# Computable Reducibility for Cantor Space

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December 17, 2016

**Abstract**

We examine various versions of Borel reducibility on equivalence relations on the Cantor space $2^\omega$, using reductions given by Turing functionals on the inputs $A \in 2^\omega$. In some versions, we vary the number of jumps of $A$ which the functional is allowed to use. In others, we do not require the reduction to succeed for all elements of the Cantor space at once, but only when applied to arbitrary finite or countable subsets of $2^\omega$. In others we allow an arbitrary oracle set in addition to the inputs. All of these versions, inspired largely by work on computable reducibility on equivalence relations on $\omega$, combine to yield a rich set of options for evaluating the precise level of difficulty of a Borel reduction, or the reasons why a Borel reduction may fail to exist.

## 1 Introduction to Reducibility

The subject of reducibility of equivalence relations has bifurcated in recent years. Much early work was devoted to the topic of *Borel reducibility*, concerning equivalence relations on the Cantor space $2^\omega$ of all subsets of the set $\omega$ of natural numbers. More recently, computability theorists have adapted the notion in order to address equivalence relations on $\omega$ itself. The principal notion here has borne several names, after being arrived at independently by several researchers; we find *computable reducibility* to be the most natural of these. The purpose of this article is to hybridize the two: we will present natural reducibilities on equivalence relations on $2^\omega$, some stronger than Borel reducibility and some weaker, through which the ideas cultivated by computability theorists can be applied.

Suppose that $E$ and $F$ are equivalence relations on the domains $S$ and $T$, respectively. A *reduction* of $E$ to $F$ is a function $g : S \to T$ satisfying the property:

$$(\forall x_0, x_1 \in S)\ [x_0\ E\ x_1 \iff g(x_0)\ F\ g(x_1)].$$

The point is that, if one has the ability to compute $g$ and to decide the relation $F$, then one can decide $E$ as well. Thus $E$ may be considered to be "no harder to decide" than $F$, at least modulo the difficulty of computing $g$. In practice, the domains $S$ and $T$ are usually equal, with the cases $S = T = 2^\omega$ and $S = T = \omega$ being by far the most widely studied. (It is clear that, in order for a reduction to exist, $F$ must have at least as many equivalence classes as $E$, and so a situation where

*The author was partially supported by Grant # DMS – 1362206 from the National Science Foundation, and by several grants from The City University of New York PSC-CUNY Research Award Program.

$T$ has lower cardinality than $S$, while allowed by the definition, would usually be uninteresting to investigate.)

The crucial question in this definition is how much computational power one allows the function $g$ to have. Of course, whenever $F$ has at least as many equivalence classes as $E$, some reduction $g$ of $E$ to $F$ must exist, unless one refuses to allow use of the Axiom of Choice. In set theory, the standard preference has been to require $g$ to be a Borel function, in which case $S$ and $T$ should be Polish spaces. We say that $E$ is *Borel-reducible to* $F$, and write $E \leq_B F$, if a Borel reduction from $E$ to $F$ exists. Research here has focused on the situation $S = T = 2^\omega$. Computability theorists seized on the same notion under the requirement that $g : \omega \to \omega$ be Turing-computable, in which case one needs $S = T = \omega$. In this situation, by analogy, we say that $E$ is *computably reducible to* $F$, and write $E \leq_{\mathbf{0}} F$, if a Turing-computable reduction from $E$ to $F$ exists.

The notation $E \leq_T F$ seemed inadvisable here, as it already denotes the existence of a Turing reduction from the set $E$ to the set $F$. One does have $E \leq_T F$ whenever $E \leq_0 F$, but the converse fails. In fact, $E \leq_0 F$ implies that there is a many-one reduction from $E$ to $F$ as sets, and for this reason $\leq_0$ has sometimes been called $m$-reducibility, as well as Fokina-Friedman reducibility. We prefer the term which actually describes the complexity of the reduction involved: $E$ is computably reducible to $F$ if there is a reduction which is a computable function, just as Borel-reducibility requires a reduction which is a Borel function. Certain other work, such as [13], has used generalizations such as $\boldsymbol{d}$-computable reductions, i.e., reductions that are $\boldsymbol{d}$-computable functions on $\omega$, for some Turing degree $\boldsymbol{d}$. Likewise, in [7], Fokina, Friedman, and Törnquist studied "effectively Borel" reductions on equivalence relations on $2^\omega$, by which they meant reductions which are $\Delta^1_1$.

The goal of this article is to enable the transfer of many of the computability-theoretic results from articles such as [3] and [13] to the context of Borel reducibility, i.e., of equivalence relations on Cantor space. We employ the very natural notion of a computable function on Cantor space, which yields the following definition.

**Definition 1.1** *Let $E$ and $F$ be equivalence relations on $2^\omega$. A* computable reduction *of $E$ to $F$ is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function $\Phi$ (that is, an oracle Turing functional) on the reals involved:*

$$(\forall A \in 2^\omega)(\forall x \in \omega)\ \chi_{g(A)}(x) = \Phi^A(x).$$

*If such a reduction exists, then $E$ is* computably reducible to $F$, *denoted $E \leq_0 F$.*

So we require, for all reals $A_0$ and $A_1$, that $A_0\ E\ A_1$ if and only if $\Phi^{A_0}\ F\ \Phi^{A_1}$. It is implicit here that, for every $A$, $\Phi^A$ should be a total function from $\omega$ into $\{0, 1\}$, equal to the characteristic function of $g(A)$, hence regarded as the set $g(A)$ itself.

The notion of a computable reduction on equivalence relations on Cantor space may be generalized using the jump operator, which maps each set $A \subseteq \omega$ to its *jump* $A' = \{\langle e, x\rangle : \Phi^A_e(x) \text{ halts}\}$. (Here $\langle x, y\rangle = \frac{(x+y+1)(x+y)+2y}{2}$ is the standard pairing function on $\omega$.) This is best seen as representing the Halting Problem relative to the set $A$.

**Definition 1.2** *Let $E$ and $F$ be equivalence relations on $2^\omega$. A* jump-reduction *of $E$ to $F$ is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function $\Phi$ (that is, an oracle Turing functional) on the jumps of the reals involved:*

$$(\forall A \in 2^\omega)\ g(A) = \Phi^{(A')}.$$

*Likewise, if for some computable ordinal $\alpha$ and some $\Phi$, we have $g(A) = \Phi^{(A^{(\alpha)})}$, then the reduction $g$ is said to be an* $\alpha$-jump reduction. *We write $E \leq_\alpha F$ if such a reduction exists, thus generalizing the notation $E \leq_0 F$ above.*

Another refinement of reducibilities on equivalence relations was introduced by Ng and the author in [13]. Studying equivalence relations on $\omega$, they defined *finitary reducibilities*. In the context of Cantor space, it is natural to extend their notion to all cardinals $\mu < 2^\omega$ (as indeed was suggested in their article), yielding the following definitions.

**Definition 1.3** *For equivalence relations $E$ and $F$ on domains $S$ and $T$, and for any cardinal $\mu < |S|$, we say that a function $g : S^\mu \to T^\mu$ is a $\mu$-*ary reduction *of $E$ to $F$ if, for every $\vec{x} = (x_\alpha)_{\alpha \in \mu} \in S^\mu$, we have*

$$(\forall \alpha < \beta < \mu)\ [x_\alpha \ E \ x_\beta \iff g_\alpha(\vec{x}) \ F \ g_\beta(\vec{x})],$$

*where $g_\alpha : S^\mu \to T$ are the component functions of $g = (g_\alpha)_{\alpha < \mu}$. For limit cardinals $\mu$, a related notion applies with $<\!\mu$ in place of $\mu$: a function $g : S^{<\mu} \to T^{<\mu}$ which restricts to a $\nu$-ary reduction of $E$ to $F$ for every cardinal $\nu < \mu$ is called a $(<\!\mu)$-*ary reduction*. (For $\mu = \omega$, an $\omega$-ary reduction is a* countable reduction*, and a $(<\!\omega)$-ary reduction is a* finitary reduction*.)*

*When $S = T = 2^\omega$ and the $\mu$-ary reduction $g$ is computable, we write $E \leq_0^\mu F$, with the natural adaptation $E \leq_\alpha^\mu F$ for $\alpha$-jump $\mu$-ary reductions. Likewise, when a $(<\mu)$-ary reduction $g$ is $\alpha$-jump computable, we write $E \leq_\alpha^{<\mu} F$, When $\alpha > 0$, it is important to note that $\Phi^{((\vec{x})^{(\alpha)})}$ is required to equal $g(\vec{x})$; this allows more information in the oracle than it would if we had required $\Phi^{((x_0^{(\alpha)} \oplus x_1^{(\alpha)} \oplus \cdots)} = g(\vec{x})$, with the jumps of the individual inputs taken separately.*

Notice that computable $\mu$-ary reductions only make sense when $\mu \leq \omega$, as we have no method for running an oracle Turing machine with uncountably many reals in the oracle. For $\mu$-ary reductions with $\mu \leq \omega$, the oracle is a single real whose columns are the $\mu$-many inputs to the reduction. In a $(<\omega)$-ary reduction, with an input $\vec{A} \in (2^\omega)^n$, the oracle is officially equal to $\{n\} \oplus A_0 \oplus \cdots \oplus A_{n-1} \oplus \emptyset \oplus \emptyset \oplus \cdots$, meaning that the oracle does specify the size of its tuple. (However, we usually gloss over this issue and just write $A = A_0 \oplus \cdots \oplus A_{n-1}$ as the oracle.)

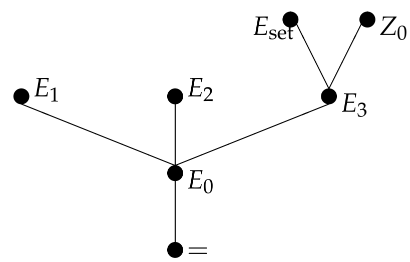
Beyond the basic intention of introducing computable reductions on equivalence relations on Cantor space, the principle goal of this article is to use these ideas to analyze the Borel reductions and non-reductions at the lowest levels in the $\leq_B$-hierarchy of Borel equivalence relations. Where Borel reductions exist, we wish to determine whether they are computable, and, if not, how many jumps away from computable they may be. Where no Borel reduction exists, we ask how close we can come to a full Borel reduction, using the notions of finitary and countable reducibility, and possibly allowing jump-reductions as well. Often our results here are based on existing results from the context of equivalence relations on $\omega$, from sources including [3, 7, 13], as well as the relevant articles [1, 2, 5, 6, 8, 9, 12].

Intuitively, results in this direction suggest where the obstacles to Borel reductions lie. If $E \not\leq_0^\omega F$, then there is an obstruction at the level of computability, i.e., at a syntactic level. If, for all $\alpha$, $E \not\leq_\alpha^\omega F$, then this obstruction extends upwards through all possible quantifier complexities, meaning that the reduction is stymied by extremely strong syntactic difficulties. On the other hand, if $E \not\leq_B F$ but $E \leq_0^\omega F$, then the obstruction is not syntactic, but rather has to do with the size of the continuum: it is simply not possible to perform the entire reduction uniformly on so many elements of the field of $E$. If $E \not\leq_B F$ and the best possible countable reduction is $E \leq_4^\omega F$, say, then 4-quantifier complexity is necessary, but beyond that the obstructions are cardinality-related rather than syntactic. Later in this article, we will also encounter situations where $E \leq_0^{<\omega} F$ but $E \not\leq_0^\omega F$. We will even meet natural cases where $E \leq_0^3 F$ but $E \not\leq_0^4 F$. These suggest more subtle syntactic issues.

A future goal is to begin to transfer the large body of knowledge about computable structures into the more general context of arbitrary countable structures on the domain $\omega$. In particular, equivalence relations such as isomorphism, bi-embeddability, and elementary equivalence have been analyzed extensively for many classes of computable structures, and may be more broadly considered as equivalence relations on the corresponding classes of countable structures, often using jumps and/or finitary or countable reducibility.

