THE DISTANCE FUNCTION ON A COMPUTABLE GRAPH

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Abstract. We apply the techniques of computable model theory to the distance function of a graph. This task leads us to adapt the definitions of several truth-table reducibilities so that they apply to functions as well as to sets, and we prove assorted theorems about the new reducibilities and about functions which have nonincreasing computable approximations. Finally, we show that the spectrum of the distance function can consist of an arbitrary single `btt`-degree which is approximable from above, or of all such `btt`-degrees at once, or of the `bT`-degrees of exactly those functions approximable from above in at most n steps.

1. Introduction

Every connected graph has a distance function, giving the length of the shortest path between any pair of nodes in the graph. Graphs appear in a wide variety of mathematical applications, and the computation of the distance function is usually crucial to these applications. Examples range from web search engine algorithms, to Erdös numbers and parlor games ("Six Degrees of Kevin Bacon"), to purely mathematical questions.

Therefore, the question of the difficulty of computing the distance function is of natural interest to mathematicians in many areas. This article is dedicated to exactly that enterprise, on infinite graphs. Assuming that the graph in question is symmetric, irreflexive, and computable – that is, that one can list out all its nodes and decide effectively which pairs of nodes have an edge between them – we investigate the Turing degree and other measures of the difficulty of computing the distance function.

We began this study by considering the spectrum of the distance function – a standard concept in computable model theory, giving the set of the Turing degrees of distance functions on all computable graphs isomorphic to the given graph. This notion is usually used for relations on a computable structure, rather than for functions, but it is certainly the natural first question one should ask. As our studies continued, however, they led us to consider finer reducibilities than ordinary Turing reducibility, and since we were studying a function instead of a relation, we often had to adapt these reducibilities to functions. The resulting concepts are likely to be of interest to pure computability theorists, as well as to those

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dealing with applications, and, writing the paper in logical rather than chronological order, we spend the first sections defining and examining these reducibilities on functions. Only in the final sections do we address the original questions about the distance function on a computable graph. Therefore, right here we will offer some further intuition about the distance function, to help the reader understand why the material in the first few sections is relevant.

For a computable connected graph G, the natural first step for approximating the distance $d(x,y)$ between two nodes $x, y \in G$ is to find some path between them. By connectedness, a systematic search is guaranteed to produce such a path sooner or later, and its length is our first approximation to the distance from x to y. The next natural step is to search for a shorter path, and then a shorter one than that, and so on. Of course, these path lengths are all just approximations to the actual distance from x to y. One of the approximations will be correct, and once we find it, its path length will never be superseded by any other approximation. That is, our (computable) approximations will *converge* to the correct answer, and so the distance function is always $\emptyset'$-computable, by the Limit Lemma (see [15, Lemma III.3.3]). However, unless $d(x,y) \leq 2$, we will never be sure that our approximation is correct, since a shorter path could always appear.

By definition $d(x,x) = 0$, and $d(x,y) = 1$ iff x and y are adjacent, so it is computable whether either of these conditions holds. It is not in general computable whether $d(x,y) = 2$, but it is Σ^0_1, since we need only find a single node adjacent to both x and y. For each $n > 2$, however, the condition $d(x,y) = n$ is given by the conjunction of a universal formula and an existential formula, hence defines a difference of computably enumerable sets, and in general cannot be expressed in any simpler way than that. Indeed, the distance function often fails to be computable, and likewise the set of pairs (x,y) with $d(x,y) = n$ often fails to be computably enumerable when $n > 2$. In what follows, however, we will show that the distance function always has computably enumerable Turing degree – which in turn will start to suggest why Turing equivalence is not the most useful measure of complexity for these purposes. Other questions immediately arise as well. For instance, must there exist a computable graph H isomorphic to G such that H has computable distance function? Or at least, must there exist a computable $H \cong G$ such that we can approximate the distance function on H with no more than one (or n) wrong answer(s)?

The approximation algorithm described above is not of arbitrary difficulty, in the pantheon of computable approximations to functions. Our approximations to $d(x,y)$ are always at least $d(x,y)$, and decrease until (at some unknown stage) they equal $d(x,y)$. Hence $d(x,y)$ is *approximable from above.* This notion already exists in the literature; the most commonly seen function of this type is probably Kolmogorov complexity, which on input n gives the shortest length of a program outputting n. Up to the present, approximability from above has not been so common in computable model theory; the more common notion there is *approximability from below*, which arises (for instance) when one tries to find the number of predecessors of a given element in a computable linear order of order type ω. In Section 3 we give full definitions of the class of functions approximable from above, which can be classified in much the same way as the Ershov hierarchy, and compare it with the class of functions approximable from below. These two classes turn out to be more different than one would expect! First, though, in Section 2, we present the exact definitions of the reducibilities we will use on our functions, so that we may refer to these reducibilities in Section 3.

Very little background in computable model theory is actually required in order to read this article, since the distance function turned out to demand a somewhat different approach than is typical in that field of study. A background in general computability theory will be useful, however, particularly in regard to several of the so-called *truth-table reducibilities*, and for this we suggest [14], to which we will refer frequently in Section 2. We try to maintain notation from [14] as we adapt the definitions of the truth-table reducibilities to deal with functions. We give the requisite definitions about graphs in Section 4, where they are first needed.

2. Reducibilities on Functions

When discussing Turing computability relative to an oracle, mathematicians have traditionally taken the oracle to be a subset of ω. To compute relative to a total or partial function from ω^n into ω, they simply substitute the graph of the function for the function itself, then apply a coding of ω^{n+1} into ω. This metonymy works admirably as far as ordinary Turing reducibility is concerned, and any alternative definition of Turing reducibility for functions should be equivalent to this one. However, *bounded* Turing reducibility (in which the use of the oracle is computably bounded; see Definition 2.6) among functions requires a new definition, and so we offer here an informal version of our notion of a function oracle.

First, to motivate this notion, consider bounded Turing (`bT`) reducibility with a function oracle. One would certainly assume that a total function f should be `bT`-reducible to itself. However, if one wishes to compute $f(x)$ for arbitrary x, using the graph of f as an oracle, and if f is not computably bounded, then there is no obvious way to compute in advance an upper bound on the codes $\langle m, n\rangle$ of pairs for which one will have to ask the oracle about membership of that pair in the graph. (This question is addressed more rigorously in Proposition 2.7 and Lemma 2.8 below.) So, when we move to reducibilities finer than $\leq_T$, there is a clear need for a notion of Turing machine having a function as the oracle.

For simplicity, we conceive of a Turing machine with a function oracle F as having three tapes: a two-way scratch tape (on which the output of the computation finally appears, should the computation halt), a one-way *question tape*, and a one-way *answer tape*. When the machine wishes to ask its function oracle for the value $F(x)$ for some specific x, it must write a sequence of exactly $(x + 1)$ 1's on the question tape, and must set every cell of the answer tape to be blank. Then it executes the *oracle instruction*. In this model of computation, the oracle instruction is no longer a forking instruction. Rather, with the tapes in this state, the oracle instruction causes exactly $(1 + F(x))$ 1's to appear on the answer tape at the next step, and the machine simply proceeds to the next line in its program (which most likely will start counting the number of 1's on the answer tape, in order to use the information provided by the oracle). Notice that it is perfectly acceptable to say that a set is computable from a function oracle (using the above notion), or that a function is computable from a set oracle (using the usual notion of oracle Turing computation). If one wishes to speak only of functions, there is no harm in replacing a set by its characteristic function. We leave it to the compulsive reader to formulate the precise definition of a Turing machine with function oracle, by analogy to the standard definition for set oracles. A more immediate (and equivalent) definition uses a different approach.

Definition 2.1. The class of *partial functions on ω computable with a function oracle F*, where $F : \omega \to \omega$ is total, is the smallest class of partial functions closed under the axiom schemes I-VI from [15, § I.2] and containing F.

Of course, this is exactly the usual definition of the partial F-recursive functions, long known to be equivalent to the definition of functions computable by a Turing machine with the graph of F as its oracle. As far as Turing reducibility is concerned, nothing has changed. Only stronger reducibilities need to be considered. We start by converting the standard definitions (for sets) of $\leq_m$ and $\leq_1$ to definitions for functions.

Definition 2.2. Let φ and ψ be partial functions from ω to ω. We say that φ is *m-reducible to ψ*, written $\varphi \leq_m \psi$, if there exists a total computable function g with $\varphi = \psi \circ g$. (For strictly partial functions, this includes the requirement that $(\forall x)[\varphi(x)\downarrow \iff \psi(g(x))\downarrow]$.)

If the m-reduction g is injective, we say that φ is *1-reducible to ψ*, written $\varphi \leq_1 \psi$.

This definition already exists in the literature on computability, having been presented as part of the theory of numberings studied by research groups in Novosibirsk and elsewhere. (See e.g. [7], [6] or [8, p. 477] for the notion of reducibility on numberings.) It is appropriate here as an example of our approach in generalizing reducibilities on sets to reducibilities on functions, for which reason we feel justified in calling it m-reducibility. For subsets $A, B \subseteq \omega$, it is quickly seen that $A \leq_m B$ iff $\chi_A \leq_m \chi_B$, where these are the characteristic functions of those sets; similarly for $A \leq_1 B$. The analogue of Myhill's Theorem for functions states that if $\varphi \leq_1 \psi$ and $\psi \leq_1 \varphi$, then in fact there is a computable permutation h of ω with $\varphi = \psi \circ h$ (and hence $\psi = \varphi \circ (h^{-1})$), in which case we would call φ and ψ *computably isomorphic*. The proof is exactly the same as that of the original theorem of Myhill (see [15, Thm. I.5.4]), and it makes no difference whether φ and ψ are both total or not.

Definition 2.3. A function ψ is *m-complete* for a class Γ of functions if $\psi \in \Gamma$ and, for every $\varphi \in \Gamma$, we have $\varphi \leq_m \psi$. We define 1-completeness similarly, but require that $\varphi \leq_1 \psi$.

For example, the universal Turing function $\psi(\langle e, x\rangle) = \varphi_e(x)$ is 1-complete partial computable, i.e. 1-complete for the class of all *partial* computable functions. (For each single φ_e in the class, the function $x \mapsto \langle e, x\rangle$ is a 1-reduction.) A more surprising result is that there does exist a function h which is 1-complete total computable: let $h(\langle n, m\rangle) = m$, so that, for every total computable f, the function $g(n) = \langle n, f(n)\rangle$ is a 1-reduction from f to h.

The notion of m-reducibility for sets has small irritating features, particularly the status of the sets $\emptyset$ and ω. Intuitively, the complexity of each of these is as simple as possible, yet they are m-incomparable to each other. (Also, no $S \neq \emptyset$ has $S \leq_m \emptyset$ and no $S \neq \omega$ has $S \leq_m \omega$; intuitively this is reasonable, but it is still strange to have two m-degrees containing only a single set each.) The same problem is magnified for m-reducibility on functions. Clearly, if $\varphi \leq_m \psi$, then $\mathrm{rg}(\varphi) \subseteq \mathrm{rg}(\psi)$. It follows that every total constant function forms an m-degree all by itself. Moreover the function $\varphi(x) = 2x$ is m-incomparable with $\psi(x) = 2x+1$, even though these seem to have very similar complexity; and assorted other pathologies can be found. To address these issues, we offer the following adaptation of m-reducibility.

