The Computable Dimension of Trees of Infinite Height

Russell Miller*

July 23, 2004

*The first four sections of this article appeared as a chapter of a Ph.D. thesis at the University of Chicago under the supervision of Robert I. Soare. Thanks are also due to Bakhadyr Khoussainov, who originally posed the question of computable categoricity of trees and aided in the research, and to Richard Shore for useful conversations. Later the author was partially supported by a VIGRE postdoc under NSF grant number 9983660 to Cornell University.

Abstract

We prove that no computable tree of infinite height is computably categorical, and indeed that all such trees have computable dimension ω. Moreover, this dimension is effectively ω, in the sense that given any effective listing of computable presentations of the same tree, we can effectively find another computable presentation of it which is not computably isomorphic to any of the presentations on the list.

1 Introduction

In a finite language, a countable structure $\mathcal{A}$ whose universe A is a subset of ω is *computable* if A is a computable set and for all functions f and relations R in the language, $f^{\mathcal{A}}$ is a computable function and $R^{\mathcal{A}}$ is a computable relation.

Any computable structure will be isomorphic to infinitely many other computable structures. It may happen, however, that two computable structures are isomorphic, yet that the only isomorphisms between them are noncomputable (as maps from one domain to the other). If so, then these structures lie in distinct *computable isomorphism classes* of the isomorphism type of the structure. On the other hand, if there exists a computable function taking one structure isomorphically to the other, then the two structures lie in the same computable isomorphism class.

The *computable dimension* of a computable structure is the number of computable isomorphism classes of that structure. The most common computable dimensions are 1 and ω, but for each $n \in \omega$, there do exist structures with computable dimension n, by a result of Goncharov ([8]). If the computable dimension of $\mathcal{A}$ is 1, we say that $\mathcal{A}$ is *computably categorical.* This notion is somewhat analogous to the concept of categoricity in ordinary model theory: a theory is *categorical* in a given power κ if all models of the theory of power κ are isomorphic. Computable categoricity is a property of structures, not of theories: a computable structure $\mathcal{A}$ is computably categorical if every other computable structure which is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.

A standard example of a categorical theory is the theory of dense linear orders without end points, which is categorical in power ω. One proves this by taking two arbitrary countable dense linear orders and building an isomorphism between them by a back-and-forth construction. The same construction allows us to prove that the structure $\mathbb{Q}$ is computably categorical. (More formally, let $(\omega, \prec)$ be a computable linear order isomorphic to $(\mathbb{Q}, <)$. Then $(\omega, \prec)$ is computably categorical.)

Characterizations of computable categoricity have been found for certain types of structures. Goncharov and Dzgoev ([9]) and Remmel ([15]) proved that a linear order is computably categorical precisely if it contains finitely many successivities (that is, if only finitely many elements have an immediate successor in the linear order). Remmel also proved that a Boolean algebra is computably categorical if and only if it contains only finitely many atoms ([16]).

In the present paper we consider computable categoricity of trees, and prove that no tree of infinite height is computably categorical. The question of computable categoricity of trees of finite height is the subject of joint work by Lempp, McCoy, Solomon, and the author, and will appear separately.

To prove that a tree T is not computably categorical, we will construct a new tree T' isomorphic to T, satisfying the following requirements $\mathcal{R}_e$:

$$\mathcal{R}_e : \varphi_e \text{ total} \implies \text{there exists } x \in T' \text{ such that } \text{level}_{T'}(x) \neq \text{level}_T(\varphi_e(x)).$$

Clearly $\mathcal{R}_e$ implies that φ_e is not an isomorphism from T' to T. If we can establish $\mathcal{R}_e$ for every e, then, we will have proven that T is not computably categorical.

Our notation is standard, but our definitions demand attention. A *tree* consists of a universe T with a strict partial order $\prec$ on T such that for every $x \in T$, the set of predecessors of x in T is well-ordered by $\prec$, and such that T contains a least element under $\prec$. (Hence the tree is computable if T is a computable set and $\prec$ a computable relation.) In this paper, T will represent the computable tree which we wish to prove not to be computably categorical.