## 2 Analyzing the Basic Borel Theory

The following diagram appeared in [3]. It shows all Borel reducibilities among the main Borel equivalence relations at the base of the hierarchy of Borel reducibility. (Their definitions will be given as we examine them in this section.)




The surprising fact about this diagram is that $E_0$ is the second-least Borel equivalence relation under $\leq_B$, even when one includes relations not shown here. Indeed, the broader principle, known as the *Glimm-Effros dichotomy*, is that for every Borel relation $E$ on $2^\omega$ to which equality is Borel-reducible, either $E$ is Borel-equivalent to equality or else $E_0 \leq_B E$. That is, no Borel equivalence relation sits strictly between $=$ and $E_0$ under $\leq_B$. This has been extended in various ways, for instance to other Polish spaces, but here we will content ourselves with this version. An excellent description of the broader topic appears in Gao's book [8, Ch. 6]. Glimm was the first to make significant progress in this area, in [10]; Effros followed in [4] with work showing that $E_0$ embeds continuously into every non-smooth equivalence relation (on a Polish space) defined by the action of a Polish group; and Harrington, Kechris, and Louveau gave the effective version that has come to be the usual meaning of "Glimm-Effros dichotomy" in the present day (see [11]).

As a first example of our style of investigation, therefore, we start with the fact that $= <_B E_0$. Of course, this is really two statements: there is a Borel reduction in one direction, but not in the other. The Borel reduction is in fact a computable reduction, given immediately by the program

$$\Phi^A(\langle n,k\rangle) = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if not.} \end{cases}$$

That is, each real $A$ is transformed into the real whose columns are all just copies of $A$, so that any difference between two sets $A$ and $B$ becomes an infinite difference between $\Phi^A$ and $\Phi^B$.

Thus $= <_0 E_0$: the strictness is clear, since there is not even any Borel reduction in the reverse direction. Nevertheless, we can analyze the reverse direction using finitary and countable reductions, and we find that there is a 2-jump countable reduction $\Gamma$ from $E_0$ to $=$, given as follows. $\Gamma$ has as its oracle the second jump of the join of countably many sets $A_0, A_1, \ldots$, and needs to output corresponding sets $B_0, B_1, \dots$ for which $B_i = B_j$ iff $A_i\ E_0\ A_j$. First it defines $B_0 = \emptyset$. Next, with the second jump of the join, $\Gamma$ can ask whether $A_0 \triangle A_1$ is finite; if so, it sets $B_1 = \emptyset$ as well, while if not, it makes $B_1 = \{1\}$. In general, for each $j$, $\Gamma$ asks its oracle whether $A_{j+1}$ has finite difference with any of $A_0, \dots, A_j$. If so, then it makes $B_{j+1}$ equal to that $B_i$ (noting that if there is more than one such $i$, then all such $B_i$ are already equal); while if not, it makes $B_{j+1} = \{j+1\}$. Clearly this is a countable reduction, as desired.

Of course, we also want to know whether this is the best we can do. There is no way to address larger cardinalities than $\omega$ with a Turing functional, even if CH fails, but one might hope for an $n$-ary or countable 1-jump or computable reduction. We now show that a 2-jump countable reduction is the best possible. Let $A$ be any 1-generic set, and $\Gamma$ any Turing functional. Writing $A = A_0 \oplus A_1$, we know from 1-genericity that $A_0$ and $A_1$ have infinite symmetric difference, so if $\Gamma$ is to be a 1-jump binary reduction of $E_0$ to $=$, then $\Gamma^{A'} = \Gamma^{(A_0 \oplus A_1)'}$ must be the characteristic function of a set $B_0 \oplus B_1$ with $B_0 \neq B_1$. Fix an $n$ in the difference, say with $n \in B_0 - B_1$, so $2n \in B$ and $(2n+1) \notin B$. Then there is an initial segment $\sigma \subseteq A'$ such $\Gamma^\sigma(2n) = 1$ and $\Gamma^\sigma(2n+1) = 0$. For every $i$ with $\sigma(i) = 1$, fix some $\rho_i \subseteq A$ for which $\Phi_i^{\rho_i}(i)\downarrow$, and for every $i$ with $\sigma(i) = 0$, fix some $\rho_i \subseteq A$ such that $(\forall \tau \supseteq \rho_i)\ \Phi_i^\tau(i)\uparrow$. (Such a $\rho_i$ must exist, by the 1-genericity of $A$.) Now let $\rho = \cup_{i<|\sigma|}\rho_i$, which is a finite initial segment of $A$. By our choice of each $\rho_i$, we see that every $C \supseteq \rho$ will have $C' \upharpoonright |\sigma| = \sigma$, and hence $\Gamma^{C'}(2n) = \Gamma^\sigma(2n) = 1$ and $\Gamma^{C'}(2n+1) = \Gamma^\sigma(2n+1) = 0$. However, there are many $C \supseteq \rho$ such that, with $C = C_0 \oplus C_1$, we will have $C_0\ E_0\ C_1$: for instance, just let $C = \rho\widehat{\ }00000\cdots$. But for these $C$ we will still have $\Gamma^{C'} = D_0 \oplus D_1$ with $n \in D_0 - D_1$, so $D_0 \neq D_1$, and thus $\Gamma$ does not compute a 1-jump binary reduction from $E_0$ to $=$.

$E_1$ is the equivalence relation defined on $2^\omega$ by sets being equal on all but finitely many of their columns:

$$A_0\ E_1\ A_1 \iff \forall^\infty m\ A_0^m = A_1^m,$$

where $A^m$ represents the $m$-th column $\{k : \langle m,k\rangle \in A\}$ of $A$. From the Borel theory we know that $E_0 <_B E_1$. Indeed, the Borel reduction is a computable reduction: with $\Phi^A = \{\langle m,0\rangle : m \in A\}$, we have $A_0\ E_0\ A_1$ if and only if $\Phi^{A_0}\ E_1\ \Phi^{A_1}$. In the reverse direction, no Borel reduction exists, but there is a computable countable reduction.

**Proposition 2.1** $E_1 \leq_0^\omega E_0$. *That is, there is a computable countable reduction $\Gamma$ from $E_1$ to $E_0$.*

*Proof.* The input to $\Gamma$ is a real $A = \oplus_n A_n$, viewed as an $\omega$-tuple of reals $A_n$, and the $m$-th column of $A_n$ is now written $A_n^m$. (In fact, $A_n$ itself is the $n$-th column of $A$, but this is not a concern.) The real computed by $\Gamma^A$ will be $B = \oplus_n B_n$, so we need $A_n\ E_1\ A_{n'}$ if and only if $B_n\ E_0\ B_{n'}$.

The idea behind $\Gamma$ is that, if $A_m$ and $A_n$ differ on a given column, that column should contribute a single element to the symmetric difference $B_m \triangle B_n$. The $\langle m,n\rangle$-th column $B_i^{\langle m,n\rangle}$ of each set $B_i$ is devoted to collecting these single elements so as to satisfy the requirement

$$\mathcal{R}_{m,n}: \quad B_m^{\langle m,n\rangle}\ E_0\ B_n^{\langle m,n\rangle} \iff A_m\ E_1\ A_n,$$

On the sets $B_i$ (for $i \notin \{m,n\}$), $\Gamma$ will try to keep $B_i^{\langle m,n\rangle}$ equal to one of $B_m^{\langle m,n\rangle}$ and $B_n^{\langle m,n\rangle}$, depending on which of $A_m$ and $A_n$ may be $E_1$-equivalent to $A_i$. (For simplicity, all columns numbered $\langle m,n\rangle$ with $m \geq n$ are empty in every set $B_i$.)

At each stage $s$ of the computation, for every $c$ and $i$, $\Gamma$ decides whether the $s$-th element $\langle c,s\rangle$ of the $c$-th column should belong to $B_i$. Suppose $c = \langle m,n\rangle$. If $m \geq n$, or if $c > s$, the answer is automatically no for all $i$, as noted above. Otherwise, we use the oracle to compare $A_m$ with $A_n$ up to $s$, looking for columns on which they have just now been discovered to differ, as defined here.

- For each $i < s$, if $A_m^i \upharpoonright s = A_n^i \upharpoonright s$ but $s \in A_m^i \triangle A_n^i$, then we say that on column $i$, $A_m$ and $A_n$ *differ at* $s$, meaning that $s$ is the first stage at which it was established that these columns are distinct.
- $A_m^s \upharpoonright (s+1) \neq A_n^s \upharpoonright (s+1)$, then we say that on column $s$, $A_m$ and $A_n$ *differ at* $s$.

So at stage $s$ we "catch up" on one new column, and keep an eye on all previous ones.

Let $s'$ be the greatest stage $< s$ at which we acted on behalf of the $\langle m, n\rangle$-th column. If there exists a partition $P \sqcup Q$ of the set $\{0, 1, \ldots, n\}$ with $m \in P$ and $n \in Q$ such that, for every $p \in P$ and $q \in Q$, there is some $t$ with $s' < t \leq s$ and some column $\leq s$ on which $A_p$ and $A_q$ differ at stage $t$, then we fix such a partition and define $\langle\, \langle m, n\rangle, s\rangle$ to lie in every $A_p$ with $p \in P$ (including $A_m$), but not in any other $A_i$ (hence not in $A_n$). Thus we have created a new difference (of one new element) between $A_m^{\langle m,n\rangle}$ and $A_n^{\langle m,n\rangle}$, and we say that we *acted on behalf of* $\mathcal{R}_{m,n}$ at this stage. If no such partition exists, then $\langle\, \langle m, n\rangle, s\rangle$ does not lie in any set $A_i$.

As a matter of convention, we also deem ourselves to have acted (for the first time) on behalf of $\mathcal{R}_s$ at stage $s$, although we did nothing specific to help satisfy it. This completes the construction.

To see that $\Gamma$ is a countable reduction, consider any $A$ and any $c = \langle m, n\rangle$ with $m < n$. If $A_m$ and $A_n$ lie in distinct $E_1$-classes, then we can partition $\{0, \ldots, n\}$ into two classes $P$ and $Q$, with $m \in P$ and $n \in Q$, so that if $A_i\ E_1\ A_j$, then $(i \in P \iff j \in P)$. Hence, if $i \in P$ and $j \in Q$, there will be infinitely many stages at which $A_i$ and $A_j$ differ on some column. Therefore, there will be infinitely many stages at which we discover either this partition $P \sqcup Q$ or some other one and act on behalf of column $c$, putting an element into $B_m$ but not into $B_n$. Thus $B_m$ and $B_n$ lie in distinct $E_0$-classes, as desired.

On the other hand, if $A_m\ E_1\ A_n$, then these two sets differ on only finitely many columns. Therefore there exists some $s_0$ such that we never act on behalf of column $c$ after that stage. Now by stage $s_0$ we have only defined $B_m$ and $B_n$ up to their $s_0$-th element of the $d$-th column (that is, up to $\langle d, s_0\rangle$), and this only for $d \leq s_0$, so there are only finitely many differences between $B_m$ and $B_n$ so far. After stage $s_0$, we may add some further differences between them, on behalf of columns $d \neq c$, but only if the previous action on behalf of column $d$ was at a stage $s' \leq s_0$, since otherwise our $m$ and $n$ would have to lie either both in $P$ or both in $Q$, according to the rules in the construction. However, only finitely many columns (those $\leq s_0$) ever had an action taken on their behalf before stage $s_0$, and so only finitely many more differences will ever be added to $B_m$ and $B_n$. Hence $B_m\ E_0\ B_n$ as desired. ∎

Next we consider $E_3$, the equivalence relation which holds of reals $A$ and $B$ if and only if, for all $k$, the $k$-th columns of the two satisfy $A^k\ E_0\ B^k$. One quickly sees a computable reduction from $E_0$ to $E_3$: just let $\Phi^A = A \oplus A \oplus \cdots$. It is known that $E_1 \not\leq_B E_3$, and in fact the two are Borel-incomparable, but there is a countable computable reduction from $E_1$ to $E_3$, since Proposition 2.1 yields $E_1 \leq_0^\omega E_0 \leq_0 E_3$. From $E_3$ to $E_0$, no Borel reduction exists, and the best we can do is to show $E_3 \leq_2^\omega E_0$. Indeed, we have $E_3 \leq_2^\omega =$, with the two-jump countable reduction to equality given by the following functional $\Gamma$. With oracle $A'' = (\oplus_n A_n)''$, $\Gamma$ makes every $\langle m, n, k\rangle \notin B_m$, and defines:

- if $(A_n^k \triangle A_m^k)$ is finite, then for all $p$, $\langle m, n, k\rangle \notin B_p$.
- if $(A_n^k \triangle A_m^k)$ is infinite, then $\langle m, n, k\rangle \in B_n$. (Here we have found that $A_n$ and $A_m$ are in distinct $E_3$-classes, so $\langle m, n, k\rangle$ establishes $B_n \neq B_m$.)