Definition 2.4. The *join* of two partial functions φ and ψ is the function which splices them together:

$$(\varphi \oplus \psi)(x) = \begin{cases} \varphi(\frac{x}{2}), & \text{if } x \text{ is even;} \\ \psi(\frac{x-1}{2}), & \text{if } x \text{ is odd.} \end{cases}$$

A partial function φ is *augmented m-reducible* to another partial function ψ, written $\varphi \leq_a \psi$, if $\varphi \leq_m (\iota \oplus \psi)$, where $\iota(0)\uparrow$ and $\iota(x+1) = x$.

The intention here is that, for any computable subset $S \subseteq \omega$ such that $\varphi \restriction S$ and $(S \cap \text{dom}(\varphi))$ are both computable, one can define the m-reduction g from φ to $\iota \oplus \psi$ by choosing $g(x) = 2 + 2\varphi(x)$ for $x \in S$, with $g(x) = 0$ if $\varphi(x)$ is known to diverge. On $(\omega - S)$, g must actually serve as a (computable) m-reduction from $\varphi \restriction (\omega - S)$ to ψ. Under this definition, there is a single a-degree consisting of all partial computable functions with computable domains. Thus, many pathologies regarding functions with seemingly similar complexity but distinct domains are avoided. (Some remain. A noncomputable function φ will generally be a-incomparable to $(1 + \varphi)$, for instance.)

Proposition 2.5. *The a-degree of the empty function λ is the least a-degree. Moreover, a function φ belongs to this a-degree iff φ is partial computable and dom(φ) is computable.*

Proof. For every partial function φ, we have $\lambda \leq_a \varphi$, since the constant function 0 serves as an m-reduction from λ to $(\iota \oplus \varphi)$. For the forwards direction of the equivalence, let g be an m-reduction from φ to $(\iota \oplus \lambda)$. Then $x \in \text{dom}(\varphi)$ iff $g(x)$ is nonzero and even, in which case $\varphi(x) = (\iota \oplus \lambda)(g(x)) = \frac{g(x)}{2} - 1$, which is computable. For the converse, we define an m-reduction h from φ to $(\iota \oplus \lambda)$ by:

$$h(x) = \begin{cases} 2 \cdot (\varphi(x) + 1), & \text{if } x \in \text{dom}(\varphi) \\ 0, & \text{if not.} \end{cases}$$

□

For characteristic functions of sets A and B, we have $\chi_A \leq_a \chi_B$ iff $A \leq_m (B \oplus \emptyset \oplus \omega)$. We will not use this concept for sets, but we suggest writing $A \leq_a B$ whenever $A \leq_m (B \oplus \emptyset \oplus \omega)$. Under this definition, the computable sets (including $\emptyset$ and ω) form the least a-degree of sets. The definition of 1-reducibility for functions could be adapted in the same way to produce a notion of *augmented 1-reducibility* for functions, and likewise for sets, but certain questions arise about the best way to adapt the definition, and we will not address them here.

We also consider reducibilities intermediate between m-reducibility and Turing reducibility, again by analogy to such reducibilities on sets.

Definition 2.6. Let α and β be total functions. We say that α is *bounded-Turing reducible* to β, or *weak truth-table reducible* to β, if there exists a Turing reduction Φ of α to β and a computable total function f such that, in computing each value $\alpha(x)$, the reduction Φ only asks the β-oracle for values $\alpha(y)$ with $y < f(x)$. Thus, for each x, $\alpha(x) = \Phi^{\beta \restriction f(x)}(x)$. Bowing to the two distinct terminologies that exist for this notion on sets, we use two notations for this concept:

$$\alpha \leq_{\texttt{bT}} \beta \quad \text{and} \quad \alpha \leq_{\texttt{wtt}} \beta.$$

If we use D_e to denote the finite set with strong index e (i.e. the index tells the size of D_e and all of its elements), then we say that α is *truth-table reducible* to β, written $\alpha \leq_{\mathtt{tt}} \beta$, if there exist total computable functions f and g such that, for every input x to α, we have $\alpha(x) = g(x, \beta\upharpoonright D_{f(x)})$. This is different from the related reducibilities on enumerations described by Degtev [4] — in particular, it is important for what follows in the present paper that α and β can have different ranges.

Finally, if $\alpha \leq_{\mathtt{tt}} \beta$ via f and g as above and there exists some $k \in \omega$ such that $|D_{f(x)}| \leq k$ for every $x \in \omega$, then we say that α is *bounded truth-table reducible* to β *with norm* k, and write $\alpha \leq_{k\text{-}\mathtt{btt}} \beta$. For α to be *bounded truth-table reducible* to β (with no norm stated) simply means that such a k exists, and is written $\alpha \leq_{\mathtt{btt}} \beta$.

It should be noted that, as with sets, the relation $\leq_{k\text{-}\mathtt{btt}}$ on function fails to be transitive, for $k > 1$. In general, if $\alpha \leq_{j\text{-}\mathtt{btt}} \beta$ and $\beta \leq_{k\text{-}\mathtt{btt}} \gamma$, then $\alpha \leq_{(jk)\text{-}\mathtt{btt}} \gamma$.

As mentioned, functions and their graphs have always been conflated for purposes of Turing-reducibility. For these finer reducibilities, the conflations no longer apply.

Proposition 2.7. *A total function h is truth-table equivalent to (the characteristic function of) its own graph iff there exists a computable function b such that, for every x, we have $h(x) \leq b(x)$. (In this case, h is said to be* computably bounded.*)*

Proof. Let $G \subset \omega^2$ be the graph of h, and suppose first that $h(x) \leq b(x)$ for all x. Then, with a G-oracle, a Turing machine on input x can simply ask which pairs (x, n) with $n \leq b(x)$ lie in G. So we have stated in advance which oracle questions will be asked, and by assumption there will be exactly one positive answer, which will be the pair (x, y) with $y = h(x)$. Thus $h \leq_{\mathtt{tt}} G$, since we can also say in advance exactly what answer the machine will give in response to each possible set of oracle values. On the other hand, to determine whether $(x, y) \in G$, an oracle Turing machine only needs to ask an h-oracle one question: the value of $h(x)$. Thus $\chi_G \leq_{\mathtt{tt}} h$ as well. This latter reduction is actually a bounded truth-table reduction of norm 1, under Definition 2.6, and holds even without the assumption of computable boundedness of h.

For the forwards direction, suppose $h \equiv_{\mathtt{tt}} \chi_G$. Then the computation of h, on input x, asks for the value $\chi_G(m, n)$ only for pairs with codes $\langle m, n\rangle \in D_{f(x)}$, and outputs $g(x, \chi_G\upharpoonright D_{f(x)})$, with f and g as in Definition 2.6. Thus $h(x)$ must be one of the finitely many values in the set $\{g(x, \sigma) : \sigma \in 2^{|D_{f(x)}|}\}$. Since f and g are computable and total, we may take $b(x)$ to be the maximum of this set, forcing $h(x) \leq b(x)$. Thus h is computably bounded. $\square$

In Proposition 2.7, $\mathtt{tt}$-equivalence cannot be replaced by $\mathtt{bT}$-equivalence. The following proof of this fact was devised in a conversation between E. Fokina and one of us, and completes the answer to the question asked at the beginning of this section.

Lemma 2.8 (Fokina-Miller)**.** *There exists a total function f which is not computably bounded, yet is $\mathtt{bT}$-equivalent to (the characteristic function of) its own graph G.*

Proof. Let $K = \{\langle e, x\rangle : \varphi_e(x)\downarrow\}$ be the halting set. Define $f(2x) = \chi_K(\langle x, 2x+1\rangle)$ on the even numbers, using the characteristic function χ_K of K, and on the odd numbers, define

$$f(2x+1) = \begin{cases} 1+\varphi_x(2x+1), & \text{if } \varphi_x(2x+1)\downarrow, \\ 0, & \text{if not.} \end{cases}$$

Of course this f is not computable, but it is total, and for each x, the input $(2x+1)$ witnesses that φ_x is not an upper bound for f. Moreover, to determine $f(2x)$ on even numbers, we need only ask a G-oracle whether $\langle 2x, 0\rangle \in G$. To determine $f(2x+1)$ on odd numbers, we again ask the oracle whether $\langle 2x, 0\rangle \in G$. If so, then $\chi_K(\langle x, 2x+1\rangle) = f(2x) = 0$, meaning that $\varphi_x(2x+1)\uparrow$, and so we know that $f(2x+1) = 0$. If $\langle 2x, 0\rangle \notin G$, then we know that $\chi_K(\langle x, 2x+1\rangle) = f(2x) = 1$, so $\langle x, 2x+1\rangle \in K$, and we simply compute $\varphi_x(2x+1)$ (knowing that it must converge) and add 1 to get $f(2x+1)$. In all cases, therefore, we can compute $f(y)$ by asking a single question of the G-oracle about whether a predetermined value lies in G. Thus $f \leq_{\mathtt{bT}} G$, and of course $G \leq_{\mathtt{1\text{-}btt}} f$. □

3. Functions Approximable from Above

Having adapted several standard reducibilities on sets to serve for functions as well, we now perform the same service for the Ershov hierarchy. Traditionally this has been a hierarchy of $\emptyset'$-computable sets, determined by computable approximations to those sets and by the number of times the approximations "change their mind" about the membership of a given element in the set. In our investigations of the distance functions on computable graphs, we found that similar concepts arose, but pertaining to functions, not to sets. Therefore, the following definitions provide total Δ^0_2-functions with their own Ershov hierarchy, and then add some further structure.

Definition 3.1. Let $f(x) = \lim_s g(x,s)$ be a total function from ω to ω, with the binary function g total and computable.

- If there is a total computable function h such that

  $$(\forall x)\ |\{s : g(x,s) \neq g(x,s+1)\}| \leq h(x),$$

  then f is *ω-approximable.*
- If the constant function $h(x) = n$ can serve as the h in the previous item, then f is *n-approximable.*
- More generally, if α is a computable ordinal and there is a total computable nonincreasing function $h : \omega^2 \to \alpha$ such that

  $$(\forall x \forall s)\ [g(x,s) \neq g(x,s+1) \implies h(x,s) \neq h(x,s+1),$$

  then f is *α-approximable.*
- If, for all x and s, we have $g(x,s+1) \leq g(x,s)$, then f is *approximable from above.* Such functions are also sometimes called *limitwise decreasing*, *semi-computable from above* or *right c.e.* functions
- If, for all x and s, we have $g(x,s+1) \geq g(x,s)$, then f is *approximable from below.* In the literature, such functions have also been called *limitwise monotonic*, *limitwise increasing*, and *subcomputable.*
- When we combine these definitions, we assume that a single function g satisfies all of them. For instance, f is 3-*approximable from above* if f is the limit of a computable function g such that, for all x, $|\{s : g(x,s) \neq g(x,s+1)\}| \leq 3$ *and* $(\forall s)\ g(x,s+1) \leq g(x,s)$.

- Following [3], we define f to be *graph-α-c.e.* if α is a computable ordinal and the graph of f is an α-c.e. set in the Ershov hierarchy.

In [3], the term *α-c.e. function* was used for the functions we are calling α-approximable. We prefer our terminology, since the phrase "c.e. function" has been used elsewhere for functions approximable from below. (For such a function, the set $\{(x, y) : y \leq f(x)\}$ is c.e. This also explains the use of the term *subcomputable* for such functions.)

A characteristic function χ_A is approximable from below iff A is c.e., and approximable from above iff A is co-c.e. The definitions of approximability from below and from above may seem dual, but in fact there are significant distinctions between them. For an example, contrast the following easy lemma with the well-known fact that there exist functions which are approximable from below, but not ω-approximable.