If two nodes x and y in T are incomparable under $\prec$, then we write $x \perp y$. For each node $x \in T$, we define the *level* of x in T to be the order type of the set of predecessors of x in T. We view our trees as growing upwards, with a single element r (the *root*, or least element under $\prec$) at the base. Thus the level of the root is 0, its immediate successors under $\prec$ are at level 1, and so on. The height of T is defined as follows:

$$\text{ht}(T) = \sup_{x \in T}(\text{level}_T(x) + 1).$$

Thus, the height of T will be the least ordinal α such that no node of T has level α. In this paper we only consider trees of infinite height. The level of a node of T is generally not a computable function on T. (For computable trees of height $\leq \omega + 1$, though, it is a Σ_1 function, since there exists a computable function $f(x, s) = |\{y < s : y \prec x\}|$ such that for all $x \in T$,

$$\text{level}_T(x) = \lim_s f(x, s).)$$

The reader should note that different definitions of subtree and tree homomorphism have been used for different purposes in the literature. In this paper a *homomorphism* from one tree $(T, \prec)$ to another tree $(T', \prec')$ will be a map $f : T \to T'$ which respects the partial orders:

$$x \prec y \iff f(x) \prec' f(y).$$

(An *embedding* is a one-to-one homomorphism.) In other papers, a tree is sometimes defined using the infimum function $\wedge$, where the infimum $x \wedge y$ of x and y is the greatest z such that $z \preceq x$ and $z \preceq y$. Any tree under one definition is also a tree under the other definition, but when the infimum function is used, all homomorphisms are required to respect the infimum function. This is a strictly stronger requirement: all maps respecting $\wedge$ respect $\prec$, because

$$x \preceq y \iff x \wedge y = x,$$

but not conversely. Kruskal's Theorem, which we use in section 2, proves the existence of the stronger type of embedding.

If the infimum fuction is computable, then the relation $\prec$ is computable, since it is definable in terms of $\wedge$ without quantifiers. Therefore, if the computable trees $(T, \prec)$ and $(T', \prec')$ are isomorphic but not computably isomorphic, then the corresponding structures $(T, \wedge)$ and $(T', \wedge')$ are also isomorphic, but not computably isomorphic. (Notice, however, that $(T, \wedge)$ and $(T', \wedge')$ need not be computable, since computability of $\prec$ does not guarantee that we can compute the infimum function.) When we build T', we will ensure that not only $\prec'$ but also $\wedge'$ are computable. Thus, our theorem suffices to prove that even when tree is defined using the infimum, no tree of infinite height is computably categorical. The definitions of tree and tree homomorphism using the infimum are probably more common in the literature. We adopt the definitions using $\prec$ because for the purposes of our proof, they will be far more useful.

Our definition of subtree arises from our definition of homomorphism. Once again, therefore, it diverges from much of the literature: for our purposes, a tree $(T', \prec')$ is a *subtree* of $(T, \prec)$ if $T' \subseteq T$ and the inclusion map respects the partial orders. Thus the infimum of two elements in T may not be the same as their infimum in T'. Also, the root of T may be distinct from the root of T', as in the case of the subtrees $T[x]$, which we will be considering frequently. If x is a node in T, then the subtree $T[x]$ is just the tree

$$T[x] = \{y \in T : x \preceq y\}.$$

The partial order on $T[x]$ is the restriction to $T[x]$ of the partial order $\prec$ on T. Therefore $T[x]$ is a subtree of T with root x. We define the *height of T above x* by:

$$\mathrm{ht}_x(T) = \mathrm{ht}(T[x]).$$

The reason for our use of $\prec$ rather than $\wedge$ to define homomorphism and subtree is twofold. First, $\prec$ is the basic relation we used to define the notion of a tree; $\wedge$ was derived from $\prec$. If $\wedge$ were the basic function, then computability questions would be very different. Second, during our proofs about a tree T we will be considering many subsets of T which we will want to regard as subtrees. Under our definition, they will be subtrees (as will any subset of T with a $\prec$-least element), but under the $\wedge$-definition some would not be subtrees.

A *path* γ through T is a maximal linearly ordered subset of T. It may be finite or infinite. Any tree containing an infinite path must have infinite height. A node is *extendible* if it lies on an infinite path through T, and *non-extendible* otherwise. The extendible nodes of a tree T (if any exist) form a subtree of T, which we denote by T_{ext}. Notice, however, that since we allow T to be infinite-branching, the height of T above a node may be ω even if the node is nonextendible.

2 Kruskal's Theorem

Although our results concern infinite trees, we will need the ability to manipulate finite subtrees. For this purpose Kruskal's Theorem is essential. All embeddings mentioned in this section are homomorphisms with respect to both $\prec$ and $\wedge$.

