha$ and each $n < \alpha$, there are only finitely many $x \in \tilde{T}_\alpha$ such that $M_n$ ever wants to put $x$ into $A_0$.

Proof. First, if $V_n$ is finite, then $M_n$ will only want to put finitely many elements into $A_0$. So we may assume that $V_n$ is infinite, and hence that $\tilde{T}_\alpha \subseteq V_n$. \[19\]
If $\mathcal{M}_n$ wants to put $x$ into $A_0$ at stage $s$, then $x \in C^s - A^s$, so $x = d_k^s$ for some $k$. Moreover, there must be an $i$ with $n < i \leq k$ and a $j > k$ such that $\sigma(n, d^s_i, s) = \sigma(n, d^s_j, s)$ and $d^s_k \not\in V^s_n$ and $d^s_j \in V^s_n$. Furthermore, $d_i$ is the leftmost marker which any $\mathcal{M}$-requirement wants to put into $A_0$ at stage $s$, and $n$ and $j$ satisfy the minimality requirements of the construction.

Now if $d^s_k \not\in V^s_n$, then $d^s_k \not\in V_m$ since $C \setminus V_n = \emptyset$, and hence $d^s_k \not\in \hat{T}_a$. Therefore we may assume $d^s_k \in V^a_m$. (This guarantees $k \neq i$). Then minimality of $n$ forces $\sigma(n, d^s_i, s) \geq \sigma(n, d^s_k, s)$, and minimality of $j$ forces $\sigma(n, d^s_i, s) > \sigma(n, d^s_j, s)$ (since $d^s_k \not\in V^a_m$). Hence there is some $m < n$ such that $\sigma(m, d^s_i, s) = \sigma(m, d^s_k, s)$ and $d^s_i \in V^a_m$ and $d^s_k \not\in V^a_m$. This forces $d^s_i \in V_m$ and $d^s_k \not\in V_m$ (since $d^s_k \not\in C^s - V^a_m$). If $V_m$ is infinite, then $d^s_k \not\in \hat{T}_a$. But if $V_m$ is finite, then $d^s_k$ lies in the finite set

$$V = \bigcup \{V_m : m < n \land V_m \text{ finite}\}.$$ 

Hence we need only find a stage $t$ so large that for every $d \in V$, either $d \in A^l_0$ or $\mathcal{M}_n$ wants to put $d$ into $A_0$ at stage $t$ or $\mathcal{M}_n$ never wants to put $d$ into $A_0$. Then $\mathcal{M}_n$ will never want to put into $A_0$ any $x > \max(C^l)$ with $x \in \hat{T}_a$.

\[\square\]

We will show that the conclusion of $\mathcal{N}_\alpha$ holds for $\hat{T}_a$:

$$(A_0 \cap S_0 \cap \hat{T}_a) \cup A^1 = (B_0 \cap S_0 \cap \hat{T}_a) \cup A^1.$$ 

Once we have established this for all $\alpha$, clearly $R(A_0, A^1)$ itself must hold, since for each $\alpha$ we can choose another $\hat{T}_a$ which excludes the (finite) difference set of the two sides and still contains $C$.

Suppose first that $x \in A_0 \cap S_0 \cap \hat{T}_a$ and $x \not\in A^1$, and assume that $x$ is sufficiently large that:

- $x > |Z_\beta|$ for every $\beta < \alpha$ such that $Z_\beta$ is finite, and
- No $\mathcal{F}_i$ with $i < \alpha$ ever tries to put $x$ into $A_0$, and
- No $\mathcal{M}_i$ with $i < \alpha$ ever tries to put $x$ into $A_0$.

The last condition is possible by Sublemma 3.6. Notice also that the first condition forces $x \not\in T_\beta$ for all $\beta < \alpha$ with $|Z_\beta|$ finite.