Lemma 3.2. *Every function approximable from above is ω-approximable from above.*

Proof. Let $f = \lim_s g$ with $g(x, s+1) \leq g(x, s)$ for all x and s. Then the computable function $h(x) = g(x, 0)$ bounds the number of changes g can make. □

On the other hand, the hierarchy of n-approximability from above does not collapse. (See also Corollary 3.10 below, which uses this lemma to show non-collapse at the ω level, as well.)

Lemma 3.3. *For every n, there is a function f which is $(n+1)$-approximable from above but not n-approximable.*

Proof. We define a computable function $g(x, s)$ and set $f(x) = \lim_s g(x, s)$. To begin, $g(x, 0) = n+1$ for every x. At stage $s+1$, for each x, compute the sequence

$$\varphi_{x,s}(x, 0), \varphi_{x,s}(x, 1), \dots, \varphi_{x,s}(x, t)$$

for the greatest $t \leq s$ such that all these computations converge. If $\varphi_x(x, t) = g(x, s) > 0$, set $g(x, s + 1) = g(x, s) - 1$; otherwise set $g(x, s + 1) = g(x, s)$. Thus $f(x) = \lim_s g(x, s)$ is $(n + 1)$-approximable from above, but if $f(x) = \lim_s \varphi_e(x, s)$, then $\varphi_e(x, s)$ must have assumed each of the values $(n + 1), n, \dots, 1, 0$, and so φ_e is not an n-approximation to f. □

The appropriate duality pairs functions approximable from above with a subclass of the functions approximable from below, as follows.

Definition 3.4. Suppose that g is computable and total, with $g(x, s+1) \leq g(x, s)$ for all x and s, so that $f(x) = \lim_s g(x, s)$ is total and approximable from above. The *dual of* g is the function

$$h(x, s) = g(x, 0) - g(x, s).$$

Thus $j(x) = \lim_s h(x, s)$ is total, approximable from below (by h), and bounded above by $g(x, 0)$. Moreover, g is an α-approximation for f iff h is an α-approximation for j.

Conversely, let j be any function which is approximable from below via $h(x, s)$ and *computably bounded*: that is, j is such that there exists a computable total function b with $j(x) \leq b(x)$ for all x. Then the function $g(x, s) = b(x) - h(x, s)$ is the *dual of h and b*.

It is natural to call j the dual of f, but in fact j depends on the choice of the approximation g: two different approximations g and $\tilde{g}$ will often yield two different duals, though these two duals always differ by a computable function, namely $(g(x,0) - \tilde{g}(x,0))$. The dual of a function approximable from below also depends on the choice of computable bound. Nevertheless, it will be clear from our results below that the class of computably bounded functions approximable from below is the natural dual for the class of functions approximable from above. The computable upper bound in the former class is the obvious counterpart of the built-in computable lower bound of 0 for the latter class.

Functions approximable from below have seen wide usage in computable model theory, for example in [1, 2, 10, 11, 12, 13]. Our interest in functions approximable from above arose from our investigations into the distance function on a computable graph. To our knowledge, this is the first significant use of such functions in computable model theory, although, as we will mention, they arise implicitly in the study of effectively algebraic structures and in certain other contexts. The best-known example of a function approximable from above does not come from computable model theory at all: it is the function of Kolmogorov complexity (for any fixed universal machine), mapping each finite binary string (coded as a natural number) to the shortest program which the fixed machine can use to output that string.

It was a theorem of Khoussainov, Nies, and Shore in [13] that there exists a Δ^0_2 set which is not the range of any function approximable from below. The following theorem contrasts with that result, giving a very concrete distinction between approximability from above and from below.

Theorem 3.5. *The range of every approximable function is the range of some function which is* 2*-approximable from above. Indeed, the ranges of the* 2*-approximable-from-above functions are precisely the* Σ^0_2 *sets.*

Proof. We prove the stronger statement. Being in the range of a 2-approximable-from-above function is clearly a Σ^0_2 condition. For the converse, let $S \in \Sigma^0_2$ have computable 1-reduction p to the Σ^0_2-complete set **Fin**, so that

$$(\forall x)[x \in S \iff |W_{p(x)}| < \infty].$$

We will assume that at each stage s, there is exactly one x such that $W_{p(x),s+1} \neq W_{p(x),s}$, and also that for every x, $W_{p(x)} \neq \emptyset$; both of these conditions are readily arranged. Fix the least $x_0 \in S$. At stage 0 we define nothing. At stage $s+1$, for each $x < s$, define

$$g(x, s+1) = \begin{cases} x_0, & \text{if } g(x,s) = x_0 \\ x_0, & \text{if } W_{g(x,s),s+1} \neq W_{g(x,s),s} \\ g(x,s), & \text{otherwise.} \end{cases}$$

Then, for the unique y such that $W_{p(y),s+1} \neq W_{p(y),s}$, let

$$g(s,0) = g(s,1) = \cdots = g(s,s+1) = \begin{cases} p(y), & \text{if } p(y) \geq x_0 \\ x_0, & \text{if not.} \end{cases}$$

This defines g effectively on all of $\omega \times \omega$, and for every x, $g(x,s)$ is either x_0 for all s, or $p(y)$ for all s (where y was chosen at stage $x+1$), or else $p(y)$ for $s = 0, 1, \ldots, n$ and then x_0 for all $s > n$. This last holds iff $p(y) > x_0$ and $W_{p(x)}$ received a new element at some stage $n+1 > x+1$. So clearly g has a limit $f(x) = \lim_s g(x,s)$ and approximates that limit from above, with at most one change. Moreover, if $y \in S$, then $p(y) \in \mathbf{Fin}$, and so when (the nonempty set) $W_{p(y)}$ receives its last element, say at stage $x+1$, then x will have $g(x,s) = p(y)$ for all $s > x$, making $S \subseteq \mathrm{rg}(f)$. Conversely, if $x \notin S$, then $p(y) \notin \mathbf{Fin}$, so every x which ever had $g(x,s) = p(y)$ will eventually get changed and will have $f(x) = x_0$; thus $\mathrm{rg}(f) \subseteq S$. □

Corollary 3.6. *There is a function* 2*-approximable from above, the range of which is not the range of any function approximable from below.*

Proof. This is immediate from Theorem 3.5 in conjunction with a result in [13] giving the existence of a Δ^0_2 set which is not the range of any function approximable from below. □

Corollary 3.7. *There exists a function that is* 2*-approximable from above whose range is* Σ^0_2*-complete.* □

Theorem 3.8. *Every* ω*-approximable function* f *is* bT*-reducible to some function approximable from above. (This approximation from above is known as the* countdown function *for* f*.)*

Proof. Let $f(x)$ be an ω-approximable function, approximated by $g(x,s)$, with computable function $h(x)$ bounding the number of mind changes of $f(x,s)$. Set $c(x,0) = h(x)$. Let $c(x,s+1) = c(x,s)$ unless $g(x,s) \neq g(x,s+1)$, and in that case set $c(x,s+1) = c(x,s)-1$. Then $\lim_s c(x,s)$ is approximable from above and $f(x)$ is computable from this limit, since $f(x) = f(x,t)$ for each t with $c(x,t) = \lim_s c(x,s)$.

Our reasons for referring to this c as the *countdown function* for f (or, strictly speaking, for g and h, since c does depend on the approximation and the computable bound) are clear. It is important to distinguish the countdown function $c(x,s)$, which is computable, from its limit $\lim_s c(x,s)$, which in general is not computable (and was just shown to Turing-compute f). Indeed, we have $f \leq_{\mathtt{bT}} \lim c$, since the only value of the limit required to compute $f(x)$ is $\lim_s c(x,s)$. □

On the other hand, this is not in general a truth-table reduction. For that, one would need to predict in advance what the value of $f(x)$ will be for every possible value of $\lim_s c(x,s)$ between 0 and $c(x,0)$. Without knowing $\lim_s c(x,s)$ in advance, one cannot be sure for how many values of s we may need to compute $g(x,s)$ to determine these answers.

The limit of c is not in general Turing-reducible to f. However, if g is either an approximation from above or an ω-approximation from below, then $f \equiv_T \lim_s c(\cdot,s)$, and indeed $f \equiv_{\mathtt{bT}} \lim_s c(\cdot,s)$, since the computation of $\lim_s c(x,s)$ only requires us to ask the oracle for the value $f(x)$. (Once an approximation from above or from below abandons a value, it cannot later return to that value, and so, once $f(x) = g(x,t)$, we know that $c(x,t) = \lim_s c(x,s)$.) However, even for approximations from above and ω-approximations from below, we have in general that $\lim c \not\leq_{\mathtt{tt}} f$, since we cannot determine the final value of the countdown without actually knowing the value $f(x)$.

Theorem 3.9. *There is a function* f *that is* 1*-complete within the class of all functions approximable from above.*

Proof. We construct f by constructing a computable function g approximating f from above. At stage 0, g is undefined on all inputs.

At stage $s+1$, find the least pair $k = \langle e, x\rangle$ (if any) such that $\varphi_{e,s}(x,0)\downarrow$ and n_k is undefined. Let n_k be the least element such that $g(n_k, 0)$ was undefined as of stage s, and set

$$g(n_k, 0) = g(n_k, 1) = \cdots = g(n_k, s+1) = \varphi_{e,s}(x, 0).$$

Then (whether or not such a k existed), for each $j = \langle e, x\rangle$ such that $g(n_j, 0)$ was defined by stage s, we consider the sequence $\varphi_{e,s}(x,0), \ldots, \varphi_{e,s}(x,t)$, for the greatest $t \leq s$ such that all these computations converge. If this finite sequence is nonincreasing, we set $g(n_j, s+1) = \varphi_e(x,t)$. Otherwise (that is, if $\varphi_e(x, t'+1) > \varphi_e(x, t')$ for some $t' < t$), we set $g(n_j, s+1) = g(n_j, s)$.

This completes the construction of g. Clearly, every n is chosen at some stage to be n_k for some k, and subsequently $g(n_k, s)$ is defined for each s, so g is total and computable. Moreover, by construction, $g(n_k, s+1) \leq g(n_k, s)$ for every k and s. So the function $f(n) = \lim_s g(n,s)$ is approximable from above.

Now let h be any function which is approximable from above, say by $h(x) = \lim_s \varphi_e(x,s)$. Then, for every x, $\varphi_e(x,0)$ converges at some finite stage, and so some n_k with $k = \langle e, x\rangle$ is eventually chosen. The function d_e mapping x to this $n_{\langle e,x\rangle}$ (for this fixed e) is computable (since we can simply wait until $n_{\langle e,x\rangle}$ is defined) and total, and also 1-1, since $n_j \neq n_k$ for $j \neq k$. Now since φ_e approximates h from above, the sequence

$$\varphi_e(x,0), \varphi_e(x,1), \varphi_e(x,2), \ldots, \varphi_e(x,t), \ldots$$

is infinite and nonincreasing. So the sequence

$$g(n_k, 0), g(n_k, 1), g(n_k, 2), \ldots, g(n_k, s), \ldots$$

is exactly the same sequence, by construction, except that numbers which occur finitely often in one sequence might occur a different finite number of times in the other sequence. This shows that

$$f(d_e(x)) = f(n_k) = \lim_s g(n_k, s) = \lim_t \varphi_e(x,t) = h(x),$$

so $f \circ d_e = h$, proving $h \leq_1 f$ via d_e. Indeed, therefore, the 1-reduction may be found uniformly in the index of a computable approximation to h from above. $\square$

Theorem 3.9 gives an easy proof of a result which we could have shown by the method from Lemma 3.3.