Theorem 2.1 (Kruskal's Theorem) *(See [12], [17].) Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees. Then there exist $i < j$ in ω such that T_i can be embedded in T_j.*

Every version of Kruskal's Theorem which we will encounter has an analogue of the following corollary:

Corollary 2.2 *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees. Then there exists $n \in \omega$ such that for every $i > n$, T_i can be embedded in some T_j with $j > i$, and some T_k with $k < i$ can be embedded in T_i.*

Proof. If the set

$$\{i \in \omega : (\forall j > i)\ T_i \text{ does not embed in } T_j\}$$

were infinite, it would itself contradict Kruskal's Theorem. The same is true of

$$\{i \in \omega : (\forall k < i)\ T_k \text{ does not embed in } T_i\}.$$

■

We can extend Kruskal's Theorem to a version dealing with infinite trees.

Corollary 2.3 *Let $\{T_i : i \in \omega\}$ be an infinite collection of trees. (These trees need not be finite, nor even finitely branching.) Then there exists an $i \in \omega$ such that for every finite subtree $T \subseteq T_i$, there exists $j > i$ for which T embeds in T_j.*

Proof. Suppose $\{T_i : i \in \omega\}$ were a collection of trees contradicting this corollary. Then for each i, we would have some finite subtree $S_i \subseteq T_i$ which did not embed into any T_j with $j > i$. In particular, for each $i < j$, S_i would not embed in S_j. Thus the collection $\{S_i : i \in \omega\}$ would contradict Kruskal's Theorem. ■

Corollary 2.4 *Let $\{T_i : i \in \omega\}$ be as in Corollary 2.3. Then there is an $n \in \omega$ such that for every $i > n$ and every finite subtree $T \subseteq T_i$, there exists $j > i$ such that T embeds into T_j.*

Proof. If not, then we could find an increasing sequence $i_0 < i_1 < i_2 < \cdots$ such that $\{T_{i_k} : k \in \omega\}$ contradicted Corollary 2.3. ∎

In this paper we will want to embed trees in such a way that nodes with p predecessors are mapped to nodes with more than p predecessors. That is, the level in the tree T of the node x should be less than the level in T' of its image under the embedding of T into T'. To map nodes to other nodes at greater levels, we need the following stronger version of Kruskal's Theorem, in which one is allowed to "label" nodes of each tree. For our purposes, a *labelling* of a tree T is simply a map from T to ω. Proofs of this result appear in [12] and [17].

Theorem 2.5 (Kruskal) *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees, each with a labelling l_i. Then there exist $i < j$ in ω and an embedding $f : T_i \to T_j$ such that for every $x \in T_i$, $l_i(x) \leq l_j(f(x))$.*

From Theorem 2.5 we derive the following result:

Corollary 2.6 *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees such that $\sup_i \mathrm{ht}(T_i) = \omega$. Then there is a number $m \in \omega$ such that for every index i and every node $x \in T_i$ with $\mathrm{level}_{T_i}(x) = m$, there exists an embedding f of T_i into some T_j with $j > i$, such that*

$$\mathrm{level}_{T_j}(f(x)) > \mathrm{level}_{T_i}(x).$$

Proof. Suppose no $m \in \omega$ satisfied the theorem. Then for every m, we would have an index i_m and a node $x_m \in T_{(i_m)}$ with $\mathrm{level}_{T_{(i_m)}}(x_m) = m$ such that:

$$\forall \text{ embeddings } f : T_{(i_m)} \to T_j \text{ with } j > i_m, \mathrm{level}_{T_j}(f(x_m)) = \mathrm{level}_{T_{(i_m)}}(x_m). \tag{1}$$

Now the set $\{i_0, i_1, i_2 \ldots\}$ will be infinite, since each T_i has finite height. Moreover, the index i_m satisfies Equation 1 not only for x_m but also for all predecessors of x_m. Therefore we can choose $i_{m+1} > i_m$ for all m.

For each m, define the labelling l_m on the tree $T_{(i_m)}$ by

$$l_m(x) = \begin{cases} 0, & \text{if } \mathrm{level}_{T_{i_m}}(x) < m \\ 1, & \text{otherwise} \end{cases}$$

Thus $l_m(x_m) = 1$ for all m. However, for any embedding $f : T_{(i_m)} \to T_{(i_k)}$ with $k > m$, we have

$$\text{level}_{T_{i_k}}(f(x_m)) = \text{level}_{T_{(i_m)}}(x_m) = m < k.$$

This forces $l_k(f(x_m)) = 0$. Thus the sequence $\{T_{i_0}, T_{i_1}, T_{i_2}, \ldots\}$ contradicts Theorem 2.5. ■

The same result holds for all y above the level m:

Corollary 2.7 *Let $\{T_i : i \in \omega\}$ be as in Corollary 2.6. Then there is a number $m \in \omega$ such that for every index i and every node $y \in T_i$ with $\text{level}_{T_i}(y) \geq m$, there exists an embedding f of T_i into some T_j with $j > i$, such that*

$$\text{level}_{T_j}(f(y)) > \text{level}_{T_i}(y).$$

Proof. The conclusion follows for every $y \in T_i$ with $\text{level}_{T_i}(y) \geq m$, simply by finding that $x \preceq y$ in T_i with $\text{level}_{T_i}(x) = m$ and applying the embedding given by Corollary 2.6 for that x. ■

Finally, we combine the version for infinite trees with the version for embedding nodes at greater levels.