In this case, for each $p \notin \{m, n\}$, we "do no harm." $\Gamma$ checks whether $(A_n^k \triangle A_p^k)$ is finite. If it is, then $\Gamma$ puts $\langle m, n, k\rangle \in B_p$ (since possibly $A_p\ E_3\ A_n$, while definitely $A_p$ and $A_m$ are in distinct $E_3$-classes). If $(A_n^k \triangle A_p^k)$ is infinite, then $\Gamma$ defines $\langle m, n, k\rangle \notin B_p$ (since here the reverse holds). Notice that if $A_{p_0}\ E_3\ A_{p_1}$ for some $p_0 < p_1$, then either the columns $A_{p_i}^k$ both have finite difference with $A_n^k$, or else both have infinite difference with it. Thus no harm has been done.

So we have $E_3 \leq_2^\omega = \leq_0 E_0 \leq_0 E_1$. It remains to show that there is no 1-jump reduction (not even a binary reduction) from $E_3$ to $E_1$. (This will imply $E_3 \not\leq_1^2 E_0$ as well, since $E_0 \leq_0 E_1$.)

Here we introduce a new technique: the complexity argument. Relative to reals $A$ and $B$, the relation $E_3$ is $\Pi_3$, whereas $E_1$ is only $\Sigma_2$. This immediately suggests that there should be no computable reduction of $E_3$ to $E_1$, and since one is $\Pi$ and the other is $\Sigma$, there should not even be a 1-jump reduction, although the indices (in "$\Sigma_2$" and "$\Pi_3$") differ by only one.

To formalize this, let $D$ be the set of pairs $\{(i,j) \in \omega^2 : W_i = \overline{W_j}\}$. $D$ itself is only a $\Pi_2$ subset of $\omega^2$, and complete at that level under 1-reducibility, but it serves our purpose, when we think of $(i,j)$ as not just representing a decidable set $W_i$, but actually providing a decision procedure for $W_i$. We abuse notation by writing $(i_0, j_0)\ E_1\ (i_1, j_1)$ to denote $W_{i_0}\ E_1\ W_{i_1}$, viewing this as the restriction of $E_1$ to the class of decidable sets; similarly for $E_3$. As noted above, this makes $E_1$ a $\Sigma_2$ relation on $D$, and $E_3$ a $\Pi_3$ relation there. We now show that $E_3$ is $\Pi_3$-complete there, using the $\Sigma_3$-complete set $\text{Cof} = \{e \in \omega : W_e \text{ is cofinite}\}$, by giving a computable function $f : \omega \to \omega^2$ such that, for all inputs $e$, we have $e \notin \text{Cof}$ if and only if $f(e) = (i,j)$ gives the $D$-index of a set $E_3$-equivalent to $\emptyset$ (that is, just if $(i,j)\ E_3\ (i_0, j_0)$, for some fixed indices with $W_{i_0} = \emptyset$ and $W_{j_0} = \omega$). This is a simple movable-marker construction: on input $e$, write $\overline{W_{e,s}} = \{a_{0,s} < a_{1,s} < \cdots\}$ for each stage $s$, and say as usual that the $n$-th marker "moves" whenever $a_{n,s+1} \neq a_{n,s}$. We enumerate the $n$-th column of $W_i$ by watching this marker. If it moves at stage $s$, then $\langle n, s\rangle \in W_i$, while if not, then $\langle n, s\rangle \in W_j$. So, if $W_e$ is coinfinite, every marker moves only finitely often, and every column of $W_i$ is finite, making $W_i\ E_3\ \emptyset$; while if $e \in \text{Cof}$, then some marker moves infinitely often, and some column of $W_i$ is infinite, destroying this $E_3$-equivalence.

Now suppose there were a 1-jump binary reduction $\Gamma$ of $E_3$ to $E_1$. Then, for every $e$, we have $e \notin \text{Cof}$ if and only if $f(e) = (i,j)\ E_3\ (i_0, j_0)$, which holds if and only if $\Gamma^{(W_i \oplus W_{i_0})'}$ outputs two sets which are $E_1$-equivalent. So, using a $\emptyset'$-oracle, we can take any $e$, give a decision procedure for $W_i$ (where $(i,j) = f(e)$), use our oracle to produce a decision procedure for $(W_i \oplus W_{i_0})'$, and thus run $\Gamma^{(W_i' \oplus W_{i_0})'}$ on any natural-number input. This enables us to compute (below our $\emptyset'$-oracle), uniformly in $(i,j)$, a set $(B_0 \oplus B_1)$, the output of $\Gamma^{(W_i' \oplus W_{i_0})'}$, with the property that $B_0\ E_1\ B_1$ if and only if $e \notin \text{Cof}$. However, this $E_1$-equivalence is $\Sigma_2^{B_0 \oplus B_1}$, hence $\Sigma_2^{\emptyset'}$, hence $\Sigma_3$, and thus the property of being coinfinite would be $\Sigma_3^0$. Since this property is known to be $\Pi_3^0$-complete, no such reduction can exist.

The preceding argument easily adapts to prove the following theorem.

**Theorem 2.2** *Suppose $E$ is an equivalence relation on $2^\omega$ whose restriction $E_D$ defined on the set $D = \{(i,j) \in \omega^2 : W_i = \overline{W_j}\}$ by*

$$(i,j)\ E_D\ (p,q) \iff W_i\ E\ W_p$$

*is $\Pi_k^0$-complete within $D$ as a set under 1-reducibility. (That is, $E_D$ is $\Pi_k^0$-definable, and every $\Pi_k^0$-definable subset $T \subseteq D$ has $T \leq_1 E_D$.) Let $F$ be any $\Sigma_n^0$-definable equivalence relation on $2^\omega$, with $n \leq k$. Then $E \not\leq_{k-n}^2 F$.* ∎

It is important here to bear in mind that elements of $D$ are given by pairs $(i,j)$, $(p,q)$, etc., not just by $i$ or $p$. Hence, when one addresses the question whether $W_i \; E \; W_p$, the statement $(x \in W_i \iff x \in W_p)$ is decidable, not just $\emptyset'$-decidable, although it might appear to be the latter. We enlarge on this comment in Section 4.

# 3 Completeness Results

As we continue with our analysis of the basic Borel equivalence relations, we come next to $E_2$. For this it will be useful to have a notion of completeness. We give a definition which is semantical rather than syntactical.

**Definition 3.1** *A (finitary) formula $\varphi(\vec{x},\vec{A})$ with free number variables $x_0,\ldots,x_k$ and free set variables $A_0,\ldots,A_m$ is said to be $\Delta^0_1$ if there is a Turing functional $\Phi$ such that for all sets $A_0,\ldots,A_m \subseteq \omega$, the truth of $\varphi(\vec{x},\vec{A})$ is decided, uniformly in $\vec{x} \in \omega^k$, by the function $\Phi^{A_0\oplus\cdots\oplus A_m}$.*

*This is then extended through the computable ordinals in the usual way. If $\varphi(\vec{x},y,\vec{A})$ is $\Sigma^0_\alpha$, then its negation is $\Pi^0_\alpha$. A computable (possibly infinitary) disjunction of $\Pi^0_\alpha$ formulas is $\Sigma^0_{\alpha+1}$. For limit ordinals $\lambda < \omega_1^{CK}$, the $\Sigma^0_\lambda$ formulas are the computable (possibly infinitary) disjunctions $\bigvee\!\!\!\bigvee_i \varphi_i$, where each $\varphi_i$ is $\Sigma^0_{\alpha_i}$ for some $\alpha_i < \lambda$. A formula is $\Delta^0_\alpha$ if it is equivalent to both a $\Sigma^0_\alpha$ and a $\Pi^0_\alpha$ formula.*

It is not difficult to give an equivalent syntactical formulation for $\Delta^0_1$, in which atomic formulas of the form $x_i \in A_j$ are allowed. If we allowed quantification over set variables, then we would use the symbols $\Sigma^1_n$ and $\Pi^1_n$, but in this article we have no need of them.

The next definition is standard, and it too has been used in the preceding sections.

**Definition 3.2** *A $\Pi^0_\alpha$ equivalence relation $E$ on $2^\omega$ is* $\Pi^0_\alpha$-complete *under a reducibility $\leq$ if every $\Pi^0_\alpha$-definable equivalence relation $F$ satisfies $F \leq E$. Here $\leq$ is usually of the form $\leq^\kappa_\beta$, where $\beta$ is a computable ordinal (most often $0$) and $\kappa$ a cardinal (or possibly $\kappa$ can be replaced by $< \kappa$, as in finitary reducibility).*

In the Borel theory, the equivalence relation known as $E_2$ is defined using the harmonic series:

$$A \; E_2 \; B \iff \sum_{n \in A \triangle B} \frac{1}{n+1} < \infty.$$

This relation is known to be Borel-incomparable with both $E_1$ and $E_3$, but it is $\Sigma^0_2$, and so the next result is not surprising.

**Proposition 3.3** *$E_2 \equiv^\omega_0 E_1$. That is, there are countable computable reductions in both directions.*

*Indeed, each of $E_0$, $E_1$, and $E_2$ is complete under $\leq^\omega_0$ among $\Sigma^0_2$-definable equivalence relations on $2^\omega$.*

*Proof.* We first prove the $\Sigma_2$-completeness statement for $E_1$. Let $E$ be any $\Sigma^0_2$-definable equivalence relation on $2^\omega$. Standard methods allow us to collapse like quantifiers, so that we may take $E$ to be defined by using a quantifier-free formula $\varphi$ with two number variables and two set variables:

$$A \; E \; B \iff (\exists x \in \omega \forall y \in \omega) \; \varphi(x,y,A,B).$$

Our reduction $\Gamma$ accepts $A = \oplus_n A_n$ as its oracle and outputs $B = \oplus_n B_n$. For each $m < n$, the columns numbered $\langle m,n,c\rangle$ of each $B_i$ are used to distinguish $B_m$ and $B_n$ under $E_1$ if necessary. At stage $s$, consider each $m < n < s$ in turn, and suppose that $\Gamma$ has aleady made each of

$B_m^{\langle m,n,0\rangle}, \ldots, B_m^{\langle m,n,c-1\rangle}$ distinct from the corresponding $B_n$-column. This will hold just if $\Gamma$ has already found witnesses $y_0, \ldots, y_{c-1}$ such that $(\forall i < c)\neg\varphi(i, y_i, A_m, A_n)$, but has not yet found such a witness $y_c$ for $c$. At this stage, we use the oracle set to check whether $\varphi(c, y, A_m, A_n)$ holds, where $y$ is the least possible witness which has not yet been checked. If indeed $\varphi(c, y, A_m, A_n)$ holds, then we define $s \notin B_i^{\langle m,n,d\rangle}$ for every $i$ and every $d$. If $\neg\varphi(c, y, A_m, A_n)$ holds, then we define $s \in B_m^{\langle m,n,c\rangle}$ but $s \notin B_n^{\langle m,n,c\rangle}$. Thus $B_m$ and $B_n$ now differ on their $\langle m, n, c\rangle$-th column (and at the next stage, the value of $c$ for this $\langle m, n\rangle$ will be incremented by 1). We also define $s \notin B_i^{\langle m,n,d\rangle}$ for every $d \neq c$ and every $i$. It remains to decide whether $s \in B_i^{\langle m,n,c\rangle}$ for $i \notin \{m, n\}$. Consider such an $i$, and find the greatest $x_m \leq s$ and the greatest $x_n \leq s$ such that

$$(\forall x < x_m)(\exists y \leq s)\ \neg\varphi(x, y, A_i, A_m) \quad \& \quad (\forall x < x_n)(\exists y \leq s)\ \neg\varphi(x, y, A_i, A_n).$$

Then $\Gamma$ defines $s \in B_i^{\langle m,n,c\rangle}$ if and only if $x_m < x_n$. (With $x_m < x_n$, we have "more" evidence that $A_i$ and $A_n$ lie in distinct $E$-classes than that $A_i$ and $A_m$ do, and so $\Gamma$ makes $B_i$ look like $B_m$ on this column. If the evidence so far points the other way, $\Gamma$ makes $B_i$ look like $B_n$ instead.)

