Definable Incompleteness and Friedberg Splittings

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Abstract

We define a property $R(A_0, A_1)$ in the partial order $\mathcal{E}$ of computably enumerable sets under inclusion, and prove that R implies that A_0 is noncomputable and incomplete. Moreover, the property is nonvacuous, and the A_0 and A_1 which we build satisfying R form a Friedberg splitting of their union A, with A_1 prompt and A promptly simple. We conclude that A_0 and A_1 lie in distinct orbits under automorphisms of $\mathcal{E}$, yielding a strong answer to a question previously explored by Downey, Stob, and Soare about whether halves of Friedberg splittings must lie in the same orbit.

1 Introduction

The computably enumerable sets form an upper semi-lattice under Turing reducibility. Under set inclusion, they form a lattice $\mathcal{E}$, as first noted by Myhill in [14], and the properties of a c.e. set as an element of $\mathcal{E}$ often help determine its properties under Turing reducibility. Even before Myhill, Post had suggested that there should be a nonvacuous property of c.e. sets, definable without reference to the Turing degrees, which would imply that the Turing degree of such a set must lie strictly between the computable degree $\mathbf{0}$ and the complete c.e. degree $\mathbf{0}'$.

Post's own attempts to find such a property failed. The properties he defined turned out to be extremely useful in computability theory, but each of them – simplicity, hypersimplicity, and hyperhypersimplicity – actually does hold of some complete set. The existence of a Turing degree between

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0 and **0′** was first proven by completely different means, namely the finite injury constructions of Friedberg and Muchnik ([6], [13]).

The term "Post's Program" eventually came to denote the search for an $\mathcal{E}$-definable property implying incompleteness. Of the properties proposed by Post, all except hypersimplicity turned out to be definable in $\mathcal{E}$, and other $\mathcal{E}$-definable properties, such as maximality, were developed and studied in their own right. Nevertheless, Post's Program remained unfinished until 1991, when Harrington and Soare ([7]) found a property $Q(A)$ definable in $\mathcal{E}$ such that every A satisfying Q must be both noncomputable and Turing-incomplete. We give their definition of $Q(A)$:

$$Q(A): \ (\exists C)_{A \subset_{\mathrm{m}} C} (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C} \left(\begin{array}{c} B \cap (S - A) = D \cap (S - A) \implies \\ (\exists T)[\overline{C} \subset T \ \& \ A \cap (S \cap T) = B \cap (S \cap T)] \end{array} \right).$$

Here $S \sqsubset C$ abbreviates $(\exists \hat{S})[S \cup \hat{S} = C \ \& \ S \cap \hat{S} = \emptyset]$. (All variables represent elements of $\mathcal{E}$, namely c.e. sets.) $A \sqcup B$ denotes the union of two disjoint sets A and B. Also, $A \subset_{\mathrm{m}} C$ abbreviates "A is a major subset of C," meaning that $A \subset C$ with $C - A$ infinite such that for every W, if $\overline{C} \subset W$, then $\overline{A} - W$ is finite. Since the property of being finite is $\mathcal{E}$-definable, the statement $A \subset_{\mathrm{m}} C$ is $\mathcal{E}$-definable as well.

In this paper we generalize the property $Q(A)$ to an $\mathcal{E}$-definable property $R(A_0, A_1)$ of two c.e. sets. The statement of R is as follows:

$$\begin{aligned} R(A_0, A_1): \ & A_0 \cap A_1 = \emptyset \ \& \\ & (\exists C)(\forall B \subseteq C)(\exists D \subseteq C)(\forall S \sqsubset C)(\exists T)\Big[A_0 \cup A_1 \subset_{\mathrm{m}} C \ \& \\ & \big[(B \cap (S - A_0)) \cup A_1 = (D \cap (S - A_0)) \cup A_1 \implies \\ & [\overline{C} \subset T \ \& \ (A_0 \cap S \cap T) \cup A_1 = (B \cap S \cap T) \cup A_1]\big]\Big]. \end{aligned}$$

This property can be read to say that A_0 satisfies the Q-property on $\overline{A_1}$. Indeed, the statement $R(A_0, \emptyset)$ is equivalent to $Q(A_0)$. In Section 2 we prove that just as with the Q-property, $R(A_0, A_1)$ implies that A_0 is not of prompt degree, and hence not Turing complete in Σ^0_1. (A set which is not of prompt degree is said to be *tardy*, and since A_0 satisfies an $\mathcal{E}$-definable property implying tardiness, we say that A_0 is "definably tardy." Since all tardy sets are incomplete, we also say that A_0 is "definably incomplete.")

Alternatively, we can interpret $R(A_0, A_1)$ in the lattice $\mathcal{E}/\mathcal{A}$, where $\mathcal{A}$ is the principal ideal in $\mathcal{E}$ generated by A_1. (See [15], p. 225.) In this lattice, $C \subseteq_{\mathcal{A}} D$ is defined to mean $C \subseteq D \cup A_1$, and $C \approx_{\mathcal{A}} D$ if $C \subseteq_{\mathcal{A}} D$

and $D \subseteq_{\mathcal{A}} C$. Essentially, $R(A_0, A_1)$ says that $Q(A_0)$ holds in $\mathcal{E}/\mathcal{A}$, with containment and equality replaced by $\subseteq_{\mathcal{A}}$ and $\approx_{\mathcal{A}}$. The only differences are that we cannot state the properties $A_0 \cap A_1 = \emptyset$ or $A_1 \subseteq C$ in $\mathcal{E}/\mathcal{A}$, and that we have left the quantifier $(\forall S \sqsubset C)$ in $R(A_0, A_1)$ just as in the original Q-property, rather than restating it to hold on $\overline{A_1}$. Choosing not to restate it makes the R-property slightly stronger, but the stronger version can still be satisfied.

In Section 3 we construct c.e. sets A_0 and A_1 satisfying R, to show that the R-property is non-vacuous. A_0 and A_1 will also be noncomputable. Thus, the following $\mathcal{E}$-definable formula is non-vacuous:

$$(\exists A_1)[A_0 >_T \emptyset \ \& \ R(A_0, A_1)]$$

This formula guarantees that A_0 is noncomputable and incomplete, just as the property $Q(A)$ does for A. (Recall that computability is equivalent to the property of having a complement in $\mathcal{E}$.)

We then consider Friedberg splittings. Two disjoint c.e. sets B_0 and B_1 form a *Friedberg splitting* of $B = B_0 \sqcup B_1$ if for every c.e. W:

$$W - B \text{ is not c.e.} \implies \text{neither } W - B_0 \text{ nor } W - B_1 \text{ is c.e.}$$

The sets B_0 and B_1 are each said to be *half* of this Friedberg splitting. The sets A_0 and A_1 which we construct will have the additional property of forming a Friedberg splitting of their union.

We use the R-property to show that A_0 and A_1 cannot lie in the same orbit under automorphisms of $\mathcal{E}$. (In the argot of this topic, we say that A_0 and A_1 are not *automorphic*. Two sets are automorphic if they lie in the same orbit.) This will follow because the A_1 we construct will be of prompt degree, hence automorphic to a complete set, by another result of Harrington and Soare in [7].

The orbits of halves of Friedberg splittings have been a subject of interest for some time, at least since the discovery of the hemimaximal sets. A set is hemimaximal if it is half of a nontrivial splitting of a maximal set. This is $\mathcal{E}$-definable, and Downey and Stob proved that the hemimaximal sets form an orbit (see [3]).

Since the maximal sets themselves form an orbit, and since few orbits are known in $\mathcal{E}$, this led to the conjecture that if $\mathcal{O}$ is any orbit in $\mathcal{E}$, then the collection of "hemi-$\mathcal{O}$" sets, i.e. halves of nontrivial splittings of sets in $\mathcal{O}$, might also be an orbit. Alternatively, it was conjectured that halves of Friedberg splittings of sets in $\mathcal{O}$ might form an orbit. (For the orbit of maximal sets, these classes coincide, since any nontrivial splitting of a maximal set is automatically a Friedberg splitting.)

Downey and Stob refuted both conjectures in [5], by producing two Friedberg splittings $B_0 \sqcup B_1 = C_0 \sqcup C_1$ of the same set B, which were definably different in $\mathcal{E}$. Hence B_0 and C_0 satisfy different 1-types in the language of inclusion and cannot be automorphic.

The present result goes a step further. Since A_0 is definably tardy, every set in its orbit must also be tardy, and hence A_1 must lie in a different orbit. This is thus the first example of a single Friedberg splitting with the two halves known to lie in different orbits in $\mathcal{E}$. It is also the first application of Harrington and Soare's Q-property to derive results about Friedberg splittings.

Our notation mostly follows that of [16]. The finite sets form an ideal $\mathcal{F} \subset \mathcal{E}$, and we write $\mathcal{E}^*$ for the lattice $\mathcal{E}/\mathcal{F}$. (Computability is definable in $\mathcal{E}$ as the property of possessing a complement, and then finiteness is definable, since a set is finite if and only if all its subsets are computable.) We write $A \subseteq^* B$ if $B - A$ is finite, and $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$.

We use the standard enumeration $\{W_e\}_{e \in \omega}$ of the computably enumerable sets, with finite approximations $\{W_{e,s}\}_{s \in \omega}$ to each. For the c.e. sets which we construct ourselves, we will also give finite approximations, usually writing $A = \cup_{s \in \omega} A^s$. If A and B are both enumerated this way, we write $A \setminus B = \{x : (\exists s)[x \in A^s - B^s]\}$, and $A \searrow B = \{x \in A \cap B : (\exists s)[x \in A^s - B^s]\}$. Thus when an element not yet in B enters A, we put it into $A \setminus B$, and if it later enters B, then we put it into $A \searrow B$ as well.