Corollary 2.8 *Let $\{T_i : i \in \omega\}$ be any collection of trees. Then there exist an n and an m with the property that for all indices $i > n$, for every finite subtree $S \subseteq T_i$, and for any node $x \in S$ with $\text{level}_S(x) \geq m$, there is an embedding $g : S \to T_j$ of S into some T_j with $j > i$, such that*

$$\text{level}_{T_j}(g(x)) > \text{level}_S(x).$$

Proof. Suppose the statement were false. Now if g is an embedding of S into T_j, it is impossible to have $\text{level}_{T_j}(g(x)) < \text{level}_S(x)$. Therefore, the negation of the statement is as follows:

$$(\forall n)(\forall m)(\exists i > n)(\exists \text{ finite } S \subseteq T_i)(\exists x \in S)$$
$$\left[\begin{array}{c} \text{level}_S(x) \geq m \;\& \\ (\forall j > i)(\forall \text{ embeddings } g : S \to T_j)[\text{level}_{T_j}(g(x)) = \text{level}_S(x)] \end{array} \right]$$

We apply this negation first with $n = 0$ and $m = 0$, yielding an index $i_0 > 0$ and a node x_0 at level ≥ 0 in some finite subtree S_0 of T_{i_0}. Inductively, we apply the negation with $n = i_k$ and $m = k+1$ to get an index $i_{k+1} > i_k$ and a corresponding node x_{k+1} at level $\geq k+1$ of a finite subtree S_{k+1} of $T_{i_{(k+1)}}$. From the negation, we see that every embedding of any S_k into any T_j with $j > i_k$ fixes the level of x_k. In particular, the same holds for any embedding of S_k into any S_j with $j > k$. However, we know that

$$\text{ht}(S_k) > \text{level}_{S_k}(x_k) \geq k,$$

so $\sup_k \text{ht}(S_k) = \omega$. Thus the set $\{S_k : k \in \omega\}$ contradicts Corollary 2.7. ∎

We remark that in fact Kruskal's Theorem proves the existence of an embedding of T_i into T_j which respects not only $\prec$ but also $\wedge$. The same follows for all our corollaries. Therefore, if one prefers to consider computable trees under $\wedge$ rather than under $\prec$, all our results in the remainder of the paper will go through unchanged.

Finally, for computability-theoretic purposes, we note that if S and T are finite trees (and we have strong indices for each, i.e. we know the number of nodes of each), then the statement

$$\exists \text{ an embedding } g : S \to T$$

is decidable, uniformly in S and T. From the decidability of this statement, we conclude further that if S is finite with known strong index and T is any computable tree, then the question of embeddability of S into T is a Σ_1 question: it asks whether there exists a finite subtree of T into which S embeds. Therefore, if we know that there exists an embedding of S into T, then we can effectively find such an embedding, via an algorithm uniform in S and T.

3 ω-Branching Nodes with $\mathbf{ht}_x(T) = \omega$

We first consider computable trees of height ω. The general theorem that no such tree is computably categorical will be proven in the next section. In this section, to prepare for that proof, we prove that a significant subclass of such trees cannot be computably categorical.

We define the limit-supremum of a sequence $\langle n_i \rangle_{i \in \omega}$ to be

$$\limsup_i (n_i) = \inf_j \sup_{i>j} (n_i)$$

T will be a given computable tree under the partial order $\prec$, with height ω, which is ω-branching at a node x_0. (That is, x_0 has infinitely many immediate successors $x_1, x_2, \ldots$.) We assume further that $\limsup_i \mathrm{ht}(T[x_i]) = \omega$. This can occur two ways: either infinitely many $T[x_i]$ have height ω, or there exist trees $T[x_i]$ of arbitrarily large finite heights.

Since the universe of T is computable, we may take it to be ω, pulling back via a 1-1 computable function if necessary to make this so. We will construct a computable tree T' isomorphic to T, such that there is no computable isomorphism between them.