For each $\langle m, n\rangle$ with $m \geq n$ or $n \geq s$, we define $s \notin B_i^{\langle m,n,d\rangle}$ for every $i$ and $d$. This completes the construction at stage $s$.

The reduction $E_1 \leq_0^\omega E_0$, from Proposition 2.1, now shows $E_0$ to be $\Sigma_2^0$-complete as well. To complete the proof of Proposition 3.3, we prove that $E_0 \leq_0 E_2$. This is easy, and was already known (if never actually stated) from the existing Borel reduction. We know that, for every $n$, $\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdot + \frac{1}{2^{n+1}} \geq 1$. So, given $A$, define $\{2^n + 1, 2^n + 2, \ldots, 2^{n+1}\} \subseteq \Phi^A$ if and only if $n \in A$, with $\{2^n + 1, 2^n + 2, \ldots, 2^{n+1}\} \cap \Phi^A = \emptyset$ otherwise. Then $\Phi$ is the desired computable reduction of $E_0$ to $E_1$. ■

To finish our analysis of $E_2$, we state the more general lemma established above.

**Lemma 3.4** $E_0 \leq_0 E_2$. ■

**Proposition 3.5** $E_3 \leq_2^\omega\ =$.

*Proof.* The reduction $\Gamma$ has as its oracle $A = (\oplus_n A_n)''$. It simply defines $\langle m, c\rangle \in B_n$ if and only if the columns $A_m^c$ and $A_n^c$ have finite symmetric difference. This works. ■

For those accustomed to the Borel theory, this could be a surprise. The strict inequalities

$$= \ <_B\ E_0\ <_B\ E_3$$

both hold, and for each of them, 2-jump countable reducibility was the best possible reducibility for the reverse. So it might seem that $E_3$ should be farther away from equality than $E_0$ is. Nevertheless, there is no contradiction here: while Borel reducibility is far stronger than any countable reducibility, its absence does not stop us from composing the reductions in the reverse direction. (In general, however, for $\alpha > 0$, the composition of two $\alpha$-jump computable reductions need not itself be $\alpha$-jump computable.)

The equivalence relation $E_{\text{set}}$ compares the columns of the two reals involved:

$$A\ E_{\text{set}}\ B \iff (\forall n \exists m)\ A^n = B^m.$$

(Notice that a given column of $A$ or $B$ is not required to appear exactly the same number of times as a column of each. $E_{\text{set}}$ simply says that every column of $A$ is equal to some column of $B$ and vice versa.) In the Borel hierarchy, we have $E_3 <_B E_{\text{set}}$, and both of these two relations are defined by $\Pi^0_3$ formulas in $A$ and $B$. We can readily give a binary computable reduction in the reverse direction.

**Lemma 3.6** *$E_{set} \leq^2_0 E_3$.*

*Proof.* The reduction $\Gamma$, with oracle $A_0 \oplus A_1$, defines the $(2m)$-th column of $B_0$ and $B_1$ by searching for a column of $A_1$ equal to $A^m_0$. To begin with, it sets $x \notin B^{2m}_0 \cup B^{2m}_1$ for every $x$ such that $(\forall y \leq x)[y \in A^0_0 \iff y \in A^0_1]$. If it ever finds an $x_0$ with $[x_0 \in A^0_0 \iff x_0 \notin A^0_1]$, then it makes $x_0 \in B^{2m}_0$ but $x_0 \notin B^{2m}_1$, creating a difference between the two columns. Starting with $x_0 + 1$, it then compares $A^0_0$ with $A^1_1$ (starting from 0), the same way, and keeps making $B^{2m}_0$ and $B^{2m}_1$ the same until it finds a difference between $A^0_0$ and $A^1_1$. If it ever finds a difference here, then it begins comparing $A^0_0$ with $A^2_1$, and so on. Thus $B^{2m}_0 \triangle B^{2m}_1$ will be finite if and only if $A^0_0$ is equal to some column in $A_1$. Each column numbered $(2m+1)$ in $B_0$ and $B_1$ is used likewise to check whether the column $A^m_1$ appears as a column in $A_0$. ■

Since $E_{\text{set}}$ and $E_3$ are $\Pi^0_3$ relations, however, it is more difficult to extend the idea of Lemma 3.6 beyond a binary computable reduction. To explain our approach, we now describe a ternary computable reduction.

**Lemma 3.7** *$E_{set} \leq^3_0 E_3$.*

*Proof.* Consider an oracle $A = A_0 \oplus A_1 \oplus A_2$. For each $(c, m, n)$ with $c \in \omega$, $m \leq 2$, $n \leq 2$, and $m \neq n$, the following procedure builds the $\langle m, n, c, d\rangle$-th columns of the output $B_0$, $B_1$, and $B_2$, for every $d$. (For all other triples $(c, m, n)$, these columns are all empty.) Pick $p$ so that $\{m, n, p\} = \{0, 1, 2\}$. The $\langle m, n, c, 0\rangle$-th column is the *base column*, and we begin with the $\langle m, n, c, 1\rangle$-th column as the first *working column*; subsequently this may switch to $\langle m, n, c, k+1\rangle$ for $k = 1, 2, \ldots$. As in Lemma 3.6, we watch first for a difference between $A^c_m$ and $A^0_n$; if we ever find one, then we watch for a difference between $A^c_m$ and $A^1_n$, then $A^2_n$, etc. As long as no new difference appears at a stage $s$, we make all these columns of $B_0$, $B_1$, and $B_2$ agree, of course. Suppose that at stage $s$ we find a difference between $A^c_m$ and the current $A^i_n$; in this case we call $s$ an *$\langle m, n, c\rangle$-stage* and write $s' < s$ for the previous $\langle m, n, c\rangle$-stage (or 0). Check whether $A^c_m$ and the current column $A^k_p$ of $A_p$ show any difference up to $s$.

1. If $A^c_m \restriction s = A^k_p \restriction s$, then we make $s \in B^{\langle m,n,c,k+1\rangle}_m$, $s \notin B^{\langle m,n,c,k+1\rangle}_n$ and $s \in B^{\langle m,n,c,k+1\rangle}_p$ where $k$ is the number of the current working column; we keep this number $k$ fixed, but increment $i$ by 1. (Since $A_p$ appears as though its $k$-th column might be a copy of $A^c_m$, we aim here to keep $B_p$ $E_3$-equivalent to $B_m$, while taking a step towards making $B_m$ and $B_n$ $E_3$-inequivalent.)

2. If $A^c_m \restriction s \neq A^k_p \restriction s$, then we make $s \in B^{\langle m,n,c,0\rangle}_m$, $s \notin B^{\langle m,n,c,0\rangle}_n$ and $s \notin B^{\langle m,n,c,0\rangle}_p$, and increment each of $i$ and $k$ by 1, thus moving to a new working column for the next stage. (Here again we take a step towards making $B_m$ and $B_n$ $E_3$-inequivalent, by adding one more distinction between them in the base column. Now we try to keep $B_p$ $E_3$-equivalent to $B_n$, since $A^k_p$ has turned out not to match $A^c_m$)

If $A_m^c = A_n^i$ for some $i$, then there are only finitely many $\langle m,n,c\rangle$-stages in all, and none of the columns numbered $\langle m,n,c,k\rangle$ makes any two of $B_m$, $B_n$, or $B_p$ $E_3$-inequivalent. If this fails for every $i$, there are two cases. If some $k$ exists with $A_m^c = A_p^j$, then we will have reached the least such $k$ and then stayed in Step (1) at all subsequent $\langle m,n,c\rangle$-stages, thus making $B_p$ $E_3$-inequivalent to $B_n$ but allowing it still to be $E_3$-equivalent to $B_m$. (With $A_p^k = A_m^c$, which does not appear as a column of $A_n$, this case has $A_n$ and $A_p$ $E_{\text{set}}$-inequivalent, so this outcome is acceptable.) If no such $k$ exists, then we executed Step (2) at infinitely many $\langle m,n,c\rangle$-stages. Each time we did, we created a further difference between $B_p^{\langle m,n,c,0\rangle}$ and $B_m^{\langle m,n,c,0\rangle}$, and also between $B_n^{\langle m,n,c,0\rangle}$ and $B_m^{\langle m,n,c,0\rangle}$, so the base column shows $B_m$ to be $E_3$-inequivalent to both the others. However, $B_p^{\langle m,n,c,0\rangle} = B_n^{\langle m,n,c,0\rangle}$, and on each working column, $B_p^{\langle m,n,c,k+1\rangle}$ and $B_m^{\langle m,n,c,k+1\rangle}$ have only a finite difference, since eventually the construction reached Step (2) again and incremented $k$. So in this case we have not done anything to make $B_p$ $E_3$-inequivalent to $B_n$.

This completes the construction, and the argument above makes it clear that for any column $c$ which appears in any $A_m$ but not in some other $A_n$, the outputs $B_m$ and $B_n$ will have infinite difference on their $\langle m,n,c,d\rangle$-th column, for some $d$. On the other hand, if $A_m$ $E_{\text{set}}$ $A_n$, then no infinite difference between corresponding columns of $B_m$ and $B_n$ will ever have been created, leaving $B_m$ $E_3$ $B_n$. Thus we have a computable ternary reduction from $E_{\text{set}}$ to $E_3$. ■

To make this into a finitary reduction (say a $(j+2)$-ary reduction), one does a similar process for each $\langle m,n,c\rangle$, assessing whether $A_m^c$ appears as a column of $A_n$. Let $\{0,1,\ldots,j+1\} = \{m,n\} \sqcup \{p_0 < p_1 < \cdots < p_{j-1}\}$. We have in each $B_i$, for each $\langle m,n,c\rangle$, one column $C_\sigma$ for each $\sigma \in \omega^j$. The column $C_\sigma$ will be the column which distinguishes $B_m$ from $B_n$ under $E_3$ (unless $A_m$ $E_{\text{set}}$ $A_n$) in the situation where

$$(\forall i < j)\ [\sigma(i) = 0 \iff A_m^c \text{ does not appear as a column in } A_{p_i}]$$

and where $\sigma(i) = k+1$ if and only if $A_{p_i}^k = A_m^c$ (with $k$ minimal). At an $\langle m,n,c\rangle$-stage $s$, the construction chooses the lexicographically-least $\sigma$ which agrees with the situation at $s$, in the sense that there exists an $\langle m,n,c\rangle$-stage $s' < s$ satisfying:

- $\sigma(i) = k+1 \implies A_m^c$ agreed with the column $A_{p_i}^k$ at stage $s'$ and still agrees with it; and
- $\sigma(i) = 0 \implies A_m^c$ no longer agrees with the column of $A_{p_i}$ with which it appeared to agree at stage $s'$; and
- $\sigma$ has not been chosen since stage $s'$.