Then for all $s$, either $p(x, s) \geq \alpha$ or $p(x, s) \uparrow$. But since $x \in A_0$, we know that some $p(x, s) \downarrow$. For the least such $s$ we have $x \in C^s$, and hence $x \in T^s_\alpha$, since $C \cap T^s_\alpha \subseteq T_\alpha \setminus C$.

\[20\]
Now $\alpha$ satisfies conditions (a)-(c) in the construction at stage $s$, since $F(\alpha)$ holds and $x \in S_\alpha$. So there must exist $\beta = \langle i, j' \rangle \leq \alpha = \langle i, j \rangle$ which satisfies (a)-(d) at stage $s$.

We claim that this $\beta$ satisfies conditions (a)-(d) at every stage after $s$ as well. Since $x \in T^s_\beta$, we know that $Z_\beta$ is infinite and $F(\beta)$ holds, by choice of $x$. Hence (a) and (c) hold at all subsequent stages. Let $t$ be the first stage at which $q(x, t)$ converged. Then $x \in C^t$, and $x \in T^t_\beta$ since $C \setminus T^t_\beta = \emptyset$. By the definition of $q$, we must have had $x \in S_\beta^t \cup \hat{S}_\beta^t$. But $x \notin \hat{S}_\beta^t$ since (b) holds at stage $s$, and because $s > t$, this forces $x \in S_\beta^t$, so (b) always holds of $\beta$.

To show that (d) always holds of $\beta$, we choose an arbitrary $\gamma < \beta$ with the same first coordinate as $\beta$. Since $\beta$ satisfies (d) at stage $s$, $\gamma$ must fail one of (a)-(c) at stage $s$. If $\gamma$ fails (a) or (b) at stage $s$, then clearly it fails that same condition at every subsequent stage. Moreover, if $\gamma$ fails (c) at stage $s$, then $x \notin T^s_\gamma$, and since $x \in C^s$, this forces $x \notin T_\gamma$. Thus $\beta$ will always satisfy condition (d).

But since $x \in A_\alpha$, there must also be a stage $s'$ with $q(x, s') = -1$. Since (a)-(d) continue to hold of $\beta$, the only way for $q(x, s') < \beta$ to occur is for $x$ to enter $B_\beta$. (Recall that for all $s$, either $p(x, s) \geq \alpha$ or $p(x, s) = \alpha$.) But $B_\beta = W_1 = A_\alpha$ since $\beta = \langle i, j' \rangle$ and $\alpha = \langle i, j \rangle$, so this forces $x \in B_\alpha$. Hence

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 \subseteq (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$  

Now suppose that $x \in B_\alpha \cap S_\alpha \cap \hat{T}_\alpha$ and $x \notin A_1$, and assume $x$ is greater than $\max(d_0, \ldots, d_n)$, and also greater than the greatest finite $|Z_\beta|$ with $\beta < \alpha$. (Thus $x \notin T^s_\beta$ for all such $\beta$.) Now $x \in C$ since $S_\alpha \subseteq C$, so at some stage $s_0$, $x$ will enter $C$ and be given a marker: say $x = d^\alpha_{s_0}$. So $x \in C^s_{s_0}$, and since $x \in C^s_{\alpha}$, this forces $x \in T^s_{s_0}$.

If $x \notin A_0$, then we must have $x \notin D_\alpha$, since $(B_\alpha \cap (S_\alpha \setminus A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha \setminus A_0)) \cup A_1$ and $x \notin A_1$. (Notice that then $x$, being in $C^s_{\alpha}$, eventually receives some permanent marker $d^\alpha_{s' \geq k}$, with $k' > \alpha$ by choice of $x$.) For $x$ to have entered $D_\alpha$, there must have been a stage $s_1 \geq s_0$ with $q(x, s_1) = \gamma = \langle i, j' \rangle$, where $\alpha = \langle i, j \rangle$. (Also, then $p(x, s_1) \uparrow$, and since $x \notin A_1$, $p(x, s) \downarrow$ for all $s \geq s_1$.) But $\alpha$ satisfies conditions (a)-(c) at all stages $s \geq s_0$, so by condition (d) on $\gamma$, we must have $\gamma \leq \alpha$. The assumption $x \notin A_0 \cup A_1$ then means that there is some $s_2 > s_1$ such that $q(x, s_2) \downarrow = q(x, s_2)$ for all $s \geq s_2$. Let $\beta = q(x, s_2) \leq \gamma$. Then $x \in D_{s_2} - B_{\beta}$, and furthermore $\beta$ satisfies the conditions (a)-(d) at all stages $s \geq s_2$.