The isomorphism f from T to T' will be a Δ^0_2 function, the limit of a computable sequence of finite partial 1-1 functions f_s, such that the domains $D_s = \mathrm{dom}(f_s) \subset T$ form a strong array of finite sets. We will ensure that $D_s \subseteq D_{s+1}$ for each s, although f_{s+1} need not agree with f_s on D_s. (If it did so for all s, then f would be a computable isomorphism, which is precisely what we wish to avoid!) Also, we will force $\mathrm{range}(f) = \omega$, so that the universe of T' will be ω. The ordering $\prec'$ on T' will be given by lifting the ordering $\prec$ from T via f, thereby guaranteeing that f is an isomorphism. To make $\prec'$ computable, we force the approximations f_s to satisfy the following condition:

Condition 3.1 *For all $a, b \in \mathrm{range}(f_s)$, we have $a, b \in \mathrm{range}(f_{s+1})$ and*

$$f^{-1}_{s+1}(a) \prec f^{-1}_{s+1}(b) \iff f^{-1}_s(a) \prec f^{-1}_s(b).$$

To ensure that T and T' are not computably isomorphic, we impose the requirements $\mathcal{R}_e$.

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T') \, [\mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x))].$$

This will suffice to prove the proposition.

Proposition 3.2 *Let T be a computable tree of height ω containing an ω-branching node x_0 with immediate successors $x_1, x_2, \ldots$, such that*

$$\limsup_i \mathrm{ht}(T[x_i]) = \omega.$$

Then T is not computably categorical.

Proof. As previously remarked, we may assume the universe of T to be ω. A *successor tree* of x_0 is a tree of the form $T[x_i]$ with $i \geq 1$. ($\{x_1, x_2, \ldots\}$ are all the immediate successors of x_0, as stated above. This set need not be computable.) Corollary 2.8, applied to the successor trees, provides m and n in ω such that for every finite subtree $S \subseteq T[x_i]$ with $i > n$ and every node $x \in S$ with $\mathrm{level}_S(x) \geq m$, there is an embedding of S into some $T[x_j]$ with $j > i$ which maps x to a node of greater level. We fix these values of m and n for the rest of the proof. (Notice that therefore the proof is not uniform in T.)

Let T_s be the subtree of T with nodes $\{r, x_0, x_1, \ldots x_n\} \cup \{0, 1, 2, \ldots s\}$, under $\prec$, where r is the root of T.

For our purposes, the finite subtrees S will generally be of the form $D_s[y]$, where $D_s \supseteq T_s$ is the domain of f_s and y is an immediate successor of x_0 in D_s (although not necessarily in T). We will call $D_s[y]$ a *successor tree* at stage s. Notice that it may happen that two successor trees which are distinct at stage s acquire a common root at stage $s+1$, e.g. if $s+1 = x_i$ for some i, and thus merge into a single successor tree at stage $s+1$. A given successor tree at stage s, however, can only be merged this way finitely often, since each of its nodes has finite level in T.

The following construction yields a computable tree T' which is isomorphic to T but satisfies every requirement $\mathcal{R}_e$, proving that T is not computably categorical. The *witness nodes* w_e will be nodes in $T[x_0]$, and will be approximated at stage s by a node $w_{e,s}$. The successor tree at stage s containing $w_{e,s}$ will be denoted $S_{e,s}$. This is the successor tree which we use in order to satisfy requirement $\mathcal{R}_e$. The sequence $\langle w_{e,s} \rangle_{s \in \omega}$ will converge to some w_e, and each successor tree in T will contain at most one w_e. The isomorphism f from T to T' will be approximated at stage s by a finite map f_s with domain D_s. If $\varphi_{e,s}(f_s(w_{e,s}))$ converges to a node at the same level of T_s as the level of $f_s(w_{e,s})$ in T'_s, then we redefine f_{s+1} and $w_{e,s+1}$ with $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$ at a higher level in T'_{s+1}. (The level of a node in T'_s is just the level of its preimage under f_s in T_s.) Doing this requires us to redefine f_{s+1} on the entire successor tree containing $w_{e,s}$, in order to satisfy Condition 3.1, and we will appeal to Corollary 2.8 to ensure that the necessary embedding exists. Thus $f(w_e)$ will be the witness required by $\mathcal{R}_e$.