We then add a difference between $B_m$ and $B_n$ on the column $C_\sigma$, by making $s \in B_m^{\ulcorner\sigma\urcorner}$ but $s \notin B_n^{\ulcorner\sigma\urcorner}$. Each $B_{p_i}$ with $\sigma(i) = k+1$ gets $s \in B_{p_i}^{\ulcorner\sigma\urcorner}$, while each $B_{p_i}$ with $\sigma(i) = 0$ has $s \notin B_{p_i}^{\ulcorner\sigma\urcorner}$. The only $\sigma$ for which any of the columns $B_m^{\ulcorner\sigma\urcorner}$, $B_n^{\ulcorner\sigma\urcorner}$, or $B_{p_i}^{\ulcorner\sigma\urcorner}$ (for any $i$) can be infinite is the "true" $\sigma$, for which $\sigma(i) = k+1$ when $k$ is least with $A_{p_i}^k = A_m^c$ and $\sigma(i) = 0$ if there is no such $k$. On this column, each $A_{p_i}$ with $\sigma(i) = 0$ has $B_n^{\ulcorner\sigma\urcorner}$ $E_0$ $B_{p_i}^{\ulcorner\sigma\urcorner}$, and the other $i$ all have $B_m^{\ulcorner\sigma\urcorner}$ $E_0$ $B_{p_i}^{\ulcorner\sigma\urcorner}$. The reader can check, using the method from Lemma 3.7, that this does give a $(j+2)$-ary reduction, uniformly in $j$, and so we have proven our next result.

**Proposition 3.8** $E_{set} \leq_0^{<\omega} E_3$. ■

However, this method does *not* generalize to a countable collection $A = \oplus_{n\in\omega} A_n$ of sets. Having a column $C_\sigma$ for every $\sigma \in \omega^j$ worked perfectly well for finite $j$, but with $\omega$ in place of $j$ we would need uncountably many columns. So this is our first example where finitary reducibility has come into focus; until now, every finitary reduction could readily be made into a countable reduction, whereas here, imitating the proof from [13, Theorem 2.6], we show that no computable countable reduction exists.

**Theorem 3.9** $E_{set} \not\leq_0^\omega E_3$.

*Proof.* We will show that no Turing functional $\Gamma$ can be a computable countable reduction. Fix such a functional $\Gamma$, which accepts any oracle $A = \oplus_{n\in\omega} A_n$ and outputs $B = \oplus_n B_n$.

Let $A_0^0 = \omega$ and $A_0^{m+1} = A_1^m = [0, m]$ for each $m$, so $A_0$ and $A_1$ are $E_{\text{set}}$-inequivalent. Also set $A_n^{2m+1} = [0, m]$ for each $n > 1$ and each $m$. It remains to define the elements of the even-numbered columns $A_n^{2m}$ for $n > 1$. We intend for each of these columns to be an initial segment of $\omega$ (possibly all of $\omega$) at the end of the construction; so far, each $A_n^{2m}$ is an initial segment of length 0, and is allowed to be extended further. We proceed according to an $\omega$-ordering $\prec$ of the pairs $\langle i, j\rangle$ with $\langle i, j\rangle \prec \langle i, j+1\rangle$ for all $i$ and $j$. For each such pair, in order, we ask whether it is possible to extend the (currently uncapped) columns $A_n^{2m}$ (for $n > 1$) further so as to make $\Gamma^A$ output a set $B$ with $(B_1^i \triangle B_{i+2}^i) \cap \{j, j+1, \ldots\} \neq \emptyset$. If so, then we extend these columns (finitely far) so as to make this happen, and "cap" the columns $A_{i+2}^{2k}$ with $k \leq j$, decreeing these columns to equal the finite initial segments of $\omega$ to which they have already been extended. If not, then we extend every currently uncapped column $A_{i+2}^{2k}$ with $k \leq j$ by one more element.

The key feature of this construction is that, at every step $\langle i, j\rangle$, every column $A_{i+2}^{2k}$ with $k \leq j$ either is capped or receives another element. Therefore, this process defines, for each $x$, whether or not $x$ lies in each even-numbered column of $A_{i+2}$, for each $i$. Thus we have now defined the entire set $A$. Moreover, for a fixed $i$, either every column $A_{i+2}^{2k}$ (for all $k$) was eventually capped – if at every step $\langle i, j\rangle$ it was always possible to extend $A$ and create one more difference between $B_1^i$ and $B_{i+2}^i$ – or else some column of $A_{i+2}$ is infinite, which occurs if we reached a step at which it was no longer possible for any extension of $A$ to create such a difference.

Suppose that, for some $i$, $A_1 \; E_{\text{set}} \; A_{i+2}$. Then every column of $A_{i+2}$ is finite, which happens just if, for every $j$, the construction always extended $A$ so as to make $B_1^i$ and $B_{i+2}^i$ differ on some number $\geq j$. If this happened, then $B_1$ and $B_{i+2}$ are $E_3$-inequivalent, and therefore $\Gamma$ was not a reduction from $E_{\text{set}}$ to $E_3$. On the other hand, suppose that, for every $i$, $A_1$ and $A_{i+2}$ are $E_{\text{set}}$-inequivalent. Since every column of $A_{i+2}$ is an iniital segment of $\omega$, and every finite initial segment occurs there, each $A_{i+2}$ must have an infinite column $A_{i+2}^{2j}$, and therefore $A_{i+2} \; E_{\text{set}} \; A_0$ for every $i$. If $\Gamma$ were a reduction, we would then have $B_{i+2} \; E_3 \; B_0$ for all $i$, so that $B_{i+2}^i \; E_0 \; B_0^i$ for all $i$. But the infinite column of $A_{i+2}$ arose because $B_1^i \triangle B_{i+2}^i$ is finite, and so $B_1^i \; E_0 \; B_0^i$ for every $i$. This ensures $B_1 \; E_3 \; B_0$, which again shows $\Gamma$ not to be a reduction, proving the theorem. ■

Finally, we consider the standard Borel equivalence relation $Z_0$. For a real $A$, the *upper density* of $A$ is defined to equal

$$\limsup_{k\to\infty} \frac{|A \cap \{0, \ldots, k\}|}{k+1},$$

found by asking what fraction of the first $k$ elements of $\omega$ lie in $A$, as $k \to \infty$. The *lower density* is defined similarly, using the liminf, and if these two are equal, then their common value is the *density* of $A$. This allows us to define

$$A \; Z_0 \; B \iff A \triangle B \text{ has density } 0.$$

$Z_0$ is another $\Pi^0_3$ equivalence relation on $2^\omega$, strictly above $E_3$ and incomparable with $E_{\text{set}}$ under Borel reducibility. The usual Borel reduction from $E_3$ to $Z_0$ is computable, and all other questions about $Z_0$ within our various hierarchies in this article are then answered by the following proposition.

**Proposition 3.10** *$Z_0$ is computably countably bireducible with $E_3$: $Z_0 \equiv^\omega_0 E_3$. Hence $Z_0 <^\omega_0 E_{set}$ and $Z_0 \equiv^{<\omega}_0 E_{set}$.*

*Proof.* As we mentioned, the usual Borel reduction from $E_3$ to $Z_0$ is computable: essentially, one converts each element $\langle m, n\rangle$ of $A$ into a finite block of elements $E_{m,n}$ in the output $B$, in such a way that every output $B$ has lower density 0 (since many elements do not belong to any block $E_{m,n}$, hence do not lie in $B$), and has upper density $\geq \frac{1}{n}$ if and only if $A^n$ was infinite. (The latter condition is ensured by the blocks $\langle m, n\rangle$, for all $m$.)

To show that $Z_0 \leq^\omega_0 E_3$, we give a functional $\Gamma$, which as usual accepts an oracle $A = \oplus_n A_n$ and outputs $B = \oplus_n B_n$. For each triple $\langle m, n, c\rangle \in \omega^3$ with $m > n$ and $c > 0$, we use column number $\langle m, n, c\rangle$ in $B_m$ and $B_n$. For those $s$ such that $\frac{(A_m \triangle A_n) \cap \{0,1,\ldots,k\}}{k+1} \geq \frac{1}{c}$, we place $s \in B_m^{\langle 0,1,c\rangle}$ but not in $B_n^{\langle 0,1,c\rangle}$, while the remaining $s$ belong to neither set. Thus all these columns in $B_n$ (for all $c$) are finite, while $B_m$ has an infinite column $B_m^{\langle m,n,c\rangle}$ if and only if there exists $c$ with $\liminf_s \frac{(A_m \triangle A_n) \cap \{0,1,\ldots,s\}}{s+1} \geq \frac{1}{c}$. It follows that, whenever $A_m$ and $A_n$ are $Z_0$-inequivalent, $B_m$ and $B_n$ will be $E_3$-inequivalent.

Next, setting $p_0 = m$, $p_1 = n$, and $\{p_2 < p_3 < \cdots\} = \omega - \{m, n\}$, we decide, for each $i \geq 2$ in turn, whether to add the number $s$ to the corresponding column $B_{p_i}^{\langle m,n,c\rangle}$ of $B_{p_i}$. Of course, if $\frac{(A_m \triangle A_n) \cap \{0,1,\ldots,s\}}{s+1} < \frac{1}{c}$, we leave $s$ out of every such column, so that if $\liminf_t \frac{(A_m \triangle A_n) \cap \{0,1,\ldots,t\}}{t+1} < \frac{1}{c}$, every column $B_p^{\langle m,n,c\rangle}$ will be finite. If $\frac{(A_m \triangle A_n) \cap \{0,1,\ldots,s\}}{s+1} \geq \frac{1}{c}$, then we go through $i = 2, 3, \ldots$ in turn. For each $i$ and each $j < i$, find the least number $d_{j,s} \leq s$ such that

$$(\forall d \in \{d_{j,s}, d_{j,s}+1, \ldots, s\}) \; \frac{(A_{p_i} \triangle A_{p_j}) \cap \{0, 1, \ldots, d\}}{d+1} \leq \frac{1}{c \cdot 2^i}.$$

Not all of these values $d_{0,s}, d_{1,s}, \ldots, d_{i-1,s}$ need exist. If none of them exist, then $s \notin B_{p_i}^{\langle m,n,c\rangle}$. If there is at least one such value, then find the least value among them: say $d_{j_{0,s}}$ (for the least $j_{0,s}$, in case some $d_{j,s} = d_{j',s}$), and set

$$s \in B_{p_i}^{\langle m,n,c\rangle} \iff s \in B_{p_{j_{0,s}}}^{\langle m,n,c\rangle},$$

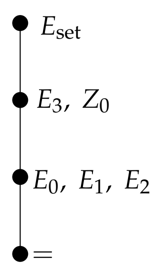
noticing that since $j_{0,s} < i$, the right-hand side here has already been defined. This completes the construction.

The goal of the second part of the construction is to ensure that, if $A_{p_i} \; Z_0 \; A_{p_j}$ for one or more $j < i$, then $B_{p_i}^{\langle m,n,c\rangle} =^* B_{p_j}^{\langle m,n,c\rangle}$ for each such $j$. (By an induction on $i$ separate from the following, one can then see that this holds for every $j < i$.) To see this, we induct on $i$. The claim we prove inductively is slightly stronger: for every $j < i$ such that $A_{p_j} \triangle A_{p_i}$ has upper density $< \frac{1}{c \cdot 2^i}$, we show that $B_{p_j}^{\langle m,n,c\rangle} =^* B_{p_i}^{\langle m,n,c\rangle}$. This is clearly enough to ensure the goal stated above. When $j = 0$ and $i = 1$, the claim is clear, so we consider $i \geq 2$, assuming inductively that the claim holds for all smaller values of $i$. For each $j$ (if any) such that $A_{p_j} \triangle A_{p_i}$ has upper density $< \frac{1}{c \cdot 2^i}$, let $d_j = \lim_s d_{j,s}$, which must exist. (For all other $j < i$, no such number $d_j$ exists, and there is nothing to prove about such a $j$.) Fix the $j$ for which $d_j$ is least; if this does not uniquely define $j$, fix the least such $j$. Then there exists a number $s_0$ such that, for all $s \geq s_0$, every $j' \neq j$ with $j < i$ will have $d_{j',s} > d_{j,s}$ in the construction for $B_{p_i}^{\langle m,n,c\rangle}$ at step $s$ and every $j' > j$ will have $d_{j',s} \geq d_{j,s}$. So, for $s \geq s_0$, the construction ensures that $s \in B_{p_i}^{\langle m,n,c\rangle}$ if and only if $s \in B_{p_j}^{\langle m,n,c\rangle}$, leaving these two columns to differ only finitely, as required.