Now $x \in T^s_{s_2}$, to satisfy condition (c), so $x < |Z_\beta|$ and $\beta \leq \gamma \leq \alpha$. If $\beta = \alpha$, then $Z_{s_2}$ is infinite since $F(\alpha)$ holds, and if $\beta < \alpha$, then $Z_{s_2}$ must
be infinite, by our choice of $x$. Therefore $F(\beta)$ holds, and in particular $S_\beta \cup \hat{S}_\beta = C$. Now $x \notin \hat{S}_\beta$ by condition (b), so $x \in S_\beta$. However, with $x \in D_\beta - B_\beta$, this contradicts $F(\beta)$. Hence $x \in A_0$, and

$$(A_0 \cap S_a \cap \hat{T}_a) \cup A_1 \subseteq^* (B_a \cap S_a \cap \hat{T}_a) \cup A_1.$$ 

This completes our proof that $\mathcal{N}_\alpha$ is satisfied.

Now we continue with the other requirements. Let $s_0$ be a stage such that no $\mathcal{P}_i$, $\mathcal{Q}_i$, $\mathcal{F}_i$, or $\mathcal{G}_i$ with $i < \epsilon$ tries to put any element into $A_0$ or $A_1$ at any stage after $s_0$. ($\mathcal{F}_i$ is different from the other requirements in that it may try to put more than one element into $A_0$. It only stops trying when one of those elements succeeds in entering $A_0$. We choose $s_0$ so that every element which $\mathcal{F}_i$ wants to put into $A_0$ either is in $A^{s_0}$ or never enters $A$.) Assume also that $s_0$ is sufficiently large that $d^{s_0}_i = d_i$ for every $i < \epsilon$.

Now if $W_\epsilon \setminus A$ is infinite, then there must be an $x$ in some $W_{\epsilon,s} - A^s$ with $s > s_0$ and \{d_0, ..., d_\epsilon\}. No requirement of higher priority will need to put this $x$ anywhere, except possibly some $\mathcal{M}_i$, and according to our construction, $\mathcal{G}_\epsilon$ does not respect the priority of the requirements $\mathcal{M}_i$, so $x \in A^{s_0+1}_1$, and $\mathcal{G}_\epsilon$ is satisfied.

Similarly, if $W_\epsilon$ is infinite, then there must be an $x$ and an $s > s_0$ such that $x \in W_{\epsilon,s} - W_{\epsilon,s-1} \cap x \in C^{w(s)}$, by prompt simplicity of $C$. If this $x$ is not already in $A^{w(s)-1}_1$, then the construction puts it into $A^{w(s)}_1$, so $\mathcal{P}_\epsilon$ holds. Also, there must be an $x$ and an $s > s_0$ with $x \in W_{\epsilon,s} - W_{\epsilon,s-1}$ such that $C^s(x) \neq C^{w(s)}\setminus x$, by promptness of $C$. Thus there is a $y < x$ which entered $C$ at some stage $t$ with $s < t \leq w(s)$. We must have $y \notin A^{t-1}$ since $A^{t-1} \subseteq C^{t-1}$. But now $y \notin \{d_0, ..., d_\epsilon\}$, since these markers had reached their limits by stage $s_0$ and $y$ only entered $C$ at stage $t$. Hence the construction will put this $y$ into $A^t_1$, and $A^{w(s)}_1 \setminus x \neq A^t_1 \setminus y$, satisfying $\mathcal{Q}_\epsilon$.

Continuing with the induction, we need a sublemma to handle $\mathcal{F}_\epsilon$.