2 The R-Property

In order to guarantee that the set A_0 is not automorphic to a complete set, we will force it to satisfy the lattice-definable property R defined in Section 1, and prove that this implies tardiness of A_0. Tardiness itself does not guarantee that a set cannot be automorphic to a complete set, of course, but satisfaction of R does, since every other set automorphic to A_0 must also satisfy R and therefore must also be tardy, hence incomplete. (A tardy set must be half of a minimal pair under $\leq_T$, as shown in [16], and therefore must be incomplete.) We restate the R-property here:

$$\begin{aligned} R(A_0, A_1):\ & A_0 \cap A_1 = \emptyset \ \& \\ & (\exists C)(\forall B \subseteq C)(\exists D \subseteq C)(\forall S \sqsubset C)(\exists T)\Big[A_0 \cup A_1 \subset_{\mathrm{m}} C \ \& \\ & \big[(B \cap (S - A_0)) \cup A_1 = (D \cap (S - A_0)) \cup A_1 \implies \\ & [\overline{C} \subset T \ \& \ (A_0 \cap S \cap T) \cup A_1 = (B \cap S \cap T) \cup A_1]\big]\Big] \end{aligned}$$

Theorem 2.1 *If A_0 and A_1 are two c.e. sets such that $R(A_0, A_1)$ holds, then A_0 is not of prompt degree.*

Proof. The proof is similar to the corresponding result for the Q-property in [7]. Given A_0 and A_1, we pick a set C as specified in $R(A_0, A_1)$ and fix enumerations $\{A_0^s\}_{s\in\omega}$ of A_0 and $\{C^s\}_{s\in\omega}$ of C such that $A_0 \subseteq C \searrow A_0$.

To prove that a given φ_e is not a promptness function for A_0, we need to find an infinite c.e. set W_i with standard enumeration $\{W_{i,s}\}_{s\in\omega}$ satisfying the tardiness requirement $\mathcal{T}_e$:

$$[(\forall s)\varphi_e(s)\downarrow \geq s] \implies (\forall x)(\forall s)[x \in W_{i,s} - W_{i,s-1} \implies A_0^s \restriction x = A_0^{\varphi_e(s)} \restriction x\ ].$$

We will prove independently for each e that $\mathcal{T}_e$ holds. Having fixed e, we will assume for the rest of this section that φ_e is total with $\varphi_e(s) \geq s$ for every s, since otherwise $\mathcal{T}_e$ is automatically fulfilled. We will build a strong array $\{V_{\langle\alpha,k\rangle,n}\}_{k,n\in\omega;\alpha\in\omega\times\omega}$ of c.e. sets with enumerations $\{V^s_{\langle\alpha,k\rangle,n}\}_{s\in\omega}$. The Slowdown Lemma then gives a computable function f such that for each $\langle\alpha, k\rangle$ and each n, $W_{f(\langle\alpha,k\rangle,n)} = V_{\langle\alpha,k\rangle,n}$ and $V_{\langle\alpha,k\rangle,n} \searrow W_{f(\langle\alpha,k\rangle,n)} = V_{\langle\alpha,k\rangle,n}$, so that no element of $V_{\langle\alpha,k\rangle,n}$ enters $W_{f(\langle\alpha,k\rangle,n)}$ until it has already entered $V_{\langle\alpha,k\rangle,n}$. Periodically the strategy for a given $\langle\alpha, k\rangle$ may be injured by a higher-priority strategy. If this happens while we are enumerating $V_{\langle\alpha,k\rangle,n}$, then we give up on $V_{\langle\alpha,k\rangle,n}$ and start enumerating $V_{\langle\alpha,k\rangle,n+1}$. There will exist an $\langle\alpha, k\rangle$ which is only injured n times (with $n < \omega$), yet receives attention at infinitely many stages, and the corresponding $V_{\langle\alpha,k\rangle,n}$ will be infinite and will be the set which proves satisfaction of $\mathcal{T}_e$.

We define the function $n(\langle\alpha,k\rangle,s)$ to keep track of which $V_{\langle\alpha,k\rangle,n}$ we are enumerating at stage s. In particular, if the $\langle\alpha,k\rangle$-strategy receives attention at stage $s+1$, then we may add an element to $V^{s+1}_{\langle\alpha,k\rangle,n(\langle\alpha,k\rangle,s+1)}$. To avoid notational chaos, however, we will write $V^{s+1}_{\langle\alpha,k\rangle,n}$ in the construction and understand $V^{s+1}_{\langle\alpha,k\rangle,n(\langle\alpha,k\rangle,s+1)}$ for it.

To ensure that one of these $W_{f(\langle\alpha,k\rangle,n)}$ will satisfy $\mathcal{T}_e$, we build a c.e. set B to which to apply the property R. When we want to preserve $A_0 \upharpoonright x$ from stage s until stage $\varphi_e(s)$ so as to satisfy $\mathcal{T}_e$, we do so by restraining all elements $< x$ from entering B until stage $\varphi_e(s)$. The R-property then prohibits such elements from entering A_0, since if they did, we would then hold them out of B forever after, thereby contradicting $R(A_0,A_1)$.

To apply the R-property, we need to know which c.e. set W_i is the D specified by the property. Of course, we do not have this information, but our strategy is to use S to cover all the possibilities. Specifically, in the construction we will split C into the disjoint union of c.e. sets:

$$C = \bigsqcup_{i\in\omega} S_i.$$

and apply the R-property to each S_i, with S_i in the role of S. (Clearly each $S_i \sqsubset C$.) We use each S_i to handle the possibility that $D = W_i$.

Of course, the R-property states that the restraints we place on elements from entering B only affect A_0 on $S \cap T \cap \overline{A_1}$. Since $R(A_0,A_1)$ also states that $A_0 \cap A_1$ is empty, we do not need to worry about elements of A_1, for they can never enter A_0. We are allowed to choose the S, since the matrix of R applies for all S, and indeed we have already done so above (namely $S = S_i$, for each i in turn). However, we can only guess at the set T.

To determine the index j such that $T = W_j$ corresponds to the set S which we choose, we use a Π^0_2 guessing procedure, since the conclusion in the matrix of R is a Π^0_2 property. The j for which $T = W_j$ will be the least j which receives infinitely many guesses under this procedure. (We ensure that the hypothesis of the matrix holds, by periodically putting all elements of $D^s \cap (S^s - A^s_0)$ into B^s.) Moreover, in the construction, we will subdivide each S_i into the disjoint union of c.e. sets $S_{i,j}$:

$$S_i = \bigsqcup_{j\in\omega} S_{i,j}.$$

$S_{i,j}$ is used to handle the possibility that $T = W_j$, so we pay attention to $S_{i,j}$

each time j is named by the guessing procedure. Thus the $S_{i,j}$ corresponding to the correct T will receive attention infinitely often.

To simplify the notation, we let the variable $\alpha = \langle i, j\rangle$ range over $\omega \times \omega$, and define:

$$D_\alpha = W_i$$
$$S_\alpha = S_{i,j}$$
$$T_\alpha = W_j.$$

We order the elements α of $\omega \times \omega$ by pulling back the usual order $<$ on ω to $\omega \times \omega$ via a standard pairing function. Thus each α has only finitely many predecessors under $<$.

For each α, let $F(\alpha)$ be the conjunction of the hypothesis and conclusion in the matrix of the R-property:

$$F(\alpha): \quad (B \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1 \ \& \quad (1)$$
$$[\overline{C} \subset T_\alpha \ \& \ (A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 = (B \cap S_\alpha \cap T_\alpha) \cup A_1] \quad (2)$$

Then $F(\alpha)$ is a Π^0_2 condition, uniformly in α, so there is a computable total function g such that $F(\alpha)$ holds just if $g^{-1}(\alpha)$ is infinite. We enumerate the c.e. set $Z_\alpha = g^1(\alpha)$ by setting $Z^s_\alpha = \{t \le s : g(t) = \alpha\}$.

Now we narrow down each T_α to a c.e. subset U_α, enumerated by:

$$U^s_\alpha = U^{s-1}_\alpha \cup \{x \in T^s_\alpha - C^s : x < |Z^s_\alpha|\}$$

Thus, if T_α actually is the T corresponding to S_i, then U_α will contain all of T_α except certain elements of C. Hence $F(\alpha)$ will hold with U_α in place of T_α. On the other hand, if $F(\alpha)$ fails, then Z_α and U_α are both finite.

If $F(\alpha)$ holds, then $\overline{C} \subseteq U_\alpha$, so $\overline{A_0} \subseteq^* U_\alpha \cup A_1$, because $A_0 \cup A_1 \subset_{\rm m} C$. For the least α such that $F(\alpha)$ holds, our construction of S^{s+1}_α will yield $C - A_0 \subseteq^* S_\alpha \cup A_1$, with S_β finite for all $\beta < \alpha$. Hence there will exist a k such that

$$C - A_0 \subseteq S_\alpha \cup A_1 \cup \{0, 1, \ldots k - 1\} \quad (3)$$

Line (3) is a Π^0_2 statement, uniformly in k and α, since our definition of S_α will be uniform in α. Therefore, there exists a total function h_α such that (3) holds if and only if $h^{-1}_\alpha(k)$ is infinite. We define:

$$h(s) = h_{g(s)}(n), \text{ where } n = |\{t < s : g(t) = g(s)\}|.$$

We will enumerate sets $V_{\langle\alpha,k\rangle,n}$ for each α, k and n. For the least α with Z_α infinite and the least k with $h^{-1}_\alpha(k)$ infinite, the set $V_{\langle\alpha,k\rangle,n}$ (for some

n) will be the W_i required by $\mathcal{T}_e$. Elements of each $V_{\langle\alpha,k\rangle,n}$ (the "witness elements" for the requirement $\mathcal{T}_e$) will be denoted $v^s_{\langle\alpha,k\rangle}$. Each $v^s_{\langle\alpha,k\rangle}$ will enter $V_{\langle\alpha,k\rangle,n}$ for at most one n.