Corollary 3.10. *There exists a function which is ω-approximable from above, but (for every $n \in \omega$) is not n-approximable.*

Proof. If the 1-complete function f from Theorem 3.9 were n-approximable, say via $g(x,s)$, then since every function approximable from above has a 1-reduction h to f, we would have that every such function is n-approximable (via $g(h(x),s)$). This contradicts Lemma 3.3. $\square$

Theorem 3.9 stands in contrast to the following results.

Theorem 3.11. *No function is m-complete for the class of functions approximable from below.*

Proof. Let f be any function with a computable approximation $g(x,s)$ of f from below. We build a computable approximation h from below to a function j which cannot be m-reducible to f. Define $h(\langle e, x\rangle, 0) = 0$ for every e and x.

At stage $s+1$, if $\varphi_{e,s}(\langle e,x\rangle)\uparrow$, set $h(\langle e,x\rangle,s+1)=0$. Otherwise, set $h(\langle e,x\rangle,s+1)=1+g(\varphi_e(\langle e,x\rangle),s+1)$. By hypothesis on g, this h is clearly nondecreasing in s, and since $\lim_s g(\varphi_e(\langle e,x\rangle,s))=f(\langle e,x\rangle)$ must exist, we see that

$$j(\langle e,x\rangle)=\lim_s h(\langle e,x\rangle,s)=1+\lim_s g(\varphi_e(\langle e,x\rangle),s)=1+f(\varphi_e(\langle e,x\rangle)).$$

However, this shows that either $j\neq f\circ\varphi_e$ or else φ_e is not total, so $j\not\leq_m f$. □

Theorem 3.12. *For every function f approximable from above, there exists a function g which is 2-approximable from below and has $g\not\leq_m f$.*

Proof. Given a function f that is approximable from above, we construct an ω-approximable function, g, that f fails to m-compute. To achieve this, we assume that $f(x)=\lim_s \varphi_f(x,s)$, and consider possible witnesses for an m-reduction, diagonalizing against them. We must meet the following requirement for each $e\in\omega$:

$$R_e:\ \varphi_e \text{ is total } \implies \exists x\ f(\varphi_e(x))\neq g(x).$$

Set $g(x,0)=0$ for each $x\in\omega$. If $\varphi_e(e)\downarrow=y_e$ at stage s, set $g(e,t)=f(y_e,s)+1$ for all $t\geq s$. For $t>s$, and any y, we have $f(y,t)<f(y,s)$, so once R_e receives attention, it is forever satisfied. Furthermore, $f(e,0)$ serves as a computable upper bound for $g(e,s)$ so g is computably bounded, and the number of mind changes is at most 2.

□

The following result suggests the insufficiency of Turing reducibility, and even of bounded Turing reducibility, to classify functions approximable from above.

Proposition 3.13. *Every function approximable from above is bounded-Turing equivalent to (the characteristic function of) a computably enumerable set, and every c.e. set is* bT*-equivalent to a function approximable from above.*

Proof. Let $f=\lim_s g(x,s)$ be an approximation of f from above. Define the c.e. set

$$V_f=\{\langle x,n\rangle : \exists^{>n}s\ [g(x,s+1)\neq g(x,s)]\}.$$

That is, $\langle x,n\rangle\in V_f$ if and only if the approximation to f changes its mind more than n times. Clearly $V_f\leq_T f$, and the computation deciding whether $\langle x,n\rangle\in V_f$ requires us only to ask the oracle for the value of $f(x)$. Conversely, to compute $f(x)$ from a V_f-oracle, we first compute $g(x,0)$, and then ask the oracle which of the values $\langle x,0\rangle,\ldots,\langle x,g(x,0)-1\rangle$ lies in V_f; the collective answer tells us exactly how many times the approximation will change its mind, and with this information we simply compute $g(x,s)$ until we have seen that many mind changes. Both of these are bounded Turing reductions. (Neither, however, is a truth-table reduction. In fact, the characteristic function of V_f is tt-equivalent to the limit of the countdown function for f, which is not in general tt-equivalent to f.)

For the second statement, note that every c.e. set is bT-equivalent (indeed tt-equivalence with norm 1) to the characteristic function of its complement. □

Corollary 3.14. *There is a 2-approximable function that is not Turing equivalent to any function approximable from above.*

Proof. Let S be a d.c.e. set which is not of c.e. degree. (See, e.g., [5] for such a construction). Then the characteristic function χ_S is 2-approximable, and Proposition 3.13 completes the result. □

4. The Distance Function in Computable Graphs

With the results of the preceding sections completed, we may now address the intended topic of this paper: the distance function on a computable graph. Computable graphs are defined by the standard computable-model-theoretic definition.

Definition 4.1. A structure $\mathcal{M}$ in a finite signature is *computable* if it has an initial segment of ω as its domain and all functions and relations on $\mathcal{M}$ are computable when viewed as functions and relations (of appropriate arities) on that domain.

Therefore, a symmetric irreflexive graph G is *computable* if its domain is either ω or a finite initial segment thereof, and if there is an algorithm which decides, for arbitrary $x, y \in G$, whether G contains an edge between x and y or not (i.e. if the algorithm computes the entries of the adjacency matrix).

The *distance function* d on a graph G maps each pair $(x, y) \in G^2$ to the length of the shortest path from x to y. Assuming G is connected, such a path must exist. By definition $d(x, x) = 0$, and this is quickly seen to be a metric on G. Moreover, if G is a computable graph, then the distance function on G is approximable from above, since for any x and y, we can simply search for the shortest path from x to y. Formally, letting G_s be the induced subgraph of G on the vertices $\{0, \ldots, s\}$, we find the least t for which G_t contains x, y, and a path between them, and let $g(x, y, 0)$ be the length of the shortest such path in G_t. Then we define $g(x, y, s + 1)$ to be the minimum of $g(x, y, s)$ and the length of the shortest path (if any) between x and y in G_s. This sequence is decreasing in s, with limit $d(x, y)$. In the language of computable model theory, we say that for every connected computable graph G, the distance function is *intrinsically approximable from above*, since it is approximable from above in every computable graph isomorphic to G. (See Definition 5.1 below.)

We will now describe a graph construction that enables us to encode arbitrary functions approximable from above into the distance function on a computable graph. The definition is best understood by looking at the subsequent diagram.

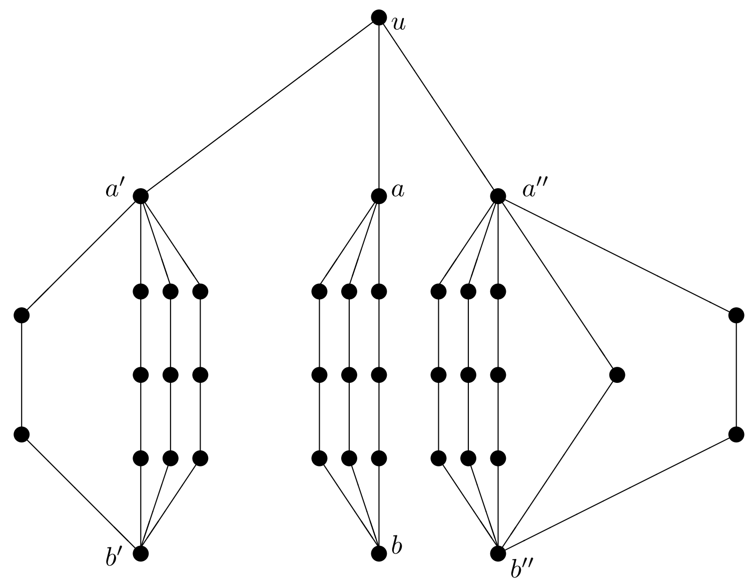
Definition 4.2. Let $\sigma \in \omega^{<\omega}$ be any strictly decreasing nonempty finite string. A *spoke* of type σ in a graph consists of the following.

- A *center node* u, which will be part of every spoke; and
- two more nodes a (adjacent to u) and b (not adjacent to u, nor to a), which belong to this particular spoke; and
- for each $n < |\sigma|$, a path from a to b consisting of $(1 + \sigma(n))$ more nodes (hence of length $2 + \sigma(n)$); and
- two more paths from a to b, of length $2 + \sigma(0)$, in addition to the one already built for $n = 0$ in the preceding instruction.

These paths do not intersect each other, except at a and b, and if x and y are nodes from two distinct paths, then x and y are not adjacent to each other, nor to u.

For any function Γ mapping each $m \in \omega$ to a finite decreasing nonempty string $\sigma_m \in \omega^{<\omega}$, the *standard graph of type* Γ consists of a center node u and, for each m, a spoke of type σ_m. (If Γ is not injective, then there are just as many spokes of type σ as there are elements in $\Gamma^{-1}(\sigma)$.)

Here is a picture of three spokes of such a graph, with $\sigma = \langle 2 \rangle$, $\sigma' = \langle 2, 1 \rangle$, and $\sigma'' = \langle 2, 1, 0 \rangle$.



If f is approximated from above by a computable g, and $g(n, 0) = 2$, for instance, then we build a spoke between a_n and b_n of type $\langle 2\rangle$ (the type between a and b in the diagram above). If, for some $s > 0$, we find $g(n, s) = 1$, then we add a path of length 3 to that spoke between a_n and b_n changing it to type $\langle 2, 1\rangle$, so that it looks like the spoke between a' and b' above. If it turns out that $g(n, t) = 0$ for some t, then we add one more path, as in the spoke between a'' and b'', leaving a spoke of type $\langle 2, 1, 0\rangle$. In all cases, the distance from the node a_n to the node b_n of this spoke turns out to be $\lim_s g(n, s)$, which is to say $f(n)$. So an arbitrary function approximable from above can be coded into a distance function in this way. However, the distance function for the entire graph needs to do more than just to determine the length of each single spoke, and complications will arise when we apply our strong reducibilities on functions.

If Γ is computable, then of course we have a computable presentation of the standard graph of type Γ. However, we will usually be interested in the situation where $\Gamma = \lim_s \Gamma_s$ and Γ_s is computable uniformly in s, with each $\Gamma_s(m)$ being an initial segment of $\Gamma_{s+1}(m)$, such that $\Gamma(m) = \cup_s \Gamma_s(m)$. Since every string $\Gamma_s(m)$ is strictly decreasing, $\Gamma(m)$ will be another finite decreasing string. Assuming that the functions Γ_s are computable uniformly in s, it is clear how to build a computable presentation of the standard graph of type Γ, as the union of nested uniform presentations of the standard graphs of type Γ_s.

As already noted, the distance function of a computable connected graph is always approximable from above. Not every function approximable from above can be a distance function, however: for one thing, the range of the distance function d of a connected graph G is always an initial segment of ω, whereas plenty of functions approximable from above do not have such ranges. Indeed, for each $x \in G$, the set $\{d(x, y) : y \in G\}$ must be an initial segment of ω. Moreover, symmetry and a triangle inequality must hold of every distance function. Nevertheless, we do have the following proposition, as well as the stronger version given in the subsequent theorem, which is the result we prove.