**Sublemma 3.7** For this $\epsilon$ and for all sufficiently large $x$, if $\mathcal{F}_\epsilon$ wants to put $x$ into $A_0$ at some stage, then $x \in A_0$.

**Proof.** Choose $x$ so large that it satisfies all of the following:

1. $x > \max\{|Z_\beta| : \beta \leq \epsilon \land Z_\beta \text{ is finite}\}$.
2. No $\mathcal{F}_i, \mathcal{G}_i, \mathcal{P}_i, \text{ or } \mathcal{Q}_i$ with $i < \epsilon$ ever wants to put $x$ into $A_0$ or $A_1$.
3. $x \notin \{d_0, ..., d_\epsilon\}$.

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Suppose $F_e$ wants $x$ to enter $A_0$ at stage $s_0$. Then $x = d^{s_0}_k$ for some $k$ and $p(x, s_0) \leq \epsilon$. Now no $G_j$, $P_j$, or $Q_j$ with $j \geq 1$ ever manages to put $x$ into $A_1$, since $F_e$ takes priority over these. (Since $x \neq d_e$, the only way to have $k \leq 1$ is for $x$ eventually to enter $A_0$. Hence we may assume $k > 1$.) Also, for every $\beta < \epsilon$, either $x \notin T_\beta$ (if $|Z_\beta| < x$) or $F(\beta)$ holds (if $Z_\beta$ is infinite). Hence there is an $s_1 \geq s_0$ such that $q(x, s_1) \downarrow$ and $q(x, s_1 + 1) \downarrow \leq \epsilon$.

Now suppose $q(x, s) = \beta$ for some $s \geq s_1$ (so $\beta \leq \epsilon$). If $F(\beta)$ failed, then $Z_\beta$ would have to be finite, so $x \notin T_\beta$ (since $|Z_\beta| < x$) and $q(x, s)$ would never equal $\beta$. Therefore, $F(\beta)$ must hold. Suppose $x \notin A_0$. If $x \notin S_\beta$, then $x \in \bar{S}_\beta$ by $F(\beta)$ and so $q(x, s_\beta) < \beta$ for some $s_\beta \geq s_1$. Otherwise $x \in D_\beta \cap (S_\beta - A_0) \subseteq B_\beta$ by $F(\beta)$, so $x \in B^{s_\beta}$ for some $s_\beta \geq s_1$, and hence $q(x, s_\beta) < \beta$. Thus, by induction on $\beta < \epsilon$, eventually we must have $q(x, s) = -1$, and so $x \in A_0^{s+1}$, proving the sublemma.

Now if $W_e \setminus A$ is infinite, then $F_e$ has infinitely many elements at its disposal to try to put into $A_0$. Hence once we find a sufficiently large $x \in W_e \setminus A$, we know by the sublemma that this $x$ will eventually enter $A_0$, thus satisfying $F_e$. This completes the induction of Lemma 3.5.

**Lemma 3.8** The requirements $M_e$ are all satisfied by our construction.

**Proof.** Suppose that $C \subseteq W_e$. To prove that $M_e$ holds, we must show $C \subseteq W_e$. By induction we assume that $M_i$ holds for all $i < e$. Let

$$\sigma = \{i < e : C \subseteq W_i\}.$$

Now if $i \in \sigma$, then also $C \subseteq V_i$, so by inductive hypothesis $C \subseteq V_i$, whereas if $i \notin \sigma$ (and $i < e$), then $V_i$ is finite. Hence for all but finitely many $k$ we have $\sigma(e, d_k) = \sigma$.

Now let $V_\epsilon = V_{\eta} \cap (\bigcap \{V_i : i \in \sigma\})$. Then $C \subseteq V_\epsilon$. But $C$, being promptly simple, is noncomputable, so $V_\epsilon \setminus C$ must be infinite. Choose $y$ so large that no element $\geq y$ can be held out of $A_0$ forever by any requirement $N_\alpha$ with $\alpha < e$, and let $s_0$ be a stage such that $C^{s_0}[y] = C[y]$.