The Slowdown Lemma (see [16], p. 284) then yields a computable function f such that, for every $\langle\alpha,k\rangle$ and every n, $V_{\langle\alpha,k\rangle,n} = W_{f(\langle\alpha,k\rangle,n)}$, and at every stage s,

$$(V^s_{\langle\alpha,k\rangle,n} - V^{s-1}_{\langle\alpha,k\rangle,n}) \cap W_{f(\langle\alpha,k\rangle,n),s} = \emptyset.$$

When a witness element $v^s_{\langle\alpha,k\rangle}$ enters $V_{\langle\alpha,k\rangle,n}$, we will find the stage $t^s_{\langle\alpha,k\rangle} > s$ at which $v^s_{\langle\alpha,k\rangle}$ enters $W_{f(\langle\alpha,k\rangle,n)}$ and restrain (with priority $\langle\alpha,k\rangle$) elements $\leq v^s_{\langle\alpha,k\rangle}$ from entering A_0 until stage $\varphi_e(t^s_{\langle\alpha,k\rangle})$. (Recall that $\mathcal{T}_e$ assumes φ_e to be total.) Thus we will have $A_0^{t^s_{\langle\alpha,k\rangle}}\restriction v^s_{\langle\alpha,k\rangle} = A_0^{\varphi_e(t^s_{\langle\alpha,k\rangle})}\restriction v^s_{\langle\alpha,k\rangle}$. If we can achieve this for all $v^s_{\langle\alpha,k\rangle}$ in the (infinite) set $V_{\langle\alpha,k\rangle,n}$ for some n, then the set $W_{f(\langle\alpha,k\rangle,n)}$ will be the set required by $\mathcal{T}_e$ to prove that φ_e is not a promptness function for A_0.

At stage 0, for all $\langle\alpha,k\rangle$, we set $n(\langle\alpha,k\rangle,0) = 0$ and $V^0_{\langle\alpha,k\rangle,0} = \emptyset$, with $v^0_{\langle\alpha,k\rangle}\uparrow$ and $t^0_{\langle\alpha,k\rangle}\uparrow$. Also, let every $S^0_\alpha = \emptyset$ and let $B^0 = \emptyset$.

At stage $s+1$, we first define each S^{s+1}_α. For each $x \in C^{s+1} - C^s$, find the least α such that $x \in U^s_\alpha$ and put x into S^{s+1}_α. If there is no such α, put x into S^{s+1}_ω. (The c.e. set S_ω simply collects elements which enter C without entering any S_α. Thus $C = \bigsqcup_{\alpha\leq\omega} S_\alpha$.)

Set $\alpha = g(s)$, and define:

$$B^{s+1} = B^s \cup \left\{ x : \begin{array}{c} x \in C^s - A^s_0 \ \& \ (\exists\beta\leq\alpha)[x \in D^{s+1}_\beta \cap S^{s+1}_\beta \ \& \\ (\forall\delta\leq\beta)(\forall k<s)[t^s_{\langle\delta,k\rangle}\downarrow \implies x \geq v^s_{\langle\delta,k\rangle}]] \end{array} \right\}$$

For each strategy which is injured at stage $s+1$, we begin enumerating a new witness set. To this end, set $n(\langle\gamma,k\rangle,s+1) = n(\langle\gamma,k\rangle,s)+1$ and $v^{s+1}_{\langle\gamma,k\rangle}\uparrow$ and $t^{s+1}_{\langle\gamma,k\rangle}\uparrow$ for each $\langle\gamma,k\rangle$ satisfying any of the following conditions:

- $\gamma > \alpha$.
- $\gamma = \alpha$ and $k > h(s)$.
- There exists $x < k$ with $x \in A^{s+1}_0 - A^s_0$.
- There exists $\beta < \gamma$ with $S^{s+1}_\beta \neq S^s_\beta$.
- There exists $\beta < \gamma$ such that U^{s+1}_β contains an element $\geq m$, where $m = \min(B^{s+1} - B^s)$.

For all other $\langle\gamma, k\rangle$, set $n(\langle\gamma, k\rangle, s+1) = n(\langle\gamma, k\rangle, s)$.

We now define the witness sets at stage $s+1$. For each $\langle\beta, k\rangle \leq \langle\alpha, h(s)\rangle$ (in the lexicographic order) which was not injured at stage $s+1$:

1. If $v^s_{\langle\beta,k\rangle}\uparrow$ and $\langle\beta, k\rangle \neq \langle\alpha, h(s)\rangle$, let $v^{s+1}_{\langle\beta,k\rangle}$ and $t^{s+1}_{\langle\beta,k\rangle}$ diverge also, with $V^{s+1}_{\langle\beta,k\rangle,n} = V^s_{\langle\beta,k\rangle,n}$.

2. If $v^s_{\langle\alpha,h(s)\rangle}\uparrow$, let $v^{s+1}_{\langle\alpha,h(s)\rangle} = s+1$, with $V^{s+1}_{\langle\alpha,h(s)\rangle,n} = V^s_{\langle\alpha,h(s)\rangle,n}$ and $t^{s+1}_{\langle\alpha,h(s)\rangle}\uparrow$.

3. If $v^s_{\langle\beta,k\rangle}\downarrow$ but $t^s_{\langle\beta,k\rangle}\uparrow$, let $v^{s+1}_{\langle\beta,k\rangle} = v^s_{\langle\beta,k\rangle}$, and ask whether the following holds:

$$(\forall y)_{k\leq y\leq v^{s+1}_{\langle\beta,k\rangle}} \left[\begin{array}{c} y \in A_0^{s+1} \vee y \in A_1^{s+1} \vee \\ y \in (U_\beta^{s+1} - C^{s+1}) \vee \\ y \in (C^{s+1} - B^{s+1}) \cap S_\beta^{s+1} \cap U_\beta^{s+1} \end{array}\right] \tag{4}$$

   If (4) holds, let $V^{s+1}_{\langle\beta,k\rangle,n} = V^s_{\langle\beta,k\rangle,n} \cup \{v^{s+1}_{\langle\beta,k\rangle}\}$ and

$$t^{s+1}_{\langle\beta,k\rangle} = \mu t[v^{s+1}_{\langle\beta,k\rangle} \in W_{f(\langle\beta,k\rangle,n),t}].$$

   (Such a t must exist, since $W_{f(\langle\beta,k\rangle,n)} = V_{\langle\beta,k\rangle,n}$.) If (4) fails, then let $V^{s+1}_{\langle\beta,k\rangle,n} = V^s_{\langle\beta,k\rangle,n}$ and $t^{s+1}_{\langle\beta,k\rangle}\uparrow$.

4. If $v^s_{\langle\beta,k\rangle}\downarrow$ and $t^s_{\langle\beta,k\rangle}\downarrow$ and $\varphi_{e,s}(t^s_{\langle\beta,k\rangle})\downarrow< s$, then let $v^{s+1}_{\langle\beta,k\rangle}\uparrow$ and $t^{s+1}_{\langle\beta,k\rangle}\uparrow$, with $V^{s+1}_{\langle\beta,k\rangle,n} = V^s_{\langle\beta,k\rangle,n}$.

5. If $v^s_{\langle\beta,k\rangle}\downarrow$ and $t^s_{\langle\beta,k\rangle}\downarrow$ but either $\varphi_{e,s}(t^s_{\langle\beta,k\rangle})\downarrow\geq s$ or $\varphi_{e,s}(t^s_{\langle\beta,k\rangle})$ diverges, then let $V^{s+1}_{\langle\beta,k\rangle,n} = V^s_{\langle\beta,k\rangle,n}$, $v^{s+1}_{\langle\beta,k\rangle} = v^s_{\langle\beta,k\rangle}$, and $t^{s+1}_{\langle\beta,k\rangle} = t^s_{\langle\beta,k\rangle}$.

This completes the construction.

We now use the sets B and S_α to prove that requirement $\mathcal{T}_e$ is satisfied.

Lemma 2.2 *If Z_β is finite, then there exists a stage s_1 such that $t^s_{\langle\beta,k\rangle}\uparrow$ for all $s \geq s_1$ and all k.*

Proof. Pick a stage s_0 such that no $s \geq s_0$ satisfies $g(s) = \beta$, and let $k' = \max\{h(s) : g(s) = \beta\}$. Then for all $k > k'$, $v^s_{\langle\beta,k\rangle}\uparrow$ for all s, and hence $t^s_{\langle\beta,k\rangle}\uparrow$ for all s. (The construction makes it clear that for any k and s, $t^s_{\langle\beta,k\rangle}$ can converge only if $v^s_{\langle\beta,k\rangle}$ converges.)

Now suppose $k \leq k'$ and $v^s_{\langle\beta,k\rangle}\downarrow$ for all $s \geq s_0$. This means that we never execute Step (4) in the construction after stage s_0, and that the $\langle\beta, k\rangle$ strategy is never injured after stage s_0. But if $t^s_{\langle\beta,k\rangle}$ ever converges after stage s_0, then eventually we must reach Step (4), since we assumed φ_e to be total. Hence $t^s_{\langle\beta,k\rangle}$ must diverge for all $s \geq s_0$.