Proposition 4.3. *Every function which is approximable from above is* $\mathtt{btt}$*-equivalent, with norm* 2*, to the distance function for some computable graph.*

Theorem 4.4. *For every function* f *which is approximable from above, there is a computable graph* G *such that the Turing degree spectrum of its distance function* d *is* $\{deg(f)\}$*. Indeed, for every computable* H *isomorphic to* G*, the distance function* d_H *on* H *satisfies*

$$d_H \equiv_1 d \leq_{2\text{-}btt} f \leq_{1\text{-}btt} d_H.$$

Proof. Write $f = \lim_s g(x,s)$, where g is computable and nonincreasing in s. For each x, let $\Gamma_s(x,s) = \langle g(x,s_0), g(x,s_1), \ldots, g(x,s_k)\rangle$, where $s_0 = 0$ and the subsequent s_i are defined so that:

$$g(x,s_0) = g(x,s_1-1) > g(x,s_1) = g(x,s_2-1) > g(x,s_2) = \cdots > g(x,s_k) = g(x,s).$$

That is, the s_i are the stages $\leq s$ at which $g(x,s)$ decreases. The uniformly computable family Γ_s then has limit Γ Turing-equivalent to f, since $\Gamma(x)$ is a decreasing string whose final value is $f(x)$. Therefore, there is a computable presentation $\tilde{G}$ of the standard graph of type Γ. Our computable graph G is simply $\tilde{G}$ with (for each n) an additional $(n+2)$ nodes which, along with a_n, form a loop of length $(n+3)$ containing a_n. This allows us to determine, for arbitrary nodes a adjacent to the top center node u_H in an arbitrary computable graph H isomorphic to G, the value n for which $a = a_n$. (The set $\{a_n : n \in \omega\}$ is defined by adjacency to the single node u_H. Then, for each a_n, the three paths of equal length between a_n and b_n allow us to identify b_n, so that we will not confuse the loop of length $n+3$ containing a_n with any loop which contains both a_n and b_n. Having picked out b_n, we then find the unique loop which contains a_n but neither b_n nor u_H, and the length of this loop determines the index n for us.) Thus, with these new loops added, the entire graph G is relatively computably categorical, having a Σ^0_1 Scott family defined using the loops. So every computable copy H of G is computably isomorphic to G.

It follows that the distance function d_H on the arbitrary computable copy H is 1-equivalent to the distance function d on G. Indeed, if $h : G \to H$ is a computable isomorphism, write $h_2 : G^2 \to H^2$ by $h_2(x,y) = (h(x), h(y))$. Then h_s is computable and $d_H \circ h_2 = d$; likewise $d_H = d \circ (h^{-1})_2$.

But f allows us to decide the shortest path from a_n to b_n, for every n, since $(2 + f(n))$ is the length of that path. It follows that the distance function d on all of G is computable in f. (This is made explicit in Lemma 4.6, which goes through for this graph with the extra loops, as well as for standard presentations.) Conversely, the distance function d allows us to compute $(2+f(n))$ for every n, just by finding $d(a_n, b_n)$. So this distance function is Turing-equivalent to f, and by the 1-equivalence above, the spectrum of the distance function is simply $\{\deg(f)\}$. More specifically, we have $f \leq_{1\text{-}\mathtt{btt}} d$ (since $f(n) = 2 + d(a_n, b_n)$) and $d \leq_{2\text{-}\mathtt{btt}} f$, using the proof of Lemma 4.6. □

Corollary 4.5. *There exists a computable graph* G *such that, for every* n *and for every computable graph* $H \cong G$*, the distance function on* H *is not* n*-approximable from above. (So the distance function on this* G *intrinsically fails to be* n*-AFA.)*

Proof. By Corollary 3.10, there is a function f which is ω-approximable from above, but not n-approximable for any n. Apply Theorem 4.4 to this f, and note that if f

were 1-reducible to a function n-approximable from above, then f itself would be n-approximable from above. □

To complete the proof of Theorem 4.4, we need the following lemma.

Lemma 4.6. *Let G be a computable copy of the standard graph of some type Γ, and let u be the center of G. Then there is are computable functions p_a and p_b which map every node $x \in G - \{u\}$ to the unique node $p_a(x)$ adjacent to u such that x and $p_a(x)$ are on the same spoke of G, and to the other end point $p_b(x)$ of that spoke. Moreover, if we define*

$$S = \{\langle a, p_b(a)\rangle \in G^2 : a \textit{ is adjacent to } u\},$$

then the distance function d for G satisfies

$$d \leq_{2\text{-}btt} (d{\restriction}S) \leq_{1\text{-}btt} d.$$

Proof. Given any $x \in G$ with $x \neq u$, we search for a path in G which goes from x to u without containing u (except as its end point). The node on this path which is adjacent to u must be the desired $p_a(x)$, simply because of the structure of G.

We then compute $p_b(x)$ by finding three paths of equal length from $p_a(x)$ to a common end point, such that none of these paths contains u or intersects another of the three paths (except at their end points). The common end point must then be $p_b(x)$. (This is the reason why a spoke of type σ has two extra paths of length $\sigma(0)$. Without those paths, there could exist computable copies of G in which this function p_b would not be computable.)

To compare d with $d{\restriction}S$, consider any $x, y \in G - \{u\}$. If x and y lie on the same spoke (that is, if $p_a(x) = p_a(y)$), then we can check whether x and y lie on the same path from $a = p_a(x)$ to $b = p_b(x)$. This gives three cases.

(1) If x and y lie on the same path between the same a and b, assume without loss of generality that on this path, x lies closer to a, say with m nodes between them, and y lies closer to b, with n nodes between them. Then $d(x, y)$ is either the distance between them along this path, or else $(m + d(a, b) + n)$, whichever is smaller. So we need only ask the $(d{\restriction}S)$-oracle for the value $d(a, b)$, and then compare these two possibilities.

(2) If x and y lie on different paths within the spoke between the same a and b, write $d'(a, x)$ and $d'(x, b)$ for the distances between those nodes along the path through x, and $d'(a, y)$ and $d'(y, b)$ likewise along the path through y. (It is possible that $d(a, x) < d'(a, x)$, if there is a much shorter separate path from a to b. However, $d'(a, x)$ is computable.) Then $d(x, y)$ is the least of the lengths of the following paths from x to y:

$[d'(x, a) + d'(a, y)]$ (via a path through a);

$[d'(x, b) + d'(b, y)]$ (via a path through b);

$[d'(x, a) + d(a, b) + d'(b, y)]$ (via a path through a, then b);

$[d'(x, b) + d(b, a) + d'(a, y)]$ (via a path through b, then a).

We can compute all of these by asking the $(d{\restriction}S)$-oracle for $d(a, b)$, and having done so, we need only take the minimum of these four values.

(3) If x and y lie on distinct spokes, find the end points $a_x = p_a(x)$, $a_y = p_a(y)$, $b_x = p_b(x)$, and $b_y = p_b(y)$ of those spokes. As in Case (2), we use $d'(x, a_x)$

to denote the distance from x to a_x along the path from a_x to a_y through x, which we can compute; similarly for $d'(y, a_y)$, etc.

First we compute $d(x, u)$, which is the minimum of

$$d'(x, a_x) + 1, \text{ (via a path through } a_x\text{), and}$$

$$d'(x, b_x) + d(b_x, a_x) + 1, \text{ (via a path through } b_x\text{, then } a_x\text{)}$$

This uses the oracle for $d{\restriction}S$. Likewise we compute $d(y, u)$. Then it is clear from the structure of G that $d(x, y) = d(x, u) + d(u, y)$. Notice that here in Case (3) we needed to ask two questions of the $(d{\restriction}S)$-oracle: the values of $d(a_x, b_x)$ and $d(a_y, b_y)$.

Finally, Case (3) showed how to compute $d(x, u)$. So we have computed $d(x, y)$ for all possible pairs $(x, y) \in G^2$, proving $d \leq_T (d{\restriction}S)$, and the only questions we asked of the oracle for $d{\restriction}S$ were the values $d(p_a(x), p_b(x))$ and $d(p_a(y), p_b(y))$. Moreover, a close reading of the proof shows that we could give in advance a formula for $d(x, y)$ based on these two values, such that the formula always converges to an answer. (The formulas are slightly different in Cases (1), (2), and (3), but we could distinguish these three cases and choose the correct one for the pair (x, y) before consulting the oracle at all.) Thus we have a `btt`-reduction of norm 2 from the function d to the function $d{\restriction}S$. (It would be of norm 1 if not for Case (3), which required two questions to be asked of the oracle.)

The 1-reduction $(d{\restriction}S) \leq_1 d$ is obvious: one reduces using the identity function. To obey the technicalities of the definition, one should define $d{\restriction}S$ to be equal to d on S, and equal to 0 everywhere else, since under a careful reading, no non-total function can 1-reduce to a total function. Fortunately, S is a computable set, so the 1-reduction can map each pair $(x, y) \notin S^2$ to a distinct pair (z, z) with $z \notin S$. □

This allows us to prove a corollary about the distance function on each computable copy of G.

Corollary 4.7. *There exists a computable connected graph G_ω whose distance function is (of course) intrinsically approximable from above, and such that for every function f approximable from above, there exists a computable graph G_f isomorphic to G whose distance function d_f satisfies $d_f \leq_{2\text{-}btt} f$ and $f \leq_{1\text{-}btt} d_f$.*

Proof. Let Γ be a computable function which enumerates all strictly decreasing sequences in $\omega^{<\omega}$. Moreover, arrange Γ so that every sequence in the range of Γ is equal to $\Gamma(n)$ for infinitely many n. Let G_ω be a computable presentation of the standard graph of type Γ, and d_ω its distance function.

Now let $f : \omega \to \omega$ be any total function which is approximable from above. Write $f = \lim_s g(x, s)$, where g is computable and nonincreasing in s. For each x, let $\Delta_s(2x) = \langle g(x, s_0), g(x, s_1), \ldots, g(x, s_k)\rangle$, where $s_0 = 0$ and s_{i+1} is the least $t \leq s$ (if any) such that $g(x, t) < g(x, s_i)$. When there is no such t, we set $k = i$, ending the sequence. Thus

$$g(x, s_0) = g(x, s_1 - 1) > g(x, s_1) = g(x, s_2 - 1) > g(x, s_2) = \cdots > g(x, s_k) = g(x, s),$$

and the s_i are the stages $\leq s$ at which $g(x, s)$ decreases. Meanwhile, let $\Delta_s(2x+1) = \Gamma(x)$ for every x and s. The uniformly computable family Δ_s then has limit Δ with $f \leq_{1\text{-btt}} \Delta$, since $\Delta(2x)$ is a decreasing string whose final value is $f(x)$, while $\Delta(2x+1) = \Gamma(x)$ is computable. Therefore, there is a computable presentation G_f of the standard graph of type Δ. We claim that $G_f \cong G_\omega$, and that the distance function d_f of G_f has $d_f \leq_{\texttt{2-btt}} F \leq_{\texttt{1-btt}} f$ (hence $d_f \leq_{\texttt{2-btt}} f$) and $f \leq_{\texttt{1-btt}} d_f$. This follows from Lemma 4.6, since $d_f(a_{2x}, b_{2x}) = f(x)$ and $d_f(a_{2x+1}, b_{2x+1})$ is computable in x. The isomorphism between G and G_ω is clear, since the range of Δ and the range of Γ each contains every strictly decreasing string in $\omega^{<\omega}$ infinitely many times: every such string equals both $\Gamma(n)$ and $\Delta(2n+1)$ for infinitely many n. This determines the isomorphism types of the standard graphs G and G_ω of these types, and they are the same. □

5. n-Approximable Distance Functions

We now turn to graphs in which the distance function is n-approximable from above, for some fixed n. Our goal here is to repeat Corollary 4.7 for the property of n-approximability from above, rather than for arbitrary approximability from above. Of course, a function which is n-approximable from above must be approximable from above, and therefore is included in statements such as Theorem 4.4. However, we would like to make the n-approximability intrinsic to the graph, in the following sense (which is standard in computable model theory).