Figure 3.3

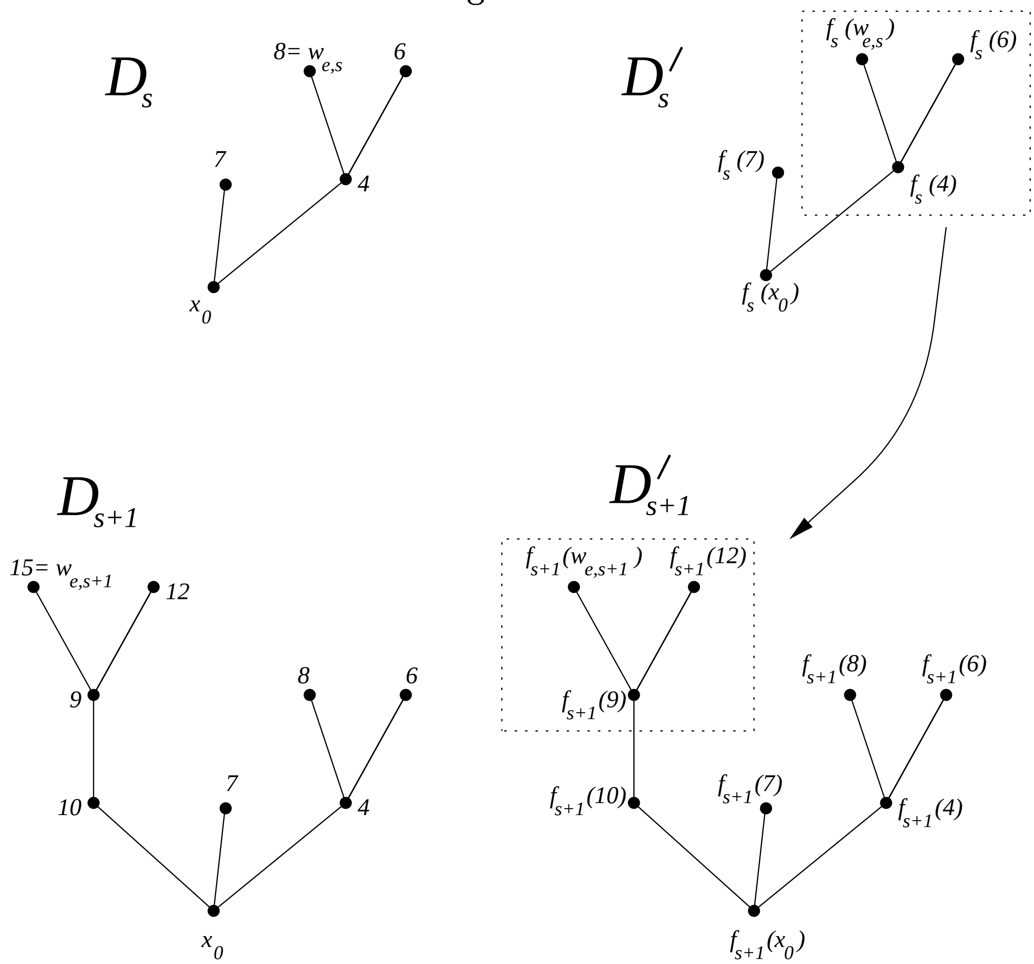

Figure 3.3 gives an example of our basic strategy. $S_{e,s}$ is the successor tree which we use to satisfy $\mathcal{R}_e$. We suppose that we have found at stage s that $\varphi_e(f_s(w_{e,s})) = 6$, which lies at level 2 in D_s. This is bad, because $f_s(w_{e,s})$ lies at level 2 in D_s', so it appears that φ_e might be an isomorphism from T' to T. $S_{e,s}$ is the successor tree above the node 4 in D_s, and we use Corollary 2.8 to find an embedding of $S_{e,s}$ upwards into the successor tree above the node 10 in D_{s+1}. (The embedding is indicated by the arrow to D'_{s+1}.) We use this embedding to make $\text{level}_{D'_{s+1}}(f_{s+1}(w_{e,s+1})) > \text{level}_{D'_s}(f_s(w_{e,s}))$, by defining

f_{s+1} so that $f_{s+1}(9) = f_s(4)$, $f_{s+1}(12) = f_s(6)$, and $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. We add new values to range(f_{s+1}) for $f_{s+1}(4)$, $f_{s+1}(6)$, $f_{s+1}(8)$, and $f_{s+1}(10)$. Thus $\text{level}_{D_s}(\varphi_e(f_{s+1}(w_{e,s+1}))) \neq \text{level}_{D'_{s+1}}(f_{s+1}(w_{e,s+1}))$.

Construction: f_0 is the identity map with $\text{dom}(f_0) = T_0$. The witness nodes $w_{e,0}$ and the successor trees $S_{e,0}$ are undefined for all e. We let $D_0 = \text{dom}(f_0)$. (At each stage s, D_s and T_s will both be subtrees of T, with $T_s \subseteq D_s$.) We immediately define the successor trees $T_0[x_i]$ with $1 \leq i \leq n$ to be frozen.