Finally, suppose $k \leq k'$ and $v^{s_{1,k}}_{\langle\beta,k\rangle}\uparrow$ for some $s_{1,k} \geq s_0$. Then $v^s_{\langle\beta,k\rangle}$ will diverge for all subsequent s, since it can only be newly defined at a stage s with $g(s) = \beta$. Thus $t^s_{\langle\beta,k\rangle}$ will diverge for all subsequent s as well. Letting $s_1 = \max_{k\leq k'} s_{1,k}$ completes the proof. ∎

Lemma 2.3 *$F(\alpha)$ holds for some α, and for the least such α, there exists a k such that $h^{-1}_\alpha(k)$ is infinite.*

Proof. First we claim that some Z_α must be infinite. Suppose not, so Z_α is finite for all α, and $F(\alpha)$ fails for all α. However, the R-property holds, so there must be some α for which line (1) fails. Choose the least such α. Then

$$(B \cap (S_\alpha - A_0)) \cup A_1 \neq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.$$

Suppose $x \in B \cap (S_\alpha - A_0)$. Pick s such that $x \in B^{s+1} - B^s$. Now to go into B^{s+1}, x must have been in $D^{s+1}_\beta \cap S^{s+1}_\beta$ for some β. Since $x \in S_\alpha$, we know $x \notin S_\beta$ for all $\beta \neq \alpha$. Hence $x \in D_\alpha$, and so

$$(B \cap (S_\alpha - A_0)) \cup A_1 \subseteq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.$$

Therefore, there must be some element $x \in \overline{A_1} \cap \overline{B} \cap D_\alpha \cap (S_\alpha - A_0)$. Assume x is the least such element. Now for every $\beta < \alpha$, line (1) must hold and line (2) must fail, since we chose α to be minimal satisfying the R-property. Hence for all $\beta < \alpha$,

$$(B \cap (S_\beta - A_0)) \cup A_1 = (D_\beta \cap (S_\beta - A_0)) \cup A_1.$$

Now since every Z_β with $\beta \leq \alpha$ is finite, there is a stage s_0 such that for all $s \geq s_0$, $g(s) > \alpha$, and we may also assume that s_0 is so large that $x \in S^{s_0}_\alpha \cap D^{s_0}_\alpha \cap C^{s_0}$. (Notice that $x \in S_\alpha$ forces $x \in C$.)

Now use Lemma 2.2 to find a stage $s_1 \geq s_0$ such that:

$$(\forall s \geq s_1)(\forall\beta \leq \alpha)(\forall k)[t^{s_1}_{\langle\beta,k\rangle}\uparrow].$$

Since φ_e is total, there must be a stage $s \geq s_1$ such that $t^s_{\langle\alpha,k\rangle}\uparrow$, and once we reach this stage s, x must go into B^{s_1+1}, contradicting our assumption that $x \notin B$.

Thus, there must be some α such that Z_α is infinite. Let α be the least such. Then every U_β with $\beta < \alpha$ is finite. Since $F(\alpha)$ holds, we have $\overline{C} \subseteq T_\alpha$, so by our construction, $\overline{C} \subseteq U_\alpha$, and by the major subset property, $\overline{A_0} \subseteq^* U_\alpha \cup A_1$.

For this α, we claim that $C - A_0 \subseteq^* S_\alpha \cup A_1$. Suppose $x \in C - A_0$. All but finitely many such x lie in $U_\alpha \cup A_1$, as noted above. If $x \in A_1$, we are done. For each sufficiently large $x \in C - A_0 - A_1$, there exists s such that $x \in U_\alpha^s - U_\alpha^{s-1}$. By definition of U_α^s, we must have $x \notin C^s$. But $x \in C$, so $x \in C^{t+1} - C^t$ for some $t \geq s$. Hence $x \in S_\alpha^{t+1}$ by definition of S_α^{t+1}, unless there exists $\beta < \alpha$ with $x \in U_\beta$. But all U_β with $\beta < \alpha$ are finite, by our choice of α, so all but finitely many of these x lie in S_α. Therefore, line (3) holds for some k, and $h_\alpha^{-1}(k)$ is infinite. ∎

Use Lemma 2.3 to take the lexicographically least $\langle \alpha, k \rangle$ such that $F(\alpha)$ holds and $h_\alpha^{-1}(k)$ is infinite. Then there are infinitely many stages s for which $g(s) = \alpha$ and $h(s) = k$, but only finitely many for which $\langle g(s), h(s) \rangle$ precedes $\langle \alpha, k \rangle$ in the lexicographic ordering. Let s_0 be the least stage with $\langle g(s_0), h(s_0) \rangle = \langle \alpha, k \rangle$ such that:

- $A_0^{s_0} \restriction k = A_0 \restriction k$, and
- $B^{s_0} \restriction m = B \restriction m$, where $m = \max \cup_{\beta < \alpha} U_\beta$, and
- for all $s \geq s_0$, $\langle g(s), h(s) \rangle \geq \langle \alpha, k \rangle$ lexicographically, and
- $S_\beta^{s_0} = S_\beta$ for all $\beta < \alpha$.

The final condition is possible since each $S_\beta \subseteq U_\beta$, which is finite for every $\beta < \alpha$. We also let $s_0 < s_1 < s_2 < \cdots$ be all the stages $s \geq s_0$ with $\langle g(s), h(s) \rangle = \langle \alpha, k \rangle$.

Now the $\langle \alpha, k \rangle$-strategy is never injured after stage s_0, so for every $s \geq s_0$, $n(\langle \alpha, k \rangle, s_0) = n(\langle \alpha, k \rangle, s)$, and we write $n = n(\langle \alpha, k \rangle, s_0)$. (Thus n is the number of times the $\langle \alpha, k \rangle$-strategy was injured during the construction.) Moreover, minimality of s_0 implies that this strategy was injured at some stage $s \leq s_0$ such that there is no s_{-1} with $s \leq s_{-1} < s_0$ and $\langle g(s_{-1}), h(s_{-1}) \rangle = \langle \alpha, k \rangle$. Therefore, $V_{\langle \alpha, k \rangle, n}^s = V_{\langle \alpha, k \rangle, n}^{s_0}$ is empty.

We claim that the subset $V_{\langle \alpha, k \rangle, n}$ satisfies requirement $\mathcal{T}_e$. For this we need:

Lemma 2.4 *For this $\langle \alpha, k \rangle$, and for each $y \geq k$, there exists an s such that the matrix of line (4) holds of y, $\langle \alpha, k \rangle$, and s.*

Proof. Let $y \geq k$. If $y \in A_0 \cup A_1$, we are done. If $y \in \overline{C}$, then $y \in T_\alpha$ since $F(\alpha)$ holds. But Z_α is infinite, so $T_\alpha - C \subseteq U_\alpha$, and y is in $U_\alpha - C$, hence in some $U_\alpha^{s+1} - C^{s+1}$.

So suppose $y \in C - A_0 - A_1$. Now since $h_\alpha^{-1}(k)$ is infinite and $y \geq k$, we know by line (3) that $y \in S_\alpha$. But $S_\alpha \subseteq U_\alpha \subseteq T_\alpha$ by definition of S_α^{s+1}. Since $y \notin (B \cap S_\alpha \cap T_\alpha) \cup A_1$ by line (2), we know $y \notin B$. Thus there is an s with $y \in (C^{s+1} - B^{s+1}) \cap S_\alpha^{s+1} \cap U_\alpha^{s+1}$. This proves the Lemma. ∎

Now $V_{\langle\alpha,k\rangle,n} = W_{f(\langle\langle\alpha,k\rangle,n\rangle)}$, and if s' is the stage at which $v_{\langle\alpha,k\rangle}^{s'}$ enters $V_{\langle\alpha,k\rangle,n}$, then $t_{\langle\alpha,k\rangle}^{s'}\downarrow > s'$ by our choice of f from the Slowdown Lemma. Let $s'' = \varphi_e(t_{\langle\alpha,k\rangle}^{s'})$. Then $s' < s''$, since we assumed φ_e to be increasing.

Lemma 2.5 *$V_{\langle\alpha,k\rangle,n}$ is infinite. Moreover, for any element $v_{\langle\alpha,k\rangle}^{s'}$ of $V_{\langle\alpha,k\rangle,n}$, with s' and s'' as above, we have:*

$$B^{s'}\restriction v_{\langle\alpha,k\rangle}^{s'} = B^{s''}\restriction v_{\langle\alpha,k\rangle}^{s'} \quad \textit{and} \quad A_0^{s'}\restriction v_{\langle\alpha,k\rangle}^{s'} = A_0^{s''}\restriction v_{\langle\alpha,k\rangle}^{s'}.$$

Proof. For each $v_{\langle\alpha,k\rangle}^{s}$ with $s \geq s_0$, Lemma 2.4 guarantees that there will be a stage at which Step (3) of the construction applies. The first such stage will be s', since at that stage $v_{\langle\alpha,k\rangle}^{s} = v_{\langle\alpha,k\rangle}^{s'}$ will enter $V_{\langle\alpha,k\rangle,n}$ and $t_{\langle\alpha,k\rangle}^{s'}$ will be defined. But since φ_e is total, we will eventually reach the stage $s'' > s'$ at which Step (4) applies, leaving $v_{\langle\alpha,k\rangle}^{s''+1}$ undefined. Then at the next $s_m > s''$, we will define $v_{\langle\alpha,k\rangle}^{s_m+1} = s_m + 1$, which is not yet in $V_{\langle\alpha,k\rangle,n}^{s_m}$. Thus, $V_{\langle\alpha,k\rangle,n}$ must be infinite.

Now pick $v_{\langle\alpha,k\rangle}^{s'} \in V_{\langle\alpha,k\rangle,n}$, with s' and s'' as above. Since $V_{\langle\alpha,k\rangle,n}^{s_0}$ is empty, we know that $s' > s_0$. If s is any stage with $s' \leq s < s''$, then we see from the definition of B^{s+1} that an element y can only enter B^{s+1} on behalf of some γ such that $y \in S_\gamma^{s+1}$. But then $y \in U_\gamma^{s+1}$. Since we chose s_0 to let $B^{s_0}\restriction m = B\restriction m$, we must have $\gamma \geq \alpha$. But $t_{\langle\alpha,k\rangle}^{s}\downarrow$, so $y \geq v_{\langle\alpha,k\rangle}^{s} = v_{\langle\alpha,k\rangle}^{s'}$ by definition of B^{s+1}. Hence $B^{s'}\restriction v_{\langle\alpha,k\rangle}^{s'} = B^{s''}\restriction v_{\langle\alpha,k\rangle}^{s'}$.