Suppose for a contradiction that $V_\epsilon \cap (C - A)$ is infinite. Then there exists $p$ such that $d^e_p \notin V_e$ with $p$ so large that $d^e_p \notin C^{s_0}$ and with $\sigma(e, d^e_p) = \sigma$. (Hence $d^e_p > y$.) Let $s_1$ be a stage with $d^e_\beta = d^e_p$ and $\sigma(e, d^e_p, s_1) = \sigma$. Now since $V_\epsilon \setminus C$ is infinite, there will be a stage $s > s_1$ at which some element $x \in V^{s-1}_\epsilon$ enters $C$, and is assigned the marker $d^e_q$ (with $q > p$ since $d^{s_0}_p = d^e_p$). Moreover, we may assume that $q$ is sufficiently large that not only is $d^e_q$ in $V_\epsilon$, but that $\sigma(e, d^e_q, s) = \sigma$, since every $V_i$ with $i < e$ and $i \notin \sigma$ is finite. Since $d^e_q \in V_\epsilon \subseteq V_e$ and $d^e_p \notin V_e$, $M_e$ will want to put $d^e_p$ into $A_0$.
at stage $s$, and since $d_p > y$, no negative requirement will keep $d_p$ out of $A_0$. Possibly $d_p$ will be diverted into $A_1$ by some requirement $G_j$, $P_j$, or $Q_j$, since these do not respect the priority of $M_\epsilon$. If so, then $d_p$ will enter $A_1$; if not, then $d_p$ will enter $A_0$. Either way, $d_p$ enters $A$, contradicting our assumption that the marker $d_p$ had reached its limit at stage $s_0$.

Hence $V_\epsilon \cap (C - A)$ is finite, and $\overline{A} \subseteq (C - A) \cup \overline{C} \subseteq * V_\epsilon \subseteq W_\epsilon$. Thus $M_\epsilon$ is satisfied, and the lemma is proven. 

Knowing that the requirements are all satisfied, we can easily complete the proof of the theorem. The construction ensured that $A_0 \cup A_1 = \emptyset$, and

the conjunction of all the $F_i$ and $G_i$ implies that $A_0 \cup A_1$ is a Friedberg splitting of $A$. (See pp. 181-182 of [16].) The requirements $P_i$ together make $A$ a promptly simple set, by definition, and the $Q_i$ together allow $A_1$ to satisfy the Promptly Simple Degree Theorem (Thm. XIII.1.6 of [16]), so that $A_1$ is of prompt degree. To prove that $R(A_0, A_1)$ holds, we note that the requirements $M_\epsilon$, along with Lemma 3.4, show that $A = A_0 \cup A_1$ is a major subset of $C$. Moreover, given a $B = W_i$ and a pair $(S_{j'}, \bar{S}_{j''})$ with $S_{j'} \cup \bar{S}_{j''} = C$, we have the $D_i$ and $T_\alpha$ (with $\alpha = \langle i, \langle j', j'' \rangle \rangle$) constructed above. If

$$(B_i \cap (S_{j'} - A_0)) \cup A_1 = (D_i \cap (S_{j'} - A_0)) \cup A_1,$$

then $F(\alpha)$ holds. Since $N_\alpha$ is satisfied, we know that there exists a $T$ with $\overline{C} \subseteq T$ such that

$$(A_0 \cap S_{j'} \cap T) \cup A_1 = * (B_i \cap S_{j'} \cap T) \cup A_1.$$

So we can pick a sufficiently large $n_\alpha$, and let

$$T' = \{x \in T : x \geq n_\alpha\} \cup \{x \in \overline{C} : x < n_\alpha\}.$$

Then $\overline{C} \subseteq T'$ and also $(A_0 \cap S_{j'} \cap T') \cup A_1 = (B_i \cap S_{j'} \cap T') \cup A_1$, since $S_{j'} \cap \overline{C} = \emptyset$. Thus $R(A_0, A_1)$ holds. Finally, since $A$ is a major subset of the set $C$, $A$ must be of high degree (see [10], page 214). 

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