Definition 5.1. The distance function on a computable graph G is *intrinsically n-approximable from above* if, for every computable graph H isomorphic to G, the distance function on H is n-approximable from above. G is *relatively intrinsically n-approximable from above* if for every graph $H \cong G$ with domain ω, the Turing degree of the edge relation on H computes an n-approximation from above to the distance function on H.

One could give the same definition with ω in place of n, but we have already remarked that the distance function of every computable graph is approximable from above (which is to say, ω-approximable from above), so this would be trivial. In a graph satisfying Definition 5.1, there is some structural reason for which the distance function is always n-approximable from above, no matter how one presents the graph. (Relative intrinsic n-approximability from above says that this holds even when we consider noncomputable presentations.) The construction given in Corollary 4.7 does not have this property: even if we have n-approximations to $d(a_x, b_x)$ and $d(a_y, b_y)$ from above, we do not get an n-approximation to $d(b_x, b_y)$: this distance will equal the sum $d(b_x, a_x) + 2 + d(a_y, b_y)$, and since either summand could decrease as many as n times, $d(b_x, b_y)$ could decrease as many as $2n$ times. (This has to do with d being 2-`btt` reducible to the original function, rather than 1-`btt` reducible.) So we must revise the format of our graphs. The next definition does so in several ways, mainly by adding a second center node and elongating the paths between the center nodes and the a- and b-nodes of each spoke. Both of these changes will turn out to be essential, and will result in useful theorems, but as we shall see, the best result one would hope for remains unproven.

Definition 5.2. Let $\sigma \in \omega^{<\omega}$ be any strictly decreasing nonempty finite string. An *elongated spoke* of type σ in a graph consists of the following.

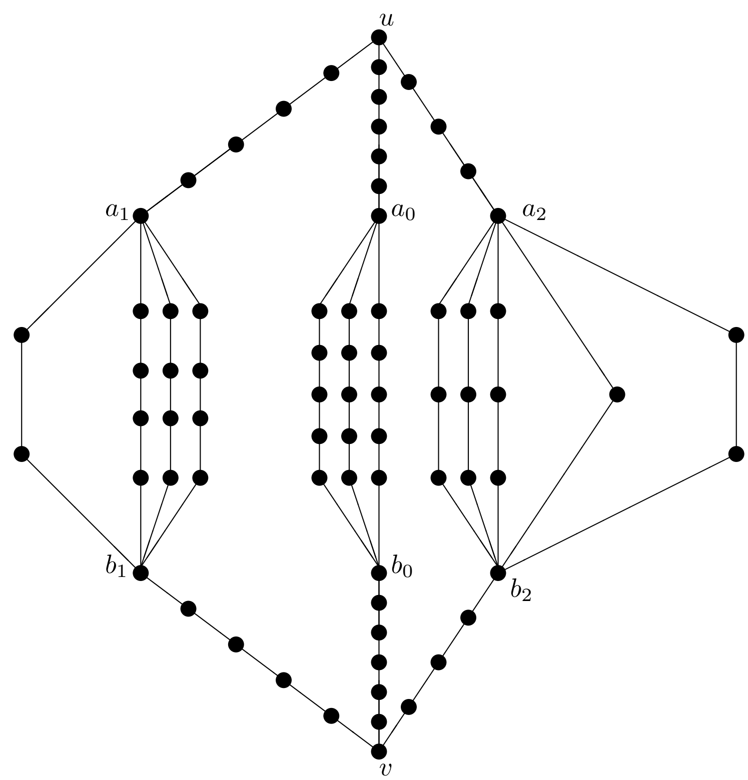
- A chain of $(3 + \sigma(0))$ nodes (hence of length $2 + \sigma(0)$), beginning with the *top center node* u and ending with a node called a; and
- another chain of $(3 + \sigma(0))$ nodes, beginning with the *bottom center node* v (usually visualized sitting below u) and ending with a node called b; and
- for each $n < |\sigma|$, a path from a to b consisting of $(1+\sigma(n))$ nodes in addition to a and b (hence of length $2 + \sigma(n)$); and

- two more paths from a to b, of length $2+\sigma(0)$, in addition to the one already built for $n=0$ in the preceding instruction.

These paths do not intersect each other, except at a and b, and if x and y are nodes from two distinct paths, then x and y are not adjacent to each other, nor to u, nor to v. The center nodes u and v will belong to every spoke in the graph, but all other nodes belong only to this spoke, and will not be adjacent to any node in any other spoke.

For any function Γ mapping each $m\in\omega$ to a finite decreasing nonempty string $\sigma_m\in\omega^{<\omega}$, the *elongated standard graph of type* Γ consists of center nodes u and v and, for each m, a spoke of type σ_m. (If Γ is not injective, then there are just as many spokes of type σ as there are elements in $\Gamma^{-1}(\sigma)$.)

Here is a picture of three spokes of such a graph, with $\sigma_0=\langle 4\rangle$, $\sigma_1=\langle 3,1\rangle$, and $\sigma_2=\langle 2,1,0\rangle$. Notice that the elongation paths emerging from u and from v have different lengths, determined by the length of the longest path from a_n to b_n.



If Γ is computable, then of course we have a computable presentation of the elongated standard graph of type Γ. As before, though, we will usually be interested in the situation where $\Gamma=\lim_s\Gamma_s$ and Γ_s is computable uniformly in s.

Lemma 5.3. *Let G be a computable copy of the elongated standard graph of some type Γ, with distance function d, and let u and v be the centers of G. Then there are computable functions p_a, p_b with domain $G-\{u,v\}$ which output the unique nodes $p_a(x)$ and $p_b(x)$ of valence >2 such that x, $p_a(x)$, and $p_b(x)$ are all on the same spoke of G with $d(p_a(x),u)<d(p_b(x),u)$. We can also compute whether x lies between $p_a(x)$ and $p_b(x)$, or above $p_a(x)$, or below $p_b(x)$.*

Moreover, the distance function d satisfies $d \leq_{2\text{-}btt} (d{\restriction}S)$ and $(d{\restriction}S) \leq_1 d$, where S is the following set:

$$S = \{\langle p_a(x), p_b(x)\rangle \in G^2 : x \notin \{u, v\}\}.$$

Proof. Given any $x \in G$ with $x \notin \{u, v\}$, we search for a path in G which goes from u through x to v without containing any node more than once. Then we enumerate G until two nodes on this path (distinct from u and v) are each adjacent to at least three other nodes. By the structure of G, these two nodes are the desired $p_a(x)$ and $p_b(x)$, with $p_a(x)$ being the one closer to u along the path we found, and from this path we can also determine whether x lies above $p_a(x)$, below $p_b(x)$, or between the two of them.

To compare d with $d{\restriction}S$, let $x, y \in G - \{u, v\}$. Find the end points $a_x = p_a(x)$, $a_y = p_a(y)$, $b_x = p_b(x)$, and $b_y = p_b(y)$ of the spokes containing x and y. We consider first the case where x lies between a_x and b_x and y lies between a_y and b_y; the subsequent cases will then be easier. These x and y lie on the same spoke in G iff $a_x = a_y$, and if so, we can then check whether x and y lie on the same path between $p_a(x)$ and $p_b(x)$. We use $d'(x, a_x)$ to denote the distance from x to a_x along the path from a_x to a_y through x, which we can compute; similarly for $d'(y, b_y)$, etc. It is important to note that $d'(x, a_x)$ may fail to equal $d(x, a_x)$, since there could be a separate path from a_x to b_x so short that $d(a_x, b_x) + d'(b_x, x) < d'(a_x, x)$. Finally, we can readily find the elongation path lengths of these spokes: the length l_x of the direct path from a_x to u (and of the path from b_x to v), and the similar length l_y for y.

From the structure of G, we see that the shortest path from x to y must be one of the following thirteen paths. (This requires a combinatorial argument, based on the valences of the nodes – which determine the number of options a path has at each point, given that the path should not go through the same node twice. Certain other paths are combinatorially possible but are not on this list; they are discussed below as $P_{13}, \ldots, P_{20}$.)

Path	Route	Length
P_0	x to a_x to u to a_y to y	$d'(x, a_x) + l_x + l_y + d'(a_y, y)$
P_1	x to a_x to u to a_y to b_y to y	$d'(x, a_x) + l_x + l_y + d(a_y, b_y) + d'(b_y, y)$
P_2	x to a_x to u to v to b_y to y	$d'(x, a_x) + l_x + d(u, v) + l_y + d'(b_y, y)$
P_3	x to a_x to b_x to v to b_y to y	$d'(x, a_x) + d(a_x, b_x) + l_x + l_y + d'(b_y, y)$
P_4	x to b_x to v to b_y to y	$d'(x, b_x) + l_x + l_y + d'(b_y, y)$
P_5	x to b_x to v to b_y to a_y to y	$d'(x, b_x) + l_x + l_y + d(b_y, a_y) + d'(a_y, y)$
P_6	x to b_x to v to u to a_y to y	$d'(x, b_x) + l_x + d(v, u) + l_y + d'(a_y, y)$
P_7	x to b_x to a_x to u to a_y to y	$d'(x, b_x) + d(b_x, a_x) + l_x + l_y + d'(a_y, y)$
P_8	x to $a_x = a_y$ to y	$d'(x, a_x) + d'(a_y, y)$
P_9	x to $b_x = b_y$ to y	$d'(x, b_x) + d'(b_y, y)$
P_{10}	x to $a_x = a_y$ to b_y to y	$d'(x, a_x) + d(a_y, b_y) + d'(b_y, y)$
P_{11}	x to $b_x = b_y$ to a_y to y	$d'(x, b_x) + d(b_y, a_y) + d'(a_y, y)$
P_{12}	x to y	$d'(x, y)$

Here paths P_8 through P_{12} are separated because they apply only if $a_x = a_y$ (that is, if x and y lie on the same spoke). P_{12} only applies if x and y lie on the same path through that spoke. Paths P_4 through P_7 may be seen as vertical reflections of paths P_0 through P_3. Every one of these thirteen paths can, under certain circumstances, be the shortest path from x to y. There are eight other paths which one can define from x to y without repeating any nodes (given that one always takes the shortest route between a_x and b_x, or between a_y and b_y, or between u and v, whenever the path route says to go from one of these nodes to the other). Here are four of them; the other four are their vertical reflections.

Path	Route
P_{13}	x to a_x to u to v to b_y to a_y to y
P_{14}	x to a_x to b_x to v to b_y to a_y to y
P_{15}	x to a_x to b_x to v to u to a_y to y
P_{16}	x to a_x to b_x to v to u to a_y to b_y to y

None of these eight paths can be the shortest path from x to y. For example, P_{13} could be shortened by going from u directly to a_y, thereby reducing a subpath of length $d(u,v)+l_y+d(b_y,a_y)$ to a subpath of length l_y. P_{14} and P_{15} also take longer routes than necessary from a_x to a_y, and P_{16} takes a longer route than necessary from v to b_y; similarly for the four vertical reflections. So, by brute force, we have seen that the shortest path must be one of $P_0, \ldots, P_{12}$.

Now, given that d' is computable and that $d(u,v)$ is a finite piece of information, we see that the length of each P_i with $i < 13$ is computable from an oracle for $d{\restriction}S$, uniformly in x and y, and in particular from just two questions to that oracle: the values $d(a_x, b_x)$ and $d(a_y, b_y)$. So this oracle also allows us to compute $d(x,y)$, by taking the minimum of those thirteen lengths. For these x and y, therefore, we have a $\texttt{btt}$-reduction of norm 2.