At stage $s+1$, we consider the successor trees of x_0 in D_s. For each successor tree S (if any) of height $\geq m$ which is not frozen and does not contain $S_{e,s}$ for any $e \leq s$, we choose the least $e \leq s$ such that $S_{e,s}$ is undefined, let $S_{e,s+1} = S$ and choose $w_{e,s+1}$ to be the $<$-least node at the highest level of S. Thus $\text{level}_{S_{e,s+1}}(w_{e,s+1}) \geq m$.

We then consider in turn each e for which $S_{e,s}$ was defined.

Step 1: If there is an $i < e$ and a $z \in T_{s+1}$ such that $x_0 \prec z \prec w_{i,s}$ and $z \prec w_{e,s}$, then we immediately make $S_{j,s+1}$ and $w_{j,s+1}$ undefined for all $j \geq e$, and declare all $S_{j,s}$ with $j > e$ frozen.

(This step ensures that if two successor trees $S_{i,s}$ and $S_{e,s}$ have acquired a common root above x_0, thus becoming the same successor tree, then we use the single new successor tree to play against requirement $\mathcal{R}_i$ only.)

Step 2: Otherwise, we consider $f_s(w_{e,s})$, the potential witness for requirement $\mathcal{R}_e$. If $\varphi_{e,s}(f_s(w_{e,s}))$ diverges, or converges to an element not in D_s, or if $\text{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))) \neq \text{level}_{D_s}(w_{e,s})$, then we define:

$$w_{e,s+1} = w_{e,s}$$

$$f_{s+1} = f_s \text{ on } S_{e,s}$$

$$S_{e,s+1} = \{y \in D_s \cup T_{s+1} : (y \wedge w_{e,s+1}) \succ x_0\}.$$

(Here $y \wedge w_{e,s+1}$ represents the infimum in $D_s \cup T_{s+1}$, which is a finite tree. Taking the infimum over all of T would not be computable.)

(This $S_{e,s+1}$ is just the same successor tree as $S_{e,s}$, along with any new elements that may have appeared in this successor tree at stage s.)

Step 3: If $\text{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))) = \text{level}_{D_s}(w_{e,s})$, then find the least stage $t > s$ with $D_s \subseteq T_t$ such that the following holds:

Condition 3.4 *There exists a $z \in T_t$ such that:*

1. *z is an immediate successor of x_0 in T_t, and*

2. $T_t[z] \cap D_s = \emptyset$, *and*

3. There is an embedding g *of* $S_{e,s}$ *into* $T_t[z]$ *with*

$$\text{level}_{T_t}(g(w_{e,s})) > \text{level}_{D_s}(w_{e,s}).$$

Let $S_{e,s+1} = S$, with $w_{e,s+1} = g(w_{e,s})$. (By our choice of g, this forces $\text{level}_{T_t}(w_{e,s+1}) > \text{level}_{D_s}(w_{e,s})$. Also, $\text{level}_{S_{e,s+1}}(w_{e,s+1}) > \text{level}_{S_{e,s}}(w_{e,s}) \geq m$.) For every $x \in S_{e,s}$, define $f_{s+1}(g(x)) = f_s(x)$, and define $f_{s+1}(x)$ to be the least element which is not yet in $\text{range}(f_{s+1}) \cup \text{range}(f_s)$. Declare $S_{e,s}$ to be frozen, so that at no subsequent stage s' will any $w_{i,s'}$ be defined in the successor tree containing $S_{e,s}$. Having executed Step 3 for e, we let $w_{j,s+1}$ and $S_{j,s+1}$ diverge and freeze $S_{j,s}$ for all $j > e$, and do not execute Steps 1, 2, or 3 for any $j > e$.

(We execute Step 3 if $\mathcal{R}_e$ is not satisfied by $f_s(w_{e,s})$. By Corollary 2.8, there must exist a successor tree $T[x_j]$ into which the required embedding g exists, because $\text{level}_{Se,s}(w_{e,s}) \geq m$ and $S_{e,s} \subseteq T[x_i]$ for some $i > n$. The successor trees $T[x_1], \ldots T[x_n]$ were all frozen right away at stage 0, so none of them contains $S_{e,s}$. Thus we have found a z such that f_s is completely undefined on the successor tree $S \subset T_t$ containing z, and $S_{e,s}$ embeds into S via a map g. We use this embedding to satisfy $\mathcal{R}_e$, as in the example of Figure 3.3. Freezing $S_{e,s}$ ensures that $f_{s'}$ will never again be redefined on $S_{e,s}$, so that $\lim_{s'} f_{s'}$ must exist.)