Having seen that no $y < v_{\langle\alpha,k\rangle}^{s'}$ can enter B between stages s' and s'', we prove that no such y can enter A_0 at those stages either. First, we know that $A_0^{s_0}\restriction k = A_0\restriction k$ by choice of s_0. So suppose $k \leq y < v_{\langle\alpha,k\rangle}^{s'}$. Now since $v_{\langle\alpha,k\rangle}^{s'}$ entered $V_{\langle\alpha,k\rangle,n}$ at stage s', we know by line (4) that

$$y \in A_0^{s'} \vee y \in A_1^{s'} \ \vee y \in (U_\alpha^{s'} - C^{s'}) \ \vee y \in (C^{s'} - B^{s'}) \cap S_\alpha^{s'} \cap U_\alpha^{s'}.$$

If $y \in A_0^{s'}$, then $A_0^{s'}(y) = A_0^{s''}(y)$, and if $y \in A_1$, then $y \notin A_0$ at all. Therefore, we will assume that $y \notin A_0^{s'} \cup A_1$ and prove that $y \notin A_0^{s''}$.

If the final clause holds, then $y \in (C^{s'} - B^{s'}) \cap S_\alpha^{s'} \cap U_\alpha^{s'}$. Hence $y \notin B^{s''}$, by the first half of the lemma. If $y \in A_0^{s''}$, then $y \notin B$, since no element that has entered A_0 can later enter B. But then

$$(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1$$

since y is on the left side and not on the right side. (Notice that $y \in U_\alpha$ implies $y \in T_\alpha$.) This contradicts line (2), which we knows holds because $F(\alpha)$ holds. Therefore $y \notin A_0^{s''}$.

So suppose the third clause holds, i.e. $y \in (U_\alpha^{s'} - C^{s'})$. Then $y \notin B^{s'}$ since $B^{s'} \subseteq C^{s'}$, and so $y \notin B^{s''}$. If $y \in A_0^{s''}$, then we must have $y \in C^{s''-1}$ since we chose enumerations such that $A_0 \subseteq C \searrow A_0$. Pick s such that $y \in C^s - C^{s-1}$; then $s' < s < s''$ and $y \notin A_0^s$. Now $y \in U_\alpha^{s'} \subseteq T_\alpha^{s'}$, and by definition of S_α^s we will have $y \in S_\alpha^s$. (Recall that s_0 was chosen so large that $S_\beta^{s_0} = S_\beta$ for all $\beta < \alpha$.) But now $y \notin A_0^{s''}$, since otherwise

$$(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1$$

just as in the preceding paragraph. ■

Hence $V_{\langle\alpha,k\rangle,n} = W_{f(\langle\alpha,k\rangle,n)}$ is an infinite c.e. set which satisfies the tardiness requirement $\mathcal{T}_e$. This completes the proof of Theorem 2.1. ■

3 Satisfaction of R

We now prove that the R-property defined in Section 2 is nontrivial. The theorem establishes several other properties of the sets A_0 and A_1 as well, in order to yield the corollaries.

Theorem 3.1 *There exists a c.e. set A with Friedberg splitting $A = A_0 \sqcup A_1$ such that all of the following hold:*

1. *A is promptly simple of high degree.*
2. *A_1 has prompt degree.*
3. *$R(A_0, A_1)$.*

Corollary 3.2 *The formula in one free variable A_0:*

$$(\exists A_1)[A_0 >_T \emptyset \ \& \ R(A_0, A_1)]$$

is definable in $\mathcal{E}$ and non-vacuous, and implies that A_0 is a noncomputable incomplete set.

Proof of Corollary. The statement $A_0 >_T \emptyset$ is equivalent to the statement that A_0 does not have a complement in $\mathcal{E}$, hence is $\mathcal{E}$-definable. The A_0 and A_1 constructed in Theorem 3.1 satisfy the matrix, since halves of a Friedberg splitting must be noncomputable. Finally, Theorem 2.1 shows that A_0 is tardy, hence incomplete. ■

Corollary 3.3 *There exists a Friedberg splitting $A = A_0 \sqcup A_1$ such that A_0 and A_1 are not automorphic in the lattice of c.e. sets.*

Proof of Corollary. Take the splitting given by Theorem 3.1. If an automorphism Φ of $\mathcal{E}$ satisfied $\Phi(A_0) = A_1$, then $R(A_1, \Phi(A_1))$ would have to hold. By Theorem 2.1, then, A_1 would be tardy, contradicting the promptness of A_1. ■

Proof of Theorem. Let C be any promptly simple set, with computable enumeration $C = \{C^s\}_{s\in\omega}$. Then C is also of prompt degree, so let v and w be the prompt-simplicity and promptness functions for this enumeration of C, satisfying for every i:

$$W_i \text{ infinite} \implies (\exists^\infty s)(\exists x \in W_{i,s} - W_{i,s-1})[x \in C^{v(s)}]$$
$$W_i \text{ infinite} \implies (\exists^\infty s)(\exists x \in W_{i,s} - W_{i,s-1})[C^{w(s)}\restriction x \neq C^s\restriction x]$$

We construct disjoint sets A_0 and A_1 and auxiliary sets D_i and $T_{i,j}$, and set $A = A_0 \sqcup A_1$. The approximations to A, A_0, and A_1 at stage s will be written A^s, A_0^s, and A_1^s, and will be defined so that $A^s = A_0^s \cup A_1^s \subseteq C^s$ for all s. The construction will satisfy the following requirements for all i and j:

$\mathcal{N}_{\langle i,j\rangle}$ *(matrix of R-property)* :
$[W_i \subseteq C \ \&\ W_j \subseteq C \ \&\ C - W_j$ c.e. $\&$
$(W_i \cap (W_j - A_0)) \cup A_1 = (D_i \cap (W_j - A_0)) \cup A_1] \implies$
$(\exists T)[\overline{C} \subseteq T \ \&\ (A_0 \cap W_j \cap T) \cup A_1 =^* (W_i \cap W_j \cap T) \cup A_1]$

$\mathcal{M}_i$ *(major subset requirement)* :
$\overline{C} \subseteq W_i \implies \overline{A} \subseteq^* W_i$

$\mathcal{P}_i$ *(prompt simplicity of A)* :
W_i infinite $\implies (\exists s)(\exists x \in W_{i,s} - W_{i,s-1})[x \in A^{v(s)}]$

$\mathcal{Q}_i$ *(promptness of A_1)* :
W_i infinite $\implies (\exists s)(\exists x \in W_{i,s} - W_{i,s-1})[A_1^{w(s)}\restriction x \neq A_1^s\restriction x]$

$\mathcal{F}_i$ *(Friedberg requirement for A_0)* :
$W_i \searrow A$ infinite $\implies W_i \cap A_0 \neq \emptyset$

$\mathcal{G}_i$ *(Friedberg requirement for A_1)* :
$W_i \searrow A$ infinite $\implies W_i \cap A_1 \neq \emptyset$

In the requirement $\mathcal{N}_{\langle i,j\rangle}$, of course, W_i plays the role of B and W_j the role of S in the matrix of the R-property. We will construct c.e. sets $T_{i,j}$ for each i and j, and then refine them to form the T demanded by each $\mathcal{N}_{\langle i,j\rangle}$ Once again we order $\omega \times \omega$ in order type ω and write $\alpha = \langle i,j\rangle$, this time with:

$$\begin{aligned} B_\alpha &= W_i \\ D_\alpha &= D_i \\ \left.\begin{aligned} S_\alpha &= W_{j'} \\ \hat{S}_\alpha &= W_{j''} \end{aligned}\right\} &\text{ where } j = \langle j', j''\rangle \\ T_\alpha &= T_{i,j} \\ \mathcal{N}_\alpha &= \mathcal{N}_{i,j}. \end{aligned}$$

Thus $\mathcal{N}_\alpha$ says:

$$\begin{gathered} [B_\alpha \subseteq C \ \&\ S_\alpha \sqcup \hat{S}_\alpha = C \ \& \\ (B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1] \\ \implies (\exists T)[\overline{C} \subseteq T \ \&\ (A_0 \cap S_\alpha \cap T) \cup A_1 =^* (B_\alpha \cap S_\alpha \cap T) \cup A_1]. \end{gathered}$$

$\mathcal{N}_\alpha$ is a negative requirement, trying to keep elements from entering A_0 until they can do so without harming the R-property (if ever). All the other requirements are positive ones, trying to put elements into A_0 or A_1. There are no negative restraints on elements of C entering A_1, except that they cannot already be in A_0.

Each element which we try to put into A_0 to satisfy some $\mathcal{F}_e$ or $\mathcal{M}_e$ must receive permission to enter A_0 from each $\mathcal{N}_\alpha$ with $\alpha \leq e$. The *restraint function* $q(x,s)$ will give the greatest $\alpha \leq e$ which has not yet given this permission as of stage s. The *priority function* $p(x,s)$ keeps track of which requirement $\mathcal{F}_e$ or $\mathcal{M}_e$ wanted x to enter A_0. This can change from stage to stage, for several reasons. If a higher-priority requirement decides at stage $s+1$ that it needs x to enter A_0, then $p(x,s+1) < p(x,s)$. Alternatively, an $\mathcal{F}_e$ could find itself satisfied by another $x' \in A_0^{s+1}$ and no longer need to put x into A_0, although in this case we leave $p(x,s+1) = p(x,s)$ so as not to disrupt the flow of elements into A_0. Finally, a higher-priority requirement could make x enter A_1^{s+1}, in which case we define $p(x,s+1)\uparrow$, removing x from the flow of elements into A_0 since we need $A_0 \cap A_1 = \emptyset$.