If x lies between a_x and a_y, as above, but $y = u$, then a similar but easier argument applies: the shortest path from u to x has length either $(l_x + d'(a_x, x))$, or $(l_x + d(a_x, b_x) + d'(b_x, x))$, or possibly $(d(u,v) + l_x + d'(b_x, x))$. When y lies on an elongation path of length l_y beginning at u, we compute $d(a_y, x)$ and $d(u,x)$ as above, and use them to determine $d(y,x)$. The reader should be able to produce a similar argument when $y = v$, and when y lies on an elongation path beginning at v. Finally, in case neither x nor y lies between a_x and a_y (resp. b_x and b_y), the argument is similar, using the two nodes at the end of the elongation path containing x and the similar two nodes for y, and using these to take the minimum of the four possible ways of going from x to y. So we have computed $d(x,y)$ for all possible pairs $(x,y) \in G^2$. Moreover, the only values we ever required from the $(d{\restriction}S)$-oracle were $d(a_x, b_x)$ and $d(a_y, b_y)$, and we were able to compute in advance the value of $d(x,y)$ for each possible answer the oracle might give. Thus $d \leq_{2\text{-}\texttt{btt}} (d{\restriction}S)$. The reverse reduction, $(d{\restriction}S) \leq_1 d$, is proven just as in Lemma 4.6. □

We now repeat Corollary 4.7 for just the functions n-approximable from above. This is where it becomes clear why we used two centers and elongated spokes in our graphs in Definition 5.2. If the graph had only one center u, then the distance $d(b_x, u)$ would be n-approximable from above, but the distance $d(b_x, b_y)$ would in general only be $(2n)$-approximable from above: it would decrease whenever either $d(u, b_x)$ and $d(u, b_y)$ decreased, since it would equal the sum of these two values. It may not be immediately clear why having a second center solves this problem: for example, the distance $d(a_x, b_y)$ now depends on both $d(a_x, v)$ and $d(b_y, u)$, each of which could decrease as many as n times. However, $d(a_x, b_y)$ is now equal to the minimum of these two values (up to a computable difference), not to their sum. With the minimum, we will eventually avoid this difficulty by turning to the countdown function for an arbitrary function which is n-approximable from above. First, though, we prove the basic result.

Theorem 5.4. *For every $n < \omega$, there exists a computable connected graph G whose distance function d_G is intrinsically $2n$-approximable from above and such that, for every function f which is n-approximable from above, there is some computable graph $H \cong G$ whose distance function d_H has $d_H \leq_{2\text{-}btt} f \leq_{1\text{-}btt} d_H$.*

Proof. For $n = 0$, just take G to be any computable graph with an intrinsically computable distance function. Since all total computable functions are btt-equivalent with norm 1, this suffices. (For example, the complete graph on the domain ω could serve as G.)

For $n > 0$, define a computable function $\Gamma : \omega \to \omega^{\leq n}$ so that the range is Γ contains exactly those tuples of length $\leq n$ which are strictly decreasing; moreover, ensure that every such tuple has infinite preimage in ω under Γ. Let G be a computable presentation of the elongated standard graph of type Γ. We claim that this G instantiates the theorem.

To see that the distance function d_H on an arbitrary computable graph $H \cong G$ is $(2n)$-approximable from above, notice that just as in Lemma 5.3, we can compute the functions p_a and p_b for H. Since every $\sigma \in \text{rg}(\Gamma)$ has $|\sigma| \leq n+1$, the paths from $p_a(x)$ to $p_a(y)$ within the spoke of any $x \in H$ have at most $(n+1)$ distinct lengths, so that $d_H(p_a(x), p_a(y))$ can be approximated from above with at most n mind changes. Then we apply Lemma 5.3 to see that the values $d_H(p_a(x), p_b(x))$ and $d_H(p_a(y), p_b(y))$ determine $d_H(x, y)$, by taking minimums of the thirteen paths as described there. So we simply approximate $d_H(p_a(x), p_b(x))$ and $d_H(p_a(y), p_b(y))$ from above, and whenever either approximation is reduced, we reduce our approximation of $d_H(x, y)$ accordingly. This gives an approximation of $d_H(x, y)$ from above with at most $2n$ changes: the $\leq n$ stages at which the approximation to $d_H(p_a(x), p_b(x))$ changed, and the $\leq n$ stages at which $d_H(p_a(y), p_b(y))$ changed.

Next, given an arbitrary function f which is n-approximated from above by g, we build a graph H by starting with G and adjoining, for each $x \in \omega$, a spoke of type

$$\sigma_x = \langle g(x, s_0), g(x, s_1), \ldots, g(x, s_k) \rangle,$$

where $s_0 = 0$ and each $s_{i+1} = \min\{s : g(x, s) < g(x, s_i)\}$. It is clear how one can do this effectively, starting with a spoke of type $\langle g(x, s_0) \rangle$ and extending it to a larger spoke each time a new value of $g(x, s)$ appears. Of course, we have $k \leq n$, and so $\sigma_x = \Gamma(m)$ for infinitely many m. Therefore, the new spokes we add do not change the isomorphism type, but leave $H \cong G$.

One new problem arises: it is no longer immediate that $d(a_m, b_m) = 2 + f(m)$. To see the problem, notice that, whereas in the standard presentation of a graph (with only one center), it was clear that the shortest path from a_m to b_m was one of the paths between them within that spoke. Now, however, there is a path from a_m to u, then through another spoke to v, then up to b_m, and the length of this path is $l_m + d(u, v) + l_m$. The elongation paths were given their length (call it l_m, for the m-th spoke) precisely to ensure that this alternative is not the shortest path from a_m to b_m, and since l_m was chosen to be the length of the longest path from a_m to b_m within the spoke, it is clear that this has been accomplished. So $f(m) + 2$ really does equal $d(a_m, b_m)$. It follows that $f \leq_{1\text{-btt}} d_H$ as required, thanks to Lemma 5.3, which now gives the reducibilities $d_H \leq_{2\text{-btt}} f \leq_{1\text{-btt}} d_H$, exactly as desired. □

To strengthen Theorem 5.4, we would like to make G have a distance function which is intrinsically n-approximable from above. The proof given above does not accomplish this. In particular, for the opposite end points a_j and b_k of two distinct spokes in H, with $j \neq k$, the formula for $d_H(a_j, b_k)$ involves a minimum of eight different values, some of which depend on $d_H(a_j, b_j)$ and others on $d_H(a_k, b_k)$. Each of these two distances could decrease as many as n times as our computable approximations to $f(j)$ and $f(k)$ decrease, and so the minimum could decrease as many as $2n$ times. This is the same problem we would have had using the simpler (single-center, non-elongated) graphs of Definition 4.2, except that there the problem involved a sum, not a minimum. One is led to wonder what purpose Definition 5.2 served. The proof of the following theorem gives the answer.

Theorem 5.5. *For every $n < \omega$, there exists a computable connected graph G whose distance function d_G is intrinsically n-approximable from above and such that, for every function f with a computable n-approximation g from above, there is some computable graph $H \cong G$ whose distance function d_H has*

$$d_H \leq_{2\text{-}btt} \lim_s c \leq_{1\text{-}btt} d_H,$$

where c is the countdown function for g with bound n.

So we have achieved intrinsic n-approximability from above for the distance function, while still allowing the distance function – under a coarser reducibility – to realize all functions n-approximable from above. Since $f \equiv_{\mathtt{bT}} \lim_s c$, the theorem shows that $f \equiv_{bt} d_H$, where H is the graph built for f, but the btt-reducibility and the specific norms have been lost. Before proving the theorem, we summarize this as a corollary.

Corollary 5.6. *For every n, there exists a computable connected graph G such that the bT-degree spectrum of the distance function on G contains exactly the bT-degrees n-approximable from above (that is, those bT-degrees which contain a function n-approximable from above).* □

Proof of Theorem 5.5. Assume $n > 0$, since otherwise the description in the proof of Theorem 5.4 suffices. Let Γ be a computable function with

$$\Gamma(0) = \langle n \rangle,\ \Gamma(1) = \langle n, n-1 \rangle, \ldots,\ \Gamma(n) = \langle n, n-1, \ldots, 0 \rangle$$

and with $\Gamma(m + n + 1) = \Gamma(m)$ for all $m \in \omega$, so that each of these strings appears infinitely often in the range of Γ. Our graph G is a computable presentation of the standard graph of type Γ.

Our first goal is to show that for every computable graph H isomorphic to G, the distance function $d(x, y)$ of G is always n-approximable from above. For $x, y \in G$, the proof of Lemma 5.3 gives $d_H(x, y)$ as the minimum of finitely many values, and that each of those values being a sum of computable values along with $d_H(p_a(x), p_b(x))$ or $d_H(p_a(y), p_b(y))$. Crucially, though, none of those values (whose minimum we take) involves either $d_H(p_a(x), p_b(x))$ or $d_H(p_a(y), p_b(y))$ more than once. Therefore, whenever $d_H(p_a(x), p_b(x))$ decreases by 1, certain of the values decrease by 1 and the rest stay unchanged; likewise, whenever $d_H(p_a(y), p_b(y))$ decreases by 1, certain other values decrease by 1 and the rest stay unchanged. Moreover, the structure of G shows that our approximations to $d_H(p_a(x), p_b(x))$ and $d_H(p_a(y), p_b(y))$ never decrease by more than 1 at any stage. (More exactly, the approximation begins at n, so it can decrease at most n times. If at stage s we thought $d_H(p_a(x), p_b(x)) = 7$, and at stage $s + 1$ we find a path of length 4 between $p_a(x)$ and $p_b(x)$, we can think of this as three separate decreases by 1. In fact, the structure of G is such that in this case there will indeed be paths of length 5 and of length 6 between $p_a(x)$ and $p_b(x)$, even though the path of length 4 appeared first.) This allows us to apply the following lemma, whose proof is a straightforward induction on n.

Lemma 5.7. *If $g(x,s)$ satisfies $g(x,s)-1 \leq g(x,s+1) \leq g(x,s)$ for all x and s, and $h(x,s)$ does the same, and if $g(x,0)=h(x,0)=n$ and C and D are arbitrary constants, then*

$$|\{s : \min(g(x,s+1)+C, h(x,s+1)+D) < \min(g(x,s)+C, h(x,s)+D)\}| \leq n.$$

(By induction, the same then holds for a minimum of arbitrarily many functions with these properties.) □

So $d(x,y)$, being a minimum of exactly this type, is also n-approximable from above. Our use of countdown functions enabled us to use the family Γ whose member strings never decrease by more than 1

Next, consider any function f which has an n-approximation g from above. Let c be the countdown function for this g and for the constant bound n on changes to g. This was defined in Theorem 3.8: $c(x,0)=n$, and $c(x,s+1)=c(x,s)-1$ iff $g(x,s+1) < g(x,s)$, with $c(x,s+1)=c(x,s)$ otherwise. So c keeps track of the number of changes g is still permitted to make as it approximates $f(x)$, and clearly c satisfies the property mentioned above of never decreasing by more than 1. Recall that $f \equiv_{\mathtt{bT}} \lim_s c$, although stronger equivalences, such as $\mathtt{tt}$-equivalence, may fail to hold.