Now consider any other $j' < i$ such that $A_{p_{j'}} \triangle A_{p_i}$ has upper density $< \frac{1}{c \cdot 2^i}$. It follows that $A_{p_{j'}} \triangle A_{p_j}$ has upper density $< \frac{1}{c \cdot 2^{i-1}} \leq \frac{1}{c \cdot 2^{\max(j,j')}}$. By induction, the greater of $j$ and $j'$ will have ensured that $B_{p_{j'}}^{\langle m,n,c\rangle} =^* B_{p_j}^{\langle m,n,c\rangle}$, and hence $B_{p_{j'}}^{\langle m,n,c\rangle} =^* B_{p_i}^{\langle m,n,c\rangle}$. This completes the induction for $i$.

It now follows that, if $A_m \; Z_0 \; A_n$, then every column of $B_m$ has only finite difference with the corresponding column of $B_n$, and so $B_m \; E_3 \; B_n$. The first part of the construction established the converse, and so this $\Gamma$ is a countable computable reduction from $Z_0$ to $E_3$. The remaining statements in Proposition 3.10 now follow from Proposition 3.8 and Theorem 3.9. ■

Under computable countable reduction, therefore, the equivalence relations discussed here realize the following much simpler diagram:




Under finitary computable reduction (that is, $\leq_0^{<\omega}$), $E_{\text{set}}$ joins the class of $E_3$ and $Z_0$.

# 4 Equivalence Relations Respecting Enumerations

The reader familiar with the works [3] and [13] will see similarities between the constructions there and those here, but also some differences. Our proofs of Theorem 3.9 and Proposition 3.10 will strike most readers as significantly less difficult than the proofs of the corresponding results (Theorems 2.5, 2.6, and 3.3) of [13]. We now explain these differences.

Both [3] and [13] focused on Turing-computable reductions among equivalence relations on $\omega$. They converted each standard Borel equivalence relation $E$ into a relation $E^{ce}$ on $\omega$ using indices of computably enumerable sets $W_e$ (from any standard enumeration of the c.e. sets), under the following definition:

$$(\forall i, j \in \omega) \; [i \; E^{ce} \; j \iff W_i \; E \; W_j].$$

This defines $=^{ce}$, $E_0^{ce}$, $E_1^{ce}$, $E_2^{ce}$, $E_3^{ce}$, $E_{\text{set}}^{ce}$, and $Z_0^{ce}$, and the combined results of [3] and [13] show that the results of Section 2 here largely hold for the c.e. versions of these relations under computable reducibility $\leq_T$. Indeed, [13] shows that $E_{\text{set}}^{ce} \equiv_T^{<\omega} Z_0^{ce}$ but also that $E_{\text{set}}^{ce} <_T^{\omega} Z_0^{ce}$, mirroring our results above for $E_{\text{set}}$ and $Z_0$ on $2^\omega$.

The differences in proofs arise from the fact that for relations $E^{ce}$, one has only enumerations of sets to work with, rather than oracles for the sets themselves. The main reason why the authors of [3] and [13] worked this way is that there is no effective enumeration of the decidable subsets of $\omega$, while there are natural effective enumerations of the c.e. subsets. Our use of oracles overcomes this difficulty, and $\leq_0^{\omega}$ in particular can be restricted to the decidable sets if one likes, or to the $X$-decidable sets for any fixed $X$, giving exactly the context one might have hoped for. Here, however, we go in the other direction, adapting our definitions from Section 1 to the context of sets which are enumerated for us, rather than decided.

**Definition 4.1** *Let $\pi_1 : \omega \to \omega$ be the projection map $\pi_1(\langle x, y\rangle) = x$, using the standard pairing function $\langle x, y\rangle = \frac{(x+y+1)(x+y)+2y}{2}$. An* enumeration *of a set $S \subseteq \omega$ is a set $A \subseteq \omega$ which projects onto $S$, i.e., $\pi_1(A) = S$.*

*An equivalence relation $E$ on $2^\omega$* respects enumerations *if*

$$(\forall A, B \in 2^\omega)\ [\pi_1(A) = \pi_1(B) \implies A\ E\ B].$$

*For each equivalence relation $E$ on $2^\omega$, we define its* enumeration analogue *$E^e$ to be the following equivalence relation on $2^\omega$, which respects enumerations:*

$$A\ E^e\ B \iff \pi_1(A)\ E\ \pi_1(B).$$

It is immediately seen, for every $E$, that $E \leq_0 E^e$, via the reduction $A \mapsto \{\langle n, 0\rangle : n \in A\}$. In the opposite direction, we have a full 1-jump reduction from $E^e$ to $E$ via $A \mapsto \pi_1(A)$, of course, but a quick diagonalization argument shows that in general there need not exist even a binary computable reduction. For a more specific example, we invite the reader to show that the equality relation $=$ is $\Pi_1^0$-complete under computable countable reducibility $\leq_0^{\omega}$, whereas its enumeration analogue $=^e$ is $\Pi_2^0$-complete under computable finitary reducibility $\leq_0^{<\omega}$, but not under computable countable reducibility. Results in [13] are relevant here, using the relation $=^{ce}$ on $\omega$, which is defined by setting $i =^{ce} j$ if and only if $W_i = W_j$, and which was shown there to be complete under computable finitary reducibility among $\Pi_2^0$ equivalence relations on $\omega$.

Enumerations of sets of natural numbers are ubiquitous in mathematics and logic, and so we believe that Definition 4.1 will prove extremely useful. Moreover, it allows us to adapt the existing results in [3] and [13] to prove the following theorems. (Since the ideas of these proofs are essentially identical to those of the original proofs for the $E^{ce}$ versions, we leave them to the reader.)

**Theorem 4.2** *For countable computable reducibility on the enumeration analogues of the usual Borel equivalence relations, we have*

$$=^e \ <_0^{\omega} E_0^e <_0^{\omega} E_3^e <_0^{\omega} Z_0^e,$$

*with $E_1^e \equiv_0^{\omega} E_0^e$. Under finitary computable reducibility, the last of these two merge:*

$$=^e \ <_0^{<\omega} E_0^e <_0^{<\omega} E_3^e \equiv_0^{<\omega} Z_0^e,$$

■

It was shown in [13] that 3-ary and 4-ary reducibility are distinguished by the equivalence relations $E^{ce}_{\max}$ and $E^{ce}_{\text{card}}$, which are the c.e. versions of the relations here.

**Definition 4.3** *For sets $A, B \subseteq \omega$, we define:*

$$(A\ E_{max}\ B \iff \max(A) = \max(B)) \qquad (A\ E_{card}\ B \iff |A| = |B|).$$

*Here we interpret* $\max(\emptyset) = -\infty$*, and* $\max(A) = +\infty$ *for infinite sets $A$. Thus all infinite sets are $E_{max}$-equivalent, and also $E_{card}$-equivalent, while $\emptyset$ forms a singleton class for each of these relations.*

$E_{\max}$ and $E_{\text{card}}$ have some properties we have not seen before in Borel equivalence relations: each has only countably many classes, including one singleton class and just one uncountable class. (Indeed $E_{\max}$ has just one infinite class.) It was shown in [13] that $E^{ce}_{\max}$ and $E^{ce}_{\text{card}}$ are equivalent under computable reducibility (on equivalence relations on $\omega$), and that $E^{ce}_{\max}$ is complete among $\Pi^0_2$ equivalence relations on $\omega$ under 3-ary computable reducibility, but not under 4-ary computable reducibility.

**Proposition 4.4** *The relations $E_{max}$, $E_{card}$, $E^e_{max}$ and $E^e_{card}$ are all $\Pi^0_2$-definable, and $E^e_{max} \equiv_0 E^e_{card}$.*

*Proof.* The $\Pi^0_2$ definability is quick. First, $A\ E_{\max}\ B$ if and only if

$$(\forall n)\ [(\exists x \in A) x \geq n \iff (\exists y \in B) y \geq n],$$

and one readily converts this to a definition of $E^e_{\max}$ as well. Likewise, $A\ E_{\text{card}}\ B$ if and only if

$$(\forall n)\ [(\exists x_1 < x_2 < \cdots < x_n)[\text{all } x_i \in A] \iff (\exists y_1 < y_2 < \cdots < y_n)[\text{all } y_i \in B]],$$

and similarly for $E^e_{\text{card}}$. Notice that, while this second formula certainly defines a $\Pi^0_2$ relation on $A$ and $B$, it is not so easy to make the defining formula strictly finitary in the usual sense, unless one allows the use of an iterated pairing function.

To see that $E^e_{\max} \leq_0 E^e_{\text{card}}$, given $A$, let $\Gamma^A = \{\langle x, \langle y, z\rangle\rangle : y \geq x \ \&\ \langle y, z\rangle \in A\}$. Thus $\pi_1(\Gamma^A) = \{x : (\exists y \geq x)\ y \in A\}$. For the reverse reduction, given $B$, use the extended pairing function $\omega^{<\omega} \to \omega$ and let

$$\Phi^B = \{\langle n, \langle x_0, y_0, \ldots, x_n, y_n\rangle\rangle : \langle x_0, y_0\rangle, \ldots, \langle x_n, y_n\rangle \in B \ \&\ x_0 < \cdots < x_n\}.$$

So $n \in \pi_1(\Phi^B)$ if and only if $|\pi_1(B)| > n$. For $E^e_{\max}$ and $E^e_{\text{card}}$, this $\Gamma$ and this $\Phi$ both work. ■

**Proposition 4.5** *$E_{max}$ is $\Pi^0_2$-complete under computable ternary reducibility: every $\Pi^0_2$-definable equivalence relation $E$ on $2^\omega$ has $E \leq^3_0 E_{max}$. However, $E^e_{max}$ is not complete (in this same sense) for $\leq^4_0$: there do exist $\Pi^0_2$-definable equivalence relations $E$ on $2^\omega$ with $E \not\leq^4_0 E^e_{max}$.*

*Since $E_{max} \leq_0 E^e_{max}$, it follows that $E^e_{max}$ is also $\Pi^0_2$-complete under $\leq^3_0$ and that $E_{max}$ is not $\Pi^0_2$-complete under $\leq^4_0$.*

In light of Proposition 4.6 below, one could prove $\Pi^0_2$-completeness under $\leq^3_0$ for either $E_{\max}$ or $E^e_{\max}$, and it would follow for the other. We will imitate the work in [13], but adjust it to the context of decidable sets. That work dealt with enumerations of sets rather than with the sets themselves, but the essence of the construction is the same, and our reason for doing the proof here for $E_{\max}$ is to demonstrate the essential similarity. Roughly speaking, the similarity reflects the fact that the replacement of a positive subformula $(\exists x \in A\ R(x))$ by a subformula $(\exists \langle x, y\rangle \in A\ R(x))$ does not change the complexity. A subformula $(\exists x \notin A\ R(x)$, with $R$ quantifier-free, would usually cause the complexity to increase when one passed from the decidable case to the case of enumerations. In Proposition 4.6 below, we will see a more subtle difference between the situations for decidability and enumerability.

*Proof.* Let $E$ be any $\Pi^0_2$ equivalence relation on $2^\omega$, given by a formula

$$A\ E\ B \iff \forall x \exists y\ R(A, B, x, y)$$

with $R(A, B, x, y)$ decidable by a functional $\Gamma^{A\oplus B}(x, y)$. We give a computable ternary reduction $\Phi$ from $E$ to $E^e_{\max}$, with oracle $A = A_0 \oplus A_1 \oplus A_2$, outputting $B_0 \oplus B_1 \oplus B_2$. First, we fit the relation $E$ into the standard $\Pi^0_2$ framework. For each $i < j$ and each $s$, we can use $\Gamma$ to determine the greatest $x_{i,j,s} \leq s$ such that

$$(\forall x \leq x_{i,j,s})(\exists y \leq s)\ R(A, B, x, y).$$

If $x_{i,j,s+1} > x_{i,j,s}$, we say that the pair $(i, j)$ *receives a chip* at stage $s + 1$. Thus $A_i\ E\ A_j$ if and only if the pair $(i, j)$ receives a chip at infinitely many different stages $s$. For convenience we consider each of the three pairs to receive a chip at stage 0.