We use the Recursion Theorem on our construction of A_0, C, and D_α to define the following Π_2^0 statement $F(\alpha)$ for each α:

$$(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1 \ \& \ B_\alpha \subseteq C \ \& \ S_\alpha \sqcup \hat{S}_\alpha = C.$$

Since $F(\alpha)$ is Π_2^0, there is a computable function $g : \omega \to \omega \times \omega$ such that $F(\alpha)$ holds if and only if the set $Z_\alpha = g^{-1}(\alpha)$ is infinite. We let $Z_\alpha^s = g^{-1}(\alpha) \cap \{0, 1, \ldots s-1\}$. Monitoring $|Z_\alpha^s|$ will help us determine for which α the hypothesis in the matrix of the R-property is satisfied. For those α for which the hypothesis fails, $|Z_\alpha|$ is finite, and $\mathcal{N}_\alpha$ will only restrain finitely many elements from entering A_0, since we need not satisfy the conclusion of the R-property for such an α.

At stage $s = 0$, we set $A_0^0 = A_1^0 = \emptyset$. Also, let all $p(x,0)$ and $q(x,0)$ diverge.

At stage $s+1$, we first define T_α^{s+1} for each α:

$$T_\alpha^{s+1} = T_\alpha^s \cup \{x \in \overline{C^{s+1}} : x < |Z_\alpha^{s+1}|\}.$$

Next we determine which elements of C^{s+1} to add to A_0^s to create A_0^{s+1}. For this, we need movable markers for elements currently in $C - A$. Write

$$C^{s+1} - A^s = \{d_0^{s+1}, d_1^{s+1}, \ldots d_{m_{s+1}}^{s+1}\}$$

preserving the order of the markers from the preceding stage. (That is, if $d_i^s = d_{i'}^{s+1}$ and $d_j^s = d_{j'}^{s+1}$, then $i < j$ iff $i' < j'$; and if $d_i^{s+1} \in C^s$ and $d_j^{s+1} \notin C^s$, then $i < j$.)

For the sake of $\mathcal{M}_e$, we define

$$V_e^{s+1} = V_e^s \cup \{x \in W_{e,s+1} - C^{s+1} : (\forall y \leq x)[y \in W_{e,s+1} \cup C^{s+1}]\}.$$

(For each e, the sets V_e^s enumerate a c.e. set V_e. If $\overline{C} \not\subseteq W_e$, then V_e will be finite, but if $\overline{C} \subseteq W_e$, then $\overline{C} \subseteq V_e \subseteq W_e$.)

For each $e \leq s$, define the e-state of each d_k^{s+1} at stage $s+1$ to be:

$$\sigma(e, d_k^{s+1}, s+1) = \{i < e : d_k^{s+1} \in V_i^{s+1}\}.$$

We order the different possible e-states by viewing them as binary strings.

Find the least $i \leq s$ such that there exist e and j with $e < i < j \leq s$ and $\sigma(e, d_i^{s+1}, s+1) = \sigma(e, d_j^{s+1}, s+1)$ and $d_i^{s+1} \notin V_e^{s+1}$ and $d_j^{s+1} \in V_e^{s+1}$. For the least such e and the least corresponding j, we say that $\mathcal{M}_e$ *wants to put into* A_0 all the elements $d_i^{s+1}, d_{i+1}^{s+1}, \ldots d_{j-1}^{s+1}$, so as to give the marker d_i a higher $(e+1)$-state at subsequent stages.

Now we consider the requirements $\mathcal{F}_e$. For each $e \leq s$ with $W_{e,s} \cap A_0^s = \emptyset$ and for each x such that

$$x \in (W_{e,s} \cap C^{s+1}) - A^s - \{d_0^{s+1}, d_1^{s+1}, \ldots d_e^{s+1}\}$$

we say that $\mathcal{F}_e$ *wants to put* x *into* A_0.

We set $p(x, s+1)\uparrow$ for all $x \notin C - A^s$. Otherwise $x = d_k^{s+1}$ for some k, and $p(x, s+1)$ is the least $e \leq k$ (if any) such that either $p(x, s)\downarrow= e$ or $\mathcal{M}_e$ or $\mathcal{F}_e$ wants to put x into A_0. Thus, the function $p(x, s+1)$ gives the priority currently assigned to putting x into A_0. If there is no such e, let $p(x, s+1)\uparrow$.

We now follow the following steps for each $x \leq s$:

1. If $p(x, s+1)\uparrow$, then $q(x, s+1)\uparrow$ also.

2. If $p(x, s+1)\downarrow$ but $q(x, s)\uparrow$, we ask if every $\alpha \leq p(x, s+1)$ satisfies either $x \in S_\alpha^{s+1} \cup \hat{S}_\alpha^{s+1}$ or $x \notin T_\alpha^{s+1}$. If so, set $q(x, s+1) = p(x, s+1) + 1$. If not, then $q(x, s+1)\uparrow$.

3. If $p(x, s+1)\downarrow$ and $q(x, s)\downarrow> p(x, s+1)$, then set $q(x, s+1)$ to be the greatest $\alpha \leq p(x, s+1)$ satisfying all four of the following conditions:

   (a) $S_\alpha^{s+1} \cap \hat{S}_\alpha^{s+1} = \emptyset$.

   (b) $x \notin \hat{S}_\alpha^{s+1}$.

   (c) $x \in T_\alpha^{s+1}$.

(d) $\forall \beta < \alpha$, either β fails one of the three conditions (a)-(c), or $\beta = \langle i', j' \rangle$ and $\alpha = \langle i, j \rangle$ with $i \neq i'$.

Also, enumerate x in $D^{s+1}_{q(x,s+1)}$. (For future reference, notice that if α satisfies (a)-(c), then some $\beta \leq \alpha$ with the same first coordinate as α must satisfy (a)-(d).)

If there is no such α, set $q(x, s+1) = -1$.

4. If $p(x, s+1)\downarrow$ and $q(x, s)\downarrow$ with $0 \leq q(x, s) \leq p(x, s+1)$, we ask whether $x \in B^{s+1}_{q(x,s)}$. If so, or if $q(x, s)$ no longer satisfies the conditions (a)-(d), set $q(x, s+1)$ to be the greatest $\alpha < q(x, s)$ satisfying the conditions (a)-(d) above, and let $x \in D^{s+1}_{q(x,s+1)}$. (If there is no such α, let $q(x, s+1) = -1$.) Otherwise, let $q(x, s+1) = q(x, s)$.

5. If $p(x, s+1)\downarrow$ and $q(x, s) \downarrow= -1$, enumerate $x \in A^{s+1}_0$, and let $q(x, s+1)\uparrow$.

This completes our enumeration of A^{s+1}_0. Next we determine which elements to add to A^{s+1}_1:

1. Find the least $e \leq s$ (if any) such that $\mathcal{Q}_e$ is not yet satisfied and there is an element $x \in W_{e,t} - W_{e,t-1}$ for some $t \leq s$ such that $w(t) > s$, and there exists $y < x$ such that $y \in C^{s+1} - A^{s+1}_0$ and $y \notin A^t_1 \cup \{d^{s+1}_0, \ldots d^{s+1}_e\}$ and no $\mathcal{F}_i$ with $i < e$ wants to put y into A_0. Put the greatest such y into A^{s+1}_1. This forces $A^{s+1}_1 \upharpoonright x \neq A^t_1 \upharpoonright x$, satisfying $\mathcal{Q}_e$ permanently. (If there is no such e, do nothing.)

2. Find the least $e \leq s$ (if any) such that $\mathcal{P}_e$ is not yet satisfied and there is an element $x \in C^{s+1} \cap (W_{e,t} - W_{e,t-1})$ for some $t \leq s$ with $v(t) > s$, such that $x \notin \{d^{s+1}_0, \ldots d^{s+1}_e\}$ and no $\mathcal{F}_i$ with $i < e$ wants to put x into A_0. If no such x lies in $A^s \cup A^{s+1}_0$, then put the least such x into A^{s+1}_1. This forces $x \in A^{s+1}$, satisfying $\mathcal{P}_e$ permanently.

3. Find the least $e \leq s$ (if any) such that $\mathcal{G}_e$ is not yet satisfied and there is an element $x \in (W_{e,s+1} \cap C^{s+1}) - A^{s+1}_0$ with $x \notin \{d^{s+1}_0, \ldots d^{s+1}_e\}$, such that no $\mathcal{F}_i$ with $i < e$ wants to put x into A_0. Put this x into A^{s+1}_1. This satisfies $\mathcal{G}_e$ forever.

Let $A^{s+1} = A^{s+1}_0 \cup A^{s+1}_1$. This completes the construction.

Lemma 3.4 *$C - A$ is infinite.*

Proof. We prove by induction on e that $d_e = \lim_s d_e^s$ exists. Assume that this holds for all markers d_i with $i < e$, and let $s_0 \geq e$ be a stage such that $d_i^{s_0} = d_i$ for all $i < e$. Now each $\mathcal{F}_j$, $\mathcal{G}_j$, $\mathcal{P}_j$, and $\mathcal{Q}_j$ with $j > e$ cannot put any of the elements $d_0^s, \ldots d_e^s$ into A_1 at stage $s+1$, so none of these requirements ever moves the marker d_e^s. Also, each $\mathcal{G}_i$, $\mathcal{P}_i$, and $\mathcal{Q}_i$ with $i \leq e$ puts at most one element into A, hence moves the markers at most once. Let $s_1 \geq s_0$ be a stage so large that no $\mathcal{G}_i$, $\mathcal{P}_i$, or $\mathcal{Q}_i$ with $i \leq e$ moves any markers at any stage $s \geq s_1$.

By the construction, d_e^s can only be moved at stage $s \geq s_1$ by a requirement $\mathcal{M}_i$ or $\mathcal{F}_i$ with $i \leq e$. Furthermore, when $\mathcal{F}_i$ $(i \leq e)$ moves a marker, it puts an element into A_0, so it is satisfied at that point. Before then it may have tried to put finitely many other elements into A_0 as well, and any of them may go into A_0 or A_1 at a later stage, moving markers in the process. However, since there are only finitely many such elements, d_e is moved only finitely many times on behalf of $\mathcal{F}_i$.