Our construction of the graph H for this f mirrors that of Theorem 5.4, only using the countdown function c in place of the computable approximation g to f itself. We simply start with the computable graph G (whose distance function is also computable) and, for each n, add a new elongated spoke of type $\langle n, n-1, \ldots, k\rangle$, where $k = \lim_s c(n,s)$. Since there were already infinitely many spokes of this type in G, the addition of one (or even infinitely many) more does not change the isomorphism type; thus $H \cong G$. Moreover, it is clear that we can add this spoke in a computable fashion: it starts as a spoke of type $\langle n\rangle$, then has a path added and becomes a spoke of type $\langle n, n-1\rangle$ when and if we find an s with $c(n,s)=n-1$, and so on. By Lemma 5.3, we have $d_H \leq_{\text{2-}\mathtt{btt}} \lim_s c \leq_{\text{1-}\mathtt{btt}} d_H$.. Unfortunately, f cannot be substituted for $\lim_s c$ in these reductions, because in general we only have $f \equiv_{\mathtt{bT}} \lim_s c$, and so we conclude, as claimed by the corollary, that $f \equiv_{\mathtt{bT}} d_H$ for this graph H. □

The same strategy could have been used in the proof of Corollary 4.7, of course. There, however, it was not necessary: the distance function on any computable graph is always ω-approximable from above. Moreover, if x and y in that graph G_ω (or a copy of it) lie on distinct spokes, then an upper bound on the number of changes in the natural approximation to $d(x,y)$ can be given just by adding the (computable) upper bounds on the number of changes in the approximations to $d(a_x,b_x)$ and to $d(a_y,b_y)$. So it seemed superfluous to convert the approximable-from-above function f given there to a function which never decreases by more than 1, and indeed, not converting it allowed us to retain stronger reducibilities between f and d.

The next natural question, which remains open, is the existence of a computable connected graph whose distance function is intrinsically n-approximable from above, but which, for every f n-approximable from above, has a computable copy H with distance function $d_H \leq_{\text{2-}\mathtt{btt}} f \leq_{\text{1-}\mathtt{btt}} d_H$.

Of course, stronger reducibilities between f and d_H would be most welcome as well. Persistently throughout these results, we have had to allow norm 2 for the btt-reduction from the distance function to the function being encoded, even when the reverse reduction could be shown to have btt-norm 1. This seems to be a condition intrinsic to the notion of the distance function. When $f \leq_{1\text{-btt}} d_H$, and m is fixed, there is a single pair (x, y) of nodes in H which determines $f(m)$. There must be some separate pair (x', y') determining some other value $f(m')$ (unless f is computable), and then, in H, one can usually find nodes w and z such that $d(w, z)$ depends on both $d(x, y)$ and $d(x', y')$, either via a sum (if the shortest path from w to z goes through x, then y, then x', then y'), or via a minimum (if there is one path from w to z which goes through x and y, and a separate path going through x' and y'). In both these cases, $d(w, z)$ requires two pieces of information from f, leading to a btt-reduction of norm 2 at best. We would be significantly interested in any way of developing this analysis into a proof of the following.

Conjecture 5.8. *If G is a computable connected graph whose distance function is intrinsically n-approximable from above, with $n > 0$, then there exists some function f which is n-approximable from above and such that, for every computable graph $H \cong G$ with distance function d_H, we have $d_H \not\equiv_{1\text{-btt}} f$.*

It would follow that the 1-btt-degree spectrum of the distance function cannot contain exactly those degrees which are n-approximable from above.

6. Directed Graphs

One can repeat the questions from this paper in the context of computable directed graphs, rather than the symmetric graphs we have used. In a *directed graph*, each edge between vertices x and y has a specific orientation: it points either from x to y, or from y to x. (It is allowed for there to exist two edges between x and y, one pointing in each direction.) Of course, the orientations of the edges must be computable. In this context, one speaks of a *directed path* from x to y as a finite sequence of nodes $x = x_0, x_1, \ldots, x_n = y$ such that for all $i < n$, there is an edge from x_i to x_{i+1}. The directed graph G is *connected* if, for every $x, y \in G$, there is a directed path from x to y, and in this case the *directed distance* from x to y is always defined: it is the length of the shortest directed path from x to y. Again, this function is intrinsically approximable from above, and it appears to us that the constructions in this paper work equally well for directed graphs (modulo a few considerations, such as adding directed paths from b to u in the standard case, and from v to u in the elongated case, so as to make the graph connected). However, with directed graphs we can accomplish more than has already been proven here for symmetric graphs. In particular, it is much easier to realize the goal we set for ourselves in Section 5.

Theorem 6.1. *There exists a computable directed graph G whose distance function is intrinsically n-approximable from above, yet such that every f which is n-approximable from above is 1-btt-equivalent to the distance function on some computable directed graph isomorphic to G.*

Proof. Fix n. The directed graph G will be a version of the standard (undirected) graph on a collection of spokes. As usual, we let Γ enumerate all strictly decreasing sequences $\sigma \in \omega^{\leq n+1}$ of length at most $n+1$, and we assume that this enumeration repeats every such σ infinitely often.

A directed spoke of type σ looks somewhat like an ordinary spoke of type σ. The two main differences are that we use u itself as the top node of the directed spoke, rather than having a top node a adjacent to u, and that we include a directed path, of length $(3 + \sigma(0))$, from the bottom node b back to u. The latter modification is necessary in order for this directed graph to be connected. The reason for the former will be explained after the proof.

In between u and b, the directed spoke contains three directed paths from u to b of length $(2 + \sigma(0))$, and, for each i with $0 < i < |\sigma|$, another directed path from u to b of length $(2 + \sigma(i))$. Thus the final value of σ is the length of the shortest directed path from u to b in this spoke (and there will be no directed paths from u to b through other spokes). With this, we have described the directed spoke entirely. The directed graph G contains one directed spoke of type $\sigma = \Gamma(m)$, with bottom node b_m, for each $m \in \omega$.

Now u is the only node anywhere in G with more than one edge coming out of it. So, for any computable $H \cong G$, the same holds for some node u_H. Therefore, there is little difficulty in choosing the shortest directed path from an arbitrary node $x \in H$ to another one y: the only choice in finding the path arises when/if one reaches u_H. Starting at x, one follows the unique directed edge emerging from x, then the one emerging from that node, etc., until one reaches either y or u_H. If one has reached y, then the length of the path so far is clearly the directed distance from x to y in H. Otherwise, one then determines on which spoke of H y lies (which is computable), and fixes the bottom node b_m of that spoke. If y lies on a directed path from u_H to b_m, the one follows that path until reaching y, and this is the shortest directed path from x to y. If $y = b_m$ or y lies on the directed path from b_m to u_H, then one follows the shortest path from u_H to b_m, and then on to y. Here arises the only ambiguity: choosing the shortest directed path from u_H to b_m. The directed distance from x to y is the length of this path, plus the lengths of the (already determined) paths from x to u_H and from b_m to y. Therefore, the distance function d_H on H is 1-`btt`-reducible to the function $f(m) = d(u_H, b_m) - 2$. Conversely, this function f has $f \leq_1 d_H$. But in any computable copy H of G, this f is n-approximable from above, since one simply finds the three paths of length $(2+\sigma(0))$ from u_H to b_m (where $\sigma = \Gamma(m)$), thereby identifying b_m, and then waits for shorter directed paths to appear – which will happen at most n times, since $|\sigma| \leq n + 1$.

Thus d is intrinsically n-approximable from above. The converse is exactly the same construction we have executed previously. Given any f which is n-approximable from above, say via some computable $g(m, s)$, we start with a computable copy of G and extend it as follows to a directed graph H. For each m, add to G a directed spoke of type σ_m, where $\sigma_m(0) = g(m, 0)$ and $\sigma_m(i + 1)$ is defined iff there is an s with $g(m, s) < \sigma_m(i)$, in which case $\sigma_m(i + 1) = g(m, s)$ for the least such s. Since g is an n-approximation, we have $|\sigma_m| \leq n + 1$, and the last value of σ is $f(m)$, so $d_H(u, b_m) = f(m)$. For bottom nodes b of directed spokes in the original graph G within H, $d_h(u, b)$ is computable, since in G we know the type σ of each such directed spoke. Hence, by the same argument as in the preceding paragraph, d_H is 1-`btt`-equivalent to the function f, as desired. □

We note here that the conflation of u with the top nodes a of the directed spokes was necessary. Had the a's been part of these directed spokes, then the computation of the distance from the a_k on spoke k to the b_m on spoke m would have required knowing both $d(a_k, b_k)$ and $d(a_m, b_m)$, hence would have required a btt-reduction of norm 2. However, the trick of eliminating the nodes a does not allow us to prove Theorem 6.1 for symmetric graphs, since with no orientation on the edges, the computation of the distance from one bottom node to another still requires questions about the distance from top to bottom on two different spokes.

7. Related Topics

For graphs with infinitely many connected components, the distance function is $(\omega + 1)$-approximable from above, assuming we allow ∞ as the distance between any two nodes in distinct components. One approximates the distance function $d(x, y)$ at stage 0 by $g(x, y, 0) = \omega$ (or ∞), which will continue to be the value as long as x and y are not known to be in the same connected component. Meanwhile, we search systematically for a path from x to y within increasing finite subgraphs of G, and if we find one, say of length l, at some stage s, then we set $g(x, y, s) = l$, and then continue exactly as in the connected case, searching for shorter paths. It is clear that this distance function d is therefore $(\omega + 1)$-approximable from above, in the obvious definition, provided that one allows ∞ as an output of the function. (If ∞ is not allowed, then no notion of $(\omega + 1)$-approximability from above makes sense and distinguishes the concept from ω-approximability from above.)

In a different context, recent work by Steiner in [16] has considered the number of realizations of various algebraic types within a computable structure, and has asked in which cases one can put a computable upper bound on the number of realizations of each algebraic type. Over the theory $\mathbf{ACF}_0$, for example, an algebraic type is generated by the formula $p(X) = 0$, where p is a polynomial irreducible over the ground field, and the degree of the polynomial is an upper bound for the number of realizations of this type in an arbitrary field (not necessarily algebraically closed). Since the minimal polynomial of the element over the prime subfield can be found effectively, one can compute such an upper bound, whereas for other computable algebraic structures considered by Steiner, no such upper bound exists. The function counting the number of realizations of each algebraic type in a computable algebraic structure is approximable from below, and when a computable upper bound exists, this function becomes the dual of a function approximable from above, exactly as described in Definition 3.4. Therefore, the theorems proven in Section 3 apply to such functions. On the other hand, when there is no computable upper bound, the standard results about functions approximable from below apply, and we saw in Section 3 that these results differ in several ways from the results when a computable bound does exist.

To close, we ask what connection, if any, there might be between distance functions on computable graphs and Kolmogorov complexity. Is it possible that Kolmogorov complexity can be presented as the distance function on some computable graph? (This is a different matter than using Kolmogorov complexity to construct structures with prescribed model-theoretic properties, as in [9].) It might be useful to fix one node $e \in G$ – call it the *Erdös node* – and to consider $d(e, x)$, a unary function on G, in place of the full distance function; in this case one could directly build a computable graph G and a computable function $f : \omega \to G$ such that $d(e, f(n))$ is exactly the Kolmogorov complexity of n. Having done so, one could then ask about other computable copies of G: does the distance function on those copies correspond to Kolmogorov complexity under some different universal (prefix-free?) machine? Right now, this question is not well-formed, and there is no obvious reason to expect to find any connections at all between these topics, except for their common use of functions approximable from above, and their common triangle inequalities. (If one knows the Kolmogorov complexity of binary strings σ and τ, one gets an upper bound on the Kolmogorov complexity of the concatenation $\sigma\hat{}\tau$.) However, any connection that might arise would be a potentially fascinating link between algorithmic complexity and computable model theory.

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