$\Phi^A$ uses this idea to build its outputs $B_i$. At stage 0, it sets $0 \in B_0$ only, $1 \in B_1$ only, and $2 \in B_2$ only. Thus all three sets currently have distinct maxima. Similarly, at each stage $s \geq 1$, if none of the three pairs $(0, 1)$, $(0, 2)$ and $(1, 2)$ receives a chip at stage $s$, then $\Phi^A$ specifies that none of the numbers $2s, 2s + 1, 2s + 2$ lies in any of $B_0$, $B_1$, and $B_2$, thus preserving the three maxima (all distinct, by induction) from the preceding stage. If one pair $(i, j)$ with $i < j \leq 2$ did receive a chip, then for the least such pair, it defines $2s + 1$ to lie in $B_i$ and $2s + 2$ to lie in $B_j$. If neither of the other two pairs among the three received a chip at stage $s$, then the third output set $B_k$ receives no new elements. If at least one of the other two pairs did receive a chip, then it defines $2s$ to lie in $B_k$. None of the elements $2s, 2s + 1, 2s + 2$ will be put into any of the three output sets at any subsequent stage, so this constitutes a decision procedure for $B_0 \oplus B_1 \oplus B_2$.

Now if $A_0$, $A_1$, and $A_2$ lie in three distinct $E$-classes, then there is a stage after which no pair received any more chips. At the end of that stage, all three output sets had distinct maxima, and no further elements were ever added, so $B_0$, $B_1$, and $B_2$ lie in distinct $E_{\max}$-classes as required. If all three of the input sets are $E$-equivalent, then all three pairs received infinitely many chips, and so each of the three output sets has maximum $+\infty$. Finally, if $A_i\ E\ A_j$ but $A_k$ lies in a different $E$-class than these two, then after some stage $B_k$ never receives any more elements, hence has a finite maximum, whereas $B_i$ and $B_j$ each get new elements at infinitely many stages, hence both have maximum $+\infty$. Thus $\Phi^A$ is a ternary reduction from $E$ to $E_{\max}$.

To address 4-ary reducibility, we apply [13, Theorems 3.4 & 4.2], which together show that $=^{ce}$ has no computable reduction to $E^{ce}_{\max}$ (that is, under Turing-computable functions from $\omega$ to $\omega$). The proof is a nice illustration of the connections between that topic and this one. Suppose $E_{\max}$ were complete under $\leq^4_0$ among $\Pi^0_2$-definable equivalence relations on $2^\omega$. We "extend" $=^{ce}$ to an $E$ on Cantor space, by setting

$$A\ E\ B \iff [A = B \text{ or } [A \neq \emptyset \neq B\ \&\ (\forall i \in A)(\forall j \in B)\ W_i = W_j].$$

Note that this $E$ is $\Pi^0_2$-definable: $A = B$ is a $\Pi^0_1$ property, nonemptiness is $\Sigma^0_1$, and equality of c.e. sets is $\Pi^0_2$. By assumption, then, there is a computable 4-ary reduction $\Gamma$ of $E$ to $E_{\max}$. But now we could use this $\Gamma$ to define a computable reduction of $=^{ce}$ to $E^{ce}_{\max}$, as follows. Given any indices

$e_0, e_1, e_2, e_3 \in \omega$, we run $\Gamma^{\{e_0\}\oplus\cdots\oplus\{e_3\}}$, which computes a set $B = B_0 \oplus \cdots \oplus B_3$ such that $B_i\ E_{\max}\ B_j$ if and only if $\{e_i\}\ E\ \{e_j\}$, which holds just if $W_{e_i} = W_{e_j}$. Let $p_i$ be the code number of the program which enumerates $B_i$ by running $\Gamma^{\{e_0\}\oplus\cdots\oplus\{e_3\}}$, so that $B_i = W_{p_i}$. (In fact, we can also enumerate the complement $\overline{B_i}$, but this is unnecessary.) Then, for each $i < j \leq 3$, we have

$$p_i\ E^{ce}_{\max}\ p_j \iff W_{p_i}\ E_{\max}\ W_{p_j} \iff B_i\ E_{\max}\ B_j \iff \{e_i\}\ E\ \{e_j\} \iff W_{e_i} = W_{e_j},$$

and so the map $\vec{e} \mapsto \vec{p}$ is a computable 4-ary reduction of $=^{ce}$ to $E^{ce}_{\max}$, which is impossible. ■

The main point of the next proposition is its contrast with Proposition 4.4. The behavior of $E_{\max}$ and $E_{\text{card}}$ relative to each other is not the same as that of $E^e_{\max}$ and $E^e_{\text{card}}$.

**Proposition 4.6** *The best reductions that hold are $E_{max} \leq_0 E_{card}$ and $E_{card} \leq_0^{<\omega} E_{max}$ (and $E_{card} \leq_1 E_{max}$).*

*Proof.* The goal of the functional $\Gamma$ giving the reduction $E_{\max} \leq_0 E_{\text{card}}$ is to ensure that, for all $n$,

$$\max(A) \geq n \iff |B| \geq n+1 \quad (\text{where } B = \Gamma^A).$$

To this end, $\Gamma$ checks whether $0 \in A$, $1 \in A, \ldots$. Each time it finds a new $n \in A$, it extends its current approximation $\sigma_{n-1}$ of $B$ by setting $\sigma_n = \sigma_{n-1}\widehat{\ }1^k$, where $k$ is chosen so that $|\sigma_n^{-1}(1)| = n+1$. Each time $n \notin A$, it sets $\sigma_n = \sigma_{n-1}\widehat{\ }0$. This yields $B = \cup_n \sigma_n$ of cardinality exactly $n+1$ if $\max(A) = n$, or $B = \emptyset$ if $A = \emptyset$, or $B$ infinite if $A$ is infinite.

In the opposite direction, we show first that no $\Phi$ can be a countable computable reduction from $E_{\text{card}}$ to $E_{\max}$. Given $\Phi$, we choose $A_0 = \{0\}$ and $A_1 = \{0,1\}$, with no elements yet in any of $A_2, A_3, \ldots$. As usual we run $\Phi^A$ with $A = \oplus_n A_n$. If $\Phi$ is to succeed, then at some stage $s$ some finite portion of this oracle must yield a partial output $B = \oplus_n B_n$ with some $k$ for which either

- $k \in B_0$ and $k \notin B_2$ and $(\forall j > k)(\forall i = 0, 2)\ [j \notin B_i$ or $\Phi^A_s(\langle i, j\rangle)\uparrow]$; or
- the same with $B_0$ and $B_2$ reversed.

We freeze that finite portion of the oracle, and now add one new element (larger than the frozen part) to $A_2$. Now running $\Phi^A$ still yields $k \in B_0 \triangle B_2$, but must eventually give $B_0$ and $B_2$ the same maximum (possibly $+\infty$), and so eventually we find some $j > k$ with $j \in B_0$. We now freeze the finite portion of $A$ which has been used so far in these computations. No further changes will be made to $A_2$.

Next we do the same process with $A_1$ and $A_3$: with $A_3 = \emptyset$, wait until $\Phi^A$ produces $B_1$ and $B_3$ with distinct maxima, then freeze $A$ on the use of that computation, add two new large elements to $A_3$ (so that now $|A_3| = |A_1| = 2$), and wait until $\Phi^A$ evens up the maxima of $B_1$ and $B_3$, which must involve adding a larger element to $B_1$. After that, no further changes will be made to $A_3$.

We continue recursively, using each $A_{2i}$ to force $B_0$ to include a new larger element, and using each $A_{2i+1}$ to force $B_1$ to do so. Therefore $\max(B_0) = \max(B_1) = +\infty$, yet $|A_0| = 1 \neq 2 = |A_1|$. Therefore $\Phi$ was not a countable computable reduction.

The full 1-jump reduction from $E_{\text{card}}$ to $E_{\max}$ is easy: let $n \in \Gamma^{A'}$ if and only if $|A| \geq n$. It remains to give a finitary computable reduction $\Psi$ from $E_{\text{card}}$ to $E_{\max}$, using an oracle $A = A_0 \oplus \cdots \oplus A_n$. For this, $\Psi^A$ goes through $s = 0, 1, 2, \ldots$, one at a time, starting with $m_{i,-1} = -\infty$ for all $i \leq n$ For each $s$, $\Psi$ determines which numbers among $s(n+1), \ldots, (s+1)(n+1) - 1$ are to be added to which sets $B_i$. To do so, it determines the size $m_{i,s} = |A_i \cap \{0, \ldots, n\}|$ for each $i$. For those $i$ with $m_{i,s-1} = \min\{m_{j,s-1} : j \leq n\}$, it checks whether $m_{i,s} = m_{i,s-1}$: if so, then it adds no new elements to $B_i$; while if not, then it adds $s(n+1)$ to $B_i$ (and possibly more elements later in this step).

If this process added no elements to any $B_i$, then it determines the second-smallest value in $\{m_{j,s-1} : j \leq n\}$. For those $i$ such that $m_{i,s-1}$ has this value, it again adds no elements to $B_i$ provided $m_{i,s} = m_{i,s-1}$, buts adds $s(n+1)$ to $B_i$ otherwise. Again, if this process still has not added any elements to any $B_i$, then it proceeds with the third-smallest value in $\{m_{j,s-1} : j \leq n\}$, and so on until either some $B_i$ is enlarged or until the values in $\{m_{j,s-1} : j \leq n\}$ run out. (Notice that, if every $A_i$ is finite, then there will be a stage after which no more elements are ever added to any $B_i$.)

Now suppose that we reached a step at which $s(n+1)$ was added to some $B_i$. By this point, some sets $B_j$ have been assured that no elements will be added to them at this stage. For the remaining $B_k$ (including those $B_i$ to which $n(s+1)$ was added, as well as all $B_k$ with $m_{k,s-1} > m_{i,s-1}$ for those $i$), we determine anew the order among the maxima $m_{k,s}$. For those $k$ such that $m_{k,s}$ is least, we add $s(n+1)$ to $B_k$ (if it was not already there). For those $k$ for which $m_{k,s}$ has the second-smallest value, we add $s(n+1)+1$ to $B_k$. Those with the third-smallest value have $s(n+1)+2$ added to $B_k$, and so on. Now there are at most $(n+1)$ different indices $k$ involved, so the greatest number that can possibly be added to any $B_k$ at stage $s$ is $s(n+1)+n = (s+1)(n+1)-1$. No number $\leq s(n+1)+n$ will be added to any set $B_i$ at any subsequent stage, so this is a decision procedure for the $B_i$. Moreover, we have ensured that that order of the maxima of the sets $B_i$ (so far) corresponds to the order of the sizes $m_{i,s}$.

It is clear that whenever a set $A_i$ is finite, the corresponding $B_i$ will eventually stop receiving new elements (once the $m_{j,s}$ have reached their limiting values $m_j$ for all $j \leq n$ with $A_j$ finite, and once the $m_{k,s}$ with $A_k$ infinite have all surpassed these limiting values). Moreover, we will have $m_i = m_j$ if and only if $B_i$ and $B_j$ have the same maximum. The infinite sets $A_k$ will all have $B_k$ infinite as well, and so they all satisfy $\max(B_k) = +\infty$ (and $|A_k| = +\infty$). Therefore, this is indeed a finitary computable reduction. ∎

**Corollary 4.7** *$E_{max} \equiv_0^{<\omega} E^e_{max}$ and $E_{card} \equiv_0^{<\omega} E^e_{card}$. Hence $E_{card}$ is $\Pi^0_2$-complete under computable ternary reducibility $\leq_0^3$, but not under $\leq_0^4$.*

*Proof.* The general reduction $E \leq_0 E^e$ was described after Definition 4.1. The reduction $\Gamma$ for $E^e_{\max} \leq_0^{<\omega} E_{\max}$ is given an oracle $A = A_0 \oplus \cdots \oplus A_n$, and proceeds much like the finitary reduction $\Psi$ from $E_{\text{card}}$ to $E_{\max}$ in Proposition 4.6. It measures the maximum $m_{i,s}$ of $\pi_1(A_i \cap \{0, \ldots, s\})$ at each stage $s$, and orders the sets $A_0, \ldots, A_n$ according to their current maxima, noting which maxima have changed since the previous stage. For those sets at the bottom of the list (with lowest maxima) whose maxima have not changed, it does not add any new elements to $B_i$. Starting with the least $A_j$ in this order whose current maximum has $m_{j,s} \neq m_{j,s-1}$, it adds a new element, larger than any previously seen, to $B_j$ and to each $B_k$ with $m_{k,s-1} > m_{j,s-1}$ (or with $m_{j,s-1} = m_{k,s-1} \neq m_{k,s}$). Then it adds further new elements to these sets $B_k$ to make sure that the current order of the $B_k$ by maxima matches the current order of the $A_k$ by their maxima $m_{k,s}$. This works, just as $\Psi$ did in the proposition.