Now $\mathcal{M}_0$ moves d_e at most 2^{e+1} times after stage s_1: once to put d_0 into V_0, possibly twice to put d_1 into V_0, and so on. Once $\mathcal{M}_0$ has finished moving d_e, $\mathcal{M}_1$ moves it at most 2^e more times, to put markers into V_1. Similarly, once each $\mathcal{M}_i$ has moved d_e for the last time, $\mathcal{M}_{i+1}$ may move it at most 2^{e-i} more times. Hence we eventually reach a stage s_2 after which d_e never is moved again. Possibly $d_e^{s_2}\uparrow$, but since C is infinite and every d_i with $i < e$ has already converged to its limit, we know that d_e^t will be defined at some stage $t > s_2$. Since it never moves again, this yields $d_e^t = \lim_s d_e^s$. ∎

Lemma 3.5 *For each e, the requirements $\mathcal{N}_e$, $\mathcal{P}_e$, $\mathcal{Q}_e$, $\mathcal{F}_e$, and $\mathcal{G}_e$ are all satisfied.*

Proof. We proceed by induction on e. Assume the lemma holds for all $i < e$. We write α for the pair coded by e, and prove first that $\mathcal{N}_\alpha$ is satisfied. Suppose $(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1$ and $B_\alpha \subseteq C$ and $S_\alpha \sqcup \hat{S}_\alpha = C$. Then $F(\alpha)$ holds and Z_α is infinite. The construction of T_α then guarantees that $\overline{C} \subseteq T_\alpha$. Let G_α be the intersection of all those V_i with $i < \alpha$ such that V_i is infinite, and let $\hat{T}_\alpha = T_\alpha \cap G_\alpha$. Thus $\overline{C} \subseteq \hat{T}_\alpha$, since $\overline{C} \subseteq V_i$ whenever V_i is infinite.

Sublemma 3.6 *For each α and each $n < \alpha$, there are only finitely many $x \in \hat{T}_\alpha$ such that $\mathcal{M}_n$ ever wants to put x into A_0.*

Proof. First, if V_n is finite, then $\mathcal{M}_n$ will only want to put finitely many elements into A_0. So we may assume that V_n is infinite, and hence that $\hat{T}_\alpha \subseteq V_n$.

If $\mathcal{M}_n$ wants to put x into A_0 at stage s, then $x \in C^s - A^s$, so $x = d_k^s$ for some k. Moreover, there must be an i with $n < i \leq k$ and a $j > k$ such that $\sigma(n, d_i^s, s) = \sigma(n, d_j^s, s)$ and $d_i^s \notin V_n^s$ and $d_j^s \in V_n^s$. Furthermore, d_i is the leftmost marker which any $\mathcal{M}$-requirement wants to put into A_0 at stage s, and n and j satisfy the minimality requirements of the construction.

Now if $d_k^s \notin V_n^s$, then $d_k^s \notin V_n$, since $C \searrow V_n = \emptyset$, and hence $d_k^s \notin \hat{T}_\alpha$. Therefore we may assume $d_k^s \in V_n^s$. (This guarantees $k \neq i$). Then minimality of n forces $\sigma(n, d_i^s, s) \geq \sigma(n, d_k^s, s)$, and minimality of j forces $\sigma(n, d_i^s, s) > \sigma(n, d_k^s, s)$ (since $d_k^s \in V_n^s$). Hence there is some $m < n$ such that $\sigma(m, d_i^s, s) = \sigma(m, d_k^s, s)$ and $d_i^s \in V_m^s$ and $d_k^s \notin V_m^s$. This forces $d_i^s \in V_m$ and $d_k^s \notin V_m$ (since $d_k^s \in C^s - V_m^s$). If V_m is infinite, then $d_k^s \notin \hat{T}_\alpha$. But if V_m is finite, then d_i^s lies in the finite set

$$V = \bigcup \{V_m : m < n \ \&\ V_m \text{ finite}\}.$$

Hence we need only find a stage t so large that for every $d \in V$, either $d \in A_0^t$ or $\mathcal{M}_n$ wants to put d into A_0 at stage t or $\mathcal{M}_n$ never wants to put d into A_0. Then $\mathcal{M}_n$ will never want to put into A_0 any $x > \max(C^t)$ with $x \in \hat{T}_\alpha$.

■

We will show that the conclusion of $\mathcal{N}_\alpha$ holds for $\hat{T}_\alpha$:

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 =^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

Once we have established this for all α, clearly $R(A_0, A_1)$ itself must hold, since for each α we can choose another $\hat{T}_\alpha$ which excludes the (finite) difference set of the two sides and still contains $\overline{C}$.

Suppose first that $x \in A_0 \cap S_\alpha \cap \hat{T}_\alpha$ and $x \notin A_1$, and assume that x is sufficiently large that:

- $x > |Z_\beta|$ for every $\beta < \alpha$ such that Z_β is finite, and
- No $\mathcal{F}_i$ with $i < \alpha$ ever tries to put x into A_0, and
- No $\mathcal{M}_i$ with $i < \alpha$ ever tries to put x into A_0.

The last condition is possible by Sublemma 3.6. Notice also that the first condition forces $x \notin T_\beta$ for all $\beta < \alpha$ with $|Z_\beta|$ finite.

Then for all s, either $p(x, s) \geq \alpha$ or $p(x, s)\uparrow$. But since $x \in A_0$, we know that some $p(x, s)\downarrow$. For the least such s we have $x \in C^s$, and hence $x \in T_\alpha^s$, since $C \cap T_\alpha \subseteq T_\alpha \searrow C$.

Now α satisfies conditions (a)-(c) in the construction at stage s, since $F(\alpha)$ holds and $x \in S_\alpha$. So there must exist $\beta = \langle i, j' \rangle \leq \alpha = \langle i, j \rangle$ which satisfies (a)-(d) at stage s.

We claim that this β satisfies conditions (a)-(d) at every stage after s as well. Since $x \in T^s_\beta$, we know that Z_β is infinite and $F(\beta)$ holds, by choice of x. Hence (a) and (c) hold at all subsequent stages. Let t be the first stage at which $q(x,t)$ converged. Then $x \in C^t$, and $x \in T^t_\beta$ since $C \searrow T_\beta = \emptyset$. By the definition of q, we must have had $x \in S^t_\beta \cup \hat{S}^t_\beta$. But $x \notin \hat{S}^s_\beta$ since (b) holds at stage s, and because $s > t$, this forces $x \in S^t_\beta$, so (b) always holds of β.

To show that (d) always holds of β, we choose an arbitrary $\gamma < \beta$ witht he same first coordinate as β. Since β satisfies (d) at stage s, γ must fail one of (a)-(c) at stage s. If γ fails (a) or (b) at stage s, then clearly it fails that same consition at every subsequent stage. Moreover, if γ fails (c) at stage s, then $x \notin T^s_\gamma$, and since $x \in C^s$, this forces $x \notin T_\gamma$. Thus β will always satisfy condition (d).

But since $x \in A_0$, there must also be a stage s' with $q(x,s') = -1$. Since (a)-(d) continue to hold of β, the only way for $q(x,s') < \beta$ to occur is for x to enter B_β. (Recall that for all s, either $p(x,s) \geq \alpha$ or $p(x,s)\uparrow$.) But $B_\beta = W_i = B_\alpha$ since $\beta = \langle i, j' \rangle$ and $\alpha = \langle i, j \rangle$, so this forces $x \in B_\alpha$. Hence

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 \subseteq^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

Now suppose that $x \in B_\alpha \cap S_\alpha \cap \hat{T}_\alpha$ and $x \notin A_1$, and assume x is greater than $\max(d_0, \ldots d_\alpha)$, and also greater than the greatest finite $|Z_\beta|$ with $\beta < \alpha$. (Thus $x \notin T_\beta$ for all such β.) Now $x \in C$ since $S_\alpha \subseteq C$, so at some stage s_0, x will enter C and be given a marker: say $x = d^{s_0}_k$. So $x \in C^{s_0}$, and since $x \in T_\alpha$, this forces $x \in T^{s_0}_\alpha$.

If $x \notin A_0$, then we must have $x \in D_\alpha$, since $(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1$ and $x \notin A_1$. (Notice that then x, being in $C - A$, eventually receives some permanent marker $d_{k'}$, with $k' > \alpha$ by choice of x.) For x to have entered D_α, there must have been a stage $s_1 \geq s_0$ with $q(x,s_1) = \gamma = \langle i, j' \rangle$, where $\alpha = \langle i, j \rangle$. (Also, then $p(x,s_1)\downarrow$, and since $x \notin A_1$, $p(x,s)\downarrow$ for all $s \geq s_1$.) But α satisfies conditions (a)-(c) at all stages $s \geq s_0$, so by condition (d) on γ, we must have $\gamma \leq \alpha$. The assumption $x \notin A_0 \cup A_1$ then means that there is some $s_2 > s_1$ such that $q(x,s)\downarrow = q(x,s_2)$ for all $s \geq s_2$. Let $\beta = q(x,s_2) \leq \gamma$. Then $x \in D_\beta - B_\beta$, and furthermore β satisfies the conditions (a)-(d) at all stages $s \geq s_2$.

Now $x \in T_\beta$, to satisfy condition (c), so $x < |Z_\beta|$ and $\beta \leq \gamma \leq \alpha$. If $\beta = \alpha$, then Z_β is infinite since $F(\alpha)$ holds, and if $\beta < \alpha$, then Z_β must be infinite, by our choice of x. Therefore $F(\beta)$ holds, and in particular $S_\beta \sqcup \hat{S}_\beta = C$. Now $x \notin \hat{S}_\beta$ by condition (b), so $x \in S_\beta$. However, with $x \in D_\beta - B_\beta$, this contradicts $F(\beta)$. Hence $x \in A_0$, and

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 \subseteq^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

This completes our proof that $\mathcal{N}_\alpha$ is satisfied.