The reduction $E^e_{\text{card}} \leq_0^{<\omega} E_{\text{card}}$ now follows from

$$E^e_{\text{card}} \leq_0 E^e_{\max} \leq_0^{<\omega} E_{\max} \leq_0 E_{\text{card}},$$

and the rest of the corollary is the result of Proposition 4.5. ∎

## 5 Noncomputable Reductions

We present here an idea for further discussion, without any immediate results. It is simple to define $X$-computable reductions on $2^\omega$, and to extend the notion to jump reductions and to finitary and countable reductions as well.

**Definition 5.1** *Let $E$ and $F$ be equivalence relations on $2^\omega$, fix $X \subseteq \omega$, and choose any $X$-computable ordinal $\alpha$. We say that $E$ is* $\alpha$-jump $X$-reducible *to $F$, written $E \ \leq_{\alpha,X} \ F$, if there exists a Turing functional $\Phi$ such that the map*

$$A \mapsto \Phi^{X \oplus A^{(\alpha)}}$$

*is a reduction of $E$ to $F$. Likewise, if the map*

$$A = \oplus_i A_i \mapsto \Phi^{X \oplus A^{(\alpha)}} = \oplus_j B_j,$$

*is a k-ary, finitary, or countable reduction of $E$ to $F$ (where the joins $\oplus_i A_i$ and $\oplus_j B_j$ are over the appropriate number of sets), then $E$ is* $k$-arily, finitarily, *or* countably $\alpha$-jump $X$-reducible *to $F$, written $E \ \leq^k_{\alpha,X} \ F$ or $E \ \leq^{<\omega}_{\alpha,X} \ F$ or $E \ \leq^{\omega}_{\alpha,X} \ F$.*

As a natural example of an equivalence relation to which this can be applied, let $X$ be any set which is not c.e. Then the equivalence relation

$$A \ =_X \ B \iff (\forall n \in X)[n \in A \iff n \in B]$$

is computably $X$-reducible to the equality relation on $2^\omega$: just write $X = \{x_0 < x_1 < \cdots\}$ and let $n \in \Gamma^{X \oplus A}$ precisely when $x_n \in A$. Indeed, any enumeration of $X$ would suffice for this purpose. However, if $\Phi$ were a computable binary reduction of $=_X$ to equality, then we would be able to enumerate $X$, since then

$$n \in X \iff \exists y \in B_0 \triangle B_1, \text{ where } B_0 \oplus B_1 = \Phi^{\emptyset \oplus \{n\}}.$$

We consider the notion of $X$-computable reducibility to be natural for further study, especially since it allows the notion of jump-reductions to be extended to arbitrary countable ordinals $\alpha$ (by taking an $X$ which can compute a copy of $\alpha$). However, we will not elaborate on this notion any further here.

## 6 The Glimm-Effros Dichotomy

Now we examine the analogue of the Glimm-Effros dichotomy for computable reducibility. Unsurprisingly, the strict analogue fails to hold. To prove this, we use the equivalence relation $=_Y$ defined in Section 5, for a set $Y$ whose complement $X$ is computably enumerable.

Here we build the set $X$, along with a partial computable injective function $\psi : \omega^2 \to \omega$ satisfying the following well-known requirements, for all $e$:

$$\begin{aligned} \mathcal{N}_e :&\ \ |\overline{X}| \geq e; \\ \mathcal{P}_e :&\ \ \overline{X} \neq W_e; \\ \mathcal{R}_e :&\ \ (\exists n \geq 0)\ \psi(e,n)\!\downarrow\notin X; \\ \mathcal{S}_e :&\ \ \text{defined below.} \end{aligned}$$

The $\mathcal{N}$- and $\mathcal{P}$-requirements are completely standard for finite-injury constructions. In our construction, $\mathcal{R}_e$ first chooses a large value $\psi(e,0)$ and protects it from entering $X$. If a higher-priority requirement puts $\psi(e,0)$ into $X$ at a stage $s$, then $\mathcal{R}_e$ defines $\psi(e,1)$ to be a new large number, greater than $s$ and protects that number instead, and so on. A standard finite-injury construction builds a computably enumerable set $X$ satisfying all these requirements. Notice that the image of $\psi$ is decidable, since $y$ can only enter it at stages $\leq y$. Building $X$ this way (without the $\mathcal{S}$-requirements) will give us a contradiction to Glimm-Effros for computable reducibility on $2^\omega$; we will explain the $\mathcal{S}$-requirements below when we wish to strengthen the contradiction.

For the complement $Y = \overline{X}$, we define $=_Y$ on $2^\omega$, the *equality relation on* $Y$, as in the previous section:

$$A =_Y B \iff (\forall n \in Y)[n \in A \iff n \in B].$$

It was noted there that $=_X \leq_0^2 =$ if and only if $X$ is a computably enumerable set. Our construction makes $Y$ properly $\Pi_1^0$, and therefore $=_Y \not\leq_0^2 =$. On the other hand, we do have a computable reduction $\Gamma$ from $=$ to $=_Y$. For each $(e,n) \in \text{range}(\psi)$, $\Gamma$ defines $\psi(e,n) \in \Gamma^A$ if and only if $e \in A$, with $\Gamma^A$ disjoint from the (decidable) complement of range$(\psi)$. By construction, if any $e$ has $e \in A \triangle B$, then every $\psi(e,n)$ defined will lie in $\Gamma^A \triangle \Gamma^B$, and at least one of these (finitely many) values $\psi(e,n)$ will lie in $Y$. Thus $= \leq_0 =_Y$ via $\Gamma$, and so $= <_0 =_Y$.

However, we also claim that $E_0$ is incomparable with $=_Y$ under even binary computable reducibility, and that therefore $=_Y$ reveals the failure of the Glimm-Effros dichotomy for computable reducibility on $2^\omega$. The essence of this claim is that $E_0$ is a $\Sigma_2^0$ relation, whereas $=_Y$ is $\Pi_2^0$:

$$A =_Y B \iff (\forall n)(\exists s)[n \in X_s \text{ or } [n \in A \iff n \in B]].$$

In particular, if $\Phi$ were a binary computable reduction from $E_0$ to $=_Y$, then, given any pair $\langle i,j \rangle$ with $W_i = \overline{W_j}$, we would have

$$i \in \text{Fin} \iff W_i \; E_0 \; \emptyset \iff (\forall n)(\exists s)[n \in X_s \text{ or } [n \in B_0 \iff n \in B_1]],$$

where $\Phi^{W_i \oplus \emptyset} = B_0 \oplus B_1$. But given the pair $(i,j)$, one could compute the function $\Phi^{W_i \oplus \emptyset}$ (which is total, by assumption), making the right-hand side a $\Pi_2^0$ predicate of $i$ and $j$, which is impossible. (The pair $(\text{Fin}, \text{Inf})$ has a natural 1-reduction to $(\text{Fin} \cap D, \text{Inf} \cap D)$, with $D = \{(i,j) : W_i = \overline{W_j}\}$ as before: just map each $e$ to an index of the c.e. set $\{s : W_{e,s+1} \neq W_{e,s}\}$. The argument above then gives a 1-reduction from $(\text{Fin} \cap D)$ to a $\Pi_2^0$ set, for a contradiction.)

The $\mathcal{S}$ requirements are used to ensure that $=_Y \not\leq_0^2 E_0$, completing the proof of incomparability. $\mathcal{S}_e$ requires that, if the $e$-th oracle Turing functional $\Phi_e$ is total on all oracles, then there should be a decidable set $D_e = W_i = \overline{W_j}$ (for some $i$ and $j$) such that

$$D_e \cap Y \neq \emptyset \iff \Phi_e^{D_e \oplus \emptyset} \text{ consists of two } E_0\text{-inequivalent columns.}$$

Of course, the left-hand side just means that $D_e \neq_Y \emptyset$, so the requirements together will show that $=_Y \not\leq_0^2 E_0$.

We break up $\mathcal{S}_e$ into countably many subrequirements, so as to preserve the finite-injury nature of the construction. Each $\mathcal{S}_{e,n+1}$ inherits from $\mathcal{S}_{e,n}$ an element $x_{e,n}$ of $D_e$, at a stage $s$ at which $\mathcal{S}_{e,n}$ has completed its action (if it ever does). We write $D_{e,s} = D_e \upharpoonright s$ for the portion of $D_e$ so far decided. This $x_{e,n}$ does not lie in $X_s$, so it currently appears to establish the left-hand side of $\mathcal{S}_e$. $\mathcal{S}_{e,n+1}$ waits until the symmetric difference between the columns of $\Phi_e^{(D_{e,s} \hat{} 000000\cdots) \oplus \emptyset}$ contains

$\geq n+1$ elements, extending $D_{e,s}$ by a zero at each step to encourage convergence. If the symmetric difference ever reaches this size, then $\mathcal{S}_{e,n}$ adjoins $x_{e,n}$ to $X$ (so that it no longer establishes the left-hand side), and defines a new $x_{e,n+1}$, larger than the current stage, to hand off to $\mathcal{S}_{e,n+2}$. If the symmetric difference contains $\leq n$ elements in total at the end of the construction, then $\mathcal{S}_{e,n+1}$ has satisfied $\mathcal{S}_e$ all by itself; whereas if the symmetric difference between the columns of $\Phi_e^{D_e \oplus \emptyset}$ is infinite, then every element $x_{e,n}$ of $D_e$ is eventually enumerated into $X$ (hence out of $Y$) by its $\mathcal{S}_{e,n+1}$. Either way, $\mathcal{S}_e$ holds.

A given $\mathcal{S}_{e,n+1}$ may be injured if a higher-priority $\mathcal{P}$-requirement adds $x_{e,n}$ to $X$. If so, $\mathcal{S}_{e,n+1}$ simply chooses a new large $x'_{e,n}$, places it in $D_e$, and continues with it. This wil only happen finitely often (for a given $e$ and $n$), so $\mathcal{S}_{e,n}$ does accomplish its goal, possibly with finitely many false starts. All the other requirements fit together according to a standard finite-injury priority construction, and so for the set $Y = \overline{X}$, the equivalence relation $=_Y$ on $2^\omega$ lies strictly above equality under computable reducibility, is is incomparable with $E_0$ even under binary computable reducibility.

It is hardly surprising that, in the far more exacting context of computable reducibility, the Glimm-Effros dichotomy should fail. The surprise, after all, was that it held for Borel reducibility in the first place. Nevertheless, the arguments here, simple though they be, demonstrate that there is much more to investigate in this topic. For example, while Glimm-Effros fails for equality and $E_0$ under computable reducibility, it remains open whether there might be some two other equivalence relations which would be (respectively) the least and the second-least among all smooth equivalence relations on $2^\omega$ under computable reducibility. Alternatively, it seems a bit more plausible that Glimm-Effros might hold for $\omega$-jump reducibility, or $\lambda$-jump reducibility for some limit ordinals $\lambda$. The notions from Section 5 would even allow one to address $\lambda$-jump reducibility for admissible ordinals $\lambda$, which appears to be the most promising ground of all.

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