Now we continue with the other requirements. Let s_0 be a stage such that no $\mathcal{P}_i$, $\mathcal{Q}_i$, $\mathcal{F}_i$, or $\mathcal{G}_i$ with $i < e$ tries to put any element into A_0 or A_1 at any stage after s_0. ($\mathcal{F}_i$ is different from the other requirements in that it may try to put more than one element into A_0. It only stops trying when one of those elements succeeds in entering A_0. We choose s_0 so that every element which $\mathcal{F}_i$ wants to put into A_0 either is in A^{s_0} or never enters A.) Assume also that s_0 is sufficiently large that $d_i^{s_0} = d_i$ for every $i \leq e$.

Now if $W_e \searrow A$ is infinite, then there must be an x in some $W_{e,s} - A^s$ with $s > s_0$ and $\{d_0, \ldots d_e\}$. No requirement of higher priority will need to put this x anywhere, except possibly some $\mathcal{M}_i$, and according to our construction, $\mathcal{G}_e$ does not respect the priority of the requirements $\mathcal{M}_i$, so $x \in A_1^{s+1}$, and $\mathcal{G}_e$ is satisfied.

Similarly, if W_e is infinite, then there must be an x and an $s > s_0$ such that $x \in W_{e,s} - W_{e,s-1}$ and $x \in C^{v(s)}$, by prompt simplicity of C. If this x is not already in $A^{v(s)-1}$, then the construction puts it into $A_1^{v(s)}$, so $\mathcal{P}_e$ holds. Also, there must be an x and an $s > s_0$ with $x \in W_{e,s} - W_{e,s-1}$ such that $C^s \restriction x \neq C^{w(s)} \restriction x$, by promptness of C. Thus there is a $y < x$ which entered C at some stage t with $s < t \leq w(s)$. We must have $y \notin A^{t-1}$ since $A^{t-1} \subseteq C^{t-1}$. But now $y \notin \{d_0^t, \ldots d_e^t\}$, since these markers had reached their limits by stage s_0 and y only entered C at stage t. Hence the construction will put this y into A_1^t, and $A_1^{w(s)} \restriction x \neq A_1^s \restriction x$, satisfying $\mathcal{Q}_e$.

Continuing with the induction, we need a sublemma to handle $\mathcal{F}_e$.

Sublemma 3.7 *For this e and for all sufficiently large x, if $\mathcal{F}_e$ wants to put x into A_0 at some stage, then $x \in A_0$.*

Proof. Choose x so large that it satisfies all of the following:

1. $x > \max\{|Z_\beta| : \beta \leq e \ \& \ Z_\beta \text{ is finite}\}$.

2. No $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{P}_i$, or $\mathcal{Q}_i$ with $i < e$ ever wants to put x into A_0 or A_1.

3. $x \notin \{d_0, \ldots d_e\}$.

Suppose $\mathcal{F}_e$ wants x to enter A_0 at stage s_0. Then $x = d_k^{s_0}$ for some k and $p(x, s_0)\downarrow \leq e$. Now no $\mathcal{G}_j$, $\mathcal{P}_j$, or $\mathcal{Q}_j$ with $j \geq e$ ever manages to put x into A_1, since $\mathcal{F}_e$ takes priority over these. (Since $x \neq d_e$, the only way to have $k \leq e$ is for x eventually to enter A_0. Hence we may assume $k > e$.) Also, for every $\beta < e$, either $x \notin T_\beta$ (if $|Z_\beta| < x$) or $F(\beta)$ holds (if Z_β is infinite). Hence there is an $s_1 \geq s_0$ such that $q(x, s_1)\downarrow$ and $q(x, s_1 + 1)\downarrow \leq e$.

Now suppose $q(x, s) = \beta$ for some $s \geq s_1$ (so $\beta \leq e$). If $F(\beta)$ failed, then Z_β would have to be finite, so $x \notin T_\beta$ (since $|Z_\beta| < x$) and $q(x, s)$ would never equal β. Therefore, $F(\beta)$ must hold. Suppose $x \notin A_0$. If $x \notin S_\beta$, then $x \in \hat{S}_\beta$ by $F(\beta)$ and so $q(x, s_\beta) < \beta$ for some $s_\beta \geq s_1$. Otherwise $x \in D_\beta \cap (S_\beta - A_0) \subseteq B_\beta$ by $F(\beta)$, so $x \in B_\beta^{s_\beta}$ for some $s_\beta \geq s_1$, and hence $q(x, s_\beta) < \beta$. Thus, by induction on $\beta < e$, eventually we must have $q(x, s) = -1$, and so $x \in A_0^{s+1}$, proving the sublemma. ■

Now if $W_e \searrow A$ is infinite, then $\mathcal{F}_e$ has infinitely many elements at its disposal to try to put into A_0. Hence once we find a sufficiently large $x \in W_e \searrow A$, we know by the sublemma that this x will eventually enter A_0, thus satisfying $\mathcal{F}_e$. This completes the induction of Lemma 3.5. ■

Lemma 3.8 *The requirements $\mathcal{M}_e$ are all satisfied by our construction.*

Proof. Suppose that $\overline{C} \subseteq W_e$. To prove that $\mathcal{M}_e$ holds, we must show $\overline{A} \subseteq^* W_e$. By induction we assume that $\mathcal{M}_i$ holds for all $i < e$. Let

$$\sigma = \{i < e : \overline{C} \subseteq W_i\}.$$

Now if $i \in \sigma$, then also $\overline{C} \subseteq V_i$, so by inductive hypothesis $\overline{A} \subseteq^* V_i$, whereas if $i \notin \sigma$ (and $i < e$), then V_i is finite. Hence for all but finitely many k we have $\sigma(e, d_k) = \sigma$.

Now let $V_\sigma = V_e \cap \big(\bigcap\{V_i : i \in \sigma\}\big)$. Then $\overline{C} \subseteq V_\sigma$. But C, being promptly simple, is noncomputable, so $V_\sigma \searrow C$ must be infinite. Choose y so large that no element $\geq y$ can be held out of A_0 forever by any requirement $\mathcal{N}_\alpha$ with $\alpha \leq e$, and let s_0 be a stage such that $C^{s_0}\restriction y = C\restriction y$.

Suppose for a contradiction that $\overline{V_e} \cap (C - A)$ is infinite. Then there exists p such that $d_p \notin V_e$ with p so large that $d_p \notin C^{s_0}$ and with $\sigma(e, d_q) = \sigma$. (Hence $d_p > y$.) Let s_1 be a stage with $d_p^{s_1} = d_p$ and $\sigma(e, d_p, s_1) = \sigma$. Now since $V_\sigma \searrow C$ is infinite, there will be a stage $s > s_1$ at which some element $x \in V_\sigma^{s-1}$ enters C, and is assigned the marker d_q^s (with $q > p$ since $d_p^{s_0} = d_p$). Moreover, we may assume that q is sufficiently large that not only is d_q^s in V_σ, but that $\sigma(e, d_q^s, s) = \sigma$, since every V_i with $i < e$ and $i \notin \sigma$ is finite. Since $d_q^s \in V_\sigma \subseteq V_e$ and $d_p \notin V_e$, $\mathcal{M}_e$ will want to put d_p into A_0

at stage s, and since $d_p > y$, no negative requirement will keep d_p out of A_0. Possibly d_p will be diverted into A_1 by some requirement $\mathcal{G}_j$, $\mathcal{P}_j$, or $\mathcal{Q}_j$, since these do not respect the priority of $\mathcal{M}_e$. If so, then d_p will enter A_1; if not, then d_p will enter A_0. Either way, d_p enters A, contradicting our assumption that the marker d_p had reached its limit at stage s_0.

Hence $\overline{V_e} \cap (C - A)$ is finite, and $\overline{A} \subseteq (C - A) \cup \overline{C} \subseteq^* V_e \subseteq W_e$. Thus $\mathcal{M}_e$ is satisfied, and the lemma is proven. ■

Knowing that the requirements are all satisfied, we can easily complete the proof of the theorem. The construction ensured that $A_0 \cap A_1 = \emptyset$, and the conjunction of all the $\mathcal{F}_i$ and $\mathcal{G}_i$ implies that $A_0 \sqcup A_1$ is a Friedberg splitting of A. (See pp. 181-182 of [16].) The requirements $\mathcal{P}_i$ together make A a promptly simple set, by definition, and the $\mathcal{Q}_i$ together allow A_1 to satisfy the Promptly Simple Degree Theorem (Thm. XIII.1.6 of [16]), so that A_1 is of prompt degree. To prove that $R(A_0, A_1)$ holds, we note that the requirements $\mathcal{M}_i$, along with Lemma 3.4, show that $A = A_0 \sqcup A_1$ is a major subset of C. Moreover, given a $B = W_i$ and a pair $(S_{j'}, \hat{S}_{j''})$ with $S_{j'} \sqcup S_{j''} = C$, we have the D_i and T_α (with $\alpha = \langle i, \langle j', j'' \rangle \rangle$) constructed above. If

$$(B_i \cap (S_{j'} - A_0)) \cup A_1 = (D_i \cap (S_{j'} - A_0)) \cup A_1,$$

then $F(\alpha)$ holds. Since $\mathcal{N}_\alpha$ is satisfied, we know that there exists a T with $\overline{C} \subseteq T$ such that

$$(A_0 \cap S_{j'} \cap T) \cup A_1 =^* (B_i \cap S_{j'} \cap T) \cup A_1.$$

So we can pick a sufficiently large n_α, and let

$$T' = \{x \in T : x \geq n_\alpha\} \cup \{x \in \overline{C} : x < n_\alpha\}.$$

Then $\overline{C} \subseteq T'$ and also $(A_0 \cap S_{j'} \cap T') \cup A_1 = (B_i \cap S_{j'} \cap T') \cup A_1$, since $S_{j'} \cap \overline{C} = \emptyset$. Thus $R(A_0, A_1)$ holds. Finally, since A is a major subset of the set C, A must be of high degree (see [10], page 214). ■

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