Having completed these three steps for each $S_{e,s}$, we now define D_{s+1} to be $\left(\bigcup_e S_{e,s+1}\right) \cup D_s \cup T_{s+1}$. For any $y \in D_s$ such that $f_{s+1}(y)$ is not yet defined, take $f_{s+1}(y) = f_s(y)$. (This includes nodes on already-frozen successor trees, nodes on successor trees of height $\leq m$, and nodes not on $T[x_0]$.) For each $y \in D_{s+1}$, if $f_{s+1}(y)$ is not yet defined, take $f_{s+1}(y)$ to be the least integer not already in $\text{range}(f_{s+1})$ Thus $D_{s+1} = \text{dom}(f_{s+1})$. This completes the construction.

We now prove that this construction really does yield a tree T' which is isomorphic to T but not computably isomorphic to it.

Lemma 3.5 *For every* e*, the sequence* $w_{e,s}$ *converges to a limit* w_e*.*

Proof. Assume by induction that the Lemma holds for every $i < e$. Notice that in our construction, once $w_{e,s}$ and $S_{e,s}$ are defined, the only way they can become undefined is in Step 1 (if a new node of $T[x_0]$ appears which is a predecessor of $w_{i,s}$ for some $i < e$) or Step 3 (if $w_{i,s} \neq w_{i,s+1}$ for some $i < e$).

Once we reach a stage s_0 such that $w_{i,s} = w_i$ for every $i < e$ and $s \geq s_0$ and every predecessor of every w_i $(i < e)$ has appeared in T_{s_0}, we know that once $w_{e,s}$ is defined for some $s \geq s_0$, it will stay defined at all subsequent stages, although its value may change. Also, $w_{e,s}$ is only defined at stages s such that $w_{i,s}$ is also defined for all $i < e$.

By induction, for every $i < e$, $\langle w_{i,s} \rangle_{s \in \omega}$ converges to some w_i. Pick a stage s_0 such that $w_{i,s} = w_i$ and $\text{level}_{T_s}(w_{i,s}) = \text{level}_T(w_i)$ for all $i < e$ and $s \geq s_0$. Now if $s \geq s_0$ and $w_{e,s}$ is not defined, then no $w_{j,s}$ with $j > e$ is defined either. But since $\limsup_i \text{ht}(T[x_i]) = \omega$, there are infinitely many successor trees of height $> m$, so a new one, S, with $S \cap D_{s_0} = \emptyset$, must appear at some stage $s > s_0$. It will not be frozen, since $w_{i,s} = w_i$ for all $i < e$, so it will be chosen as $S_{e,s}$, and one of its nodes of maximal height will be $w_{e,s}$. Then $w_{e,t}$ is defined for every $t > s$, since every predecessor of every w_i with $i < e$ is already in T_s. Thus, by induction, for every e, $w_{e,s}$ is defined for all sufficiently large s.

Once it is defined at a stage beyond s_0, $w_{e,s}$ will only be redefined at a subsequent stage $t+1$ if $\text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) = \text{level}_{T_t}(w_{e,t})$ and Condition 3.4 holds. Moreover, even when it is redefined, we will still have $f_{t+1}(w_{e,t+1}) = f_t(w_{e,t})$. Since the tree T has height ω, we know that for all t,

$$\text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) \leq \text{level}_T(\varphi_e(f_t(w_{e,t}))) < \omega.$$

But $\langle \text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) \rangle_{t \in \omega}$ is a non-decreasing sequence, so it can only change value finitely often. Thus, once defined, $w_{e,s}$ will only be redefined finitely often, so it must converge. ■

Lemma 3.6 *For every x, $\lim_s f_s(x)$ exists.*

Proof. We know $x \in T_s \subseteq D_s = \text{dom}(f_s)$ for all $s > x$. Furthermore, once $f_s(x)$ is defined, the only way we can have $f_s(x) \neq f_{s+1}(x)$ is if x lies on a successor tree $S_{e,s}$ for which $w_{e,s}$ is redefined or undefined at stage $s+1$. Once this happens, $S_{e,s}$ is declared frozen, and $f_t \restriction S_{e,s} = f_{s+1} \restriction S_{e,s}$ for all $t \geq s+1$. Thus, not only does $\langle f_s(x) \rangle_{s \in \omega}$ converge, but in fact it changes value at most once. ■

We define the function $f = \lim_s f_s$.

Lemma 3.7 *The functions f_s satisfy Condition 3.1. (Hence the relation $\prec'$ defined on $T' = \text{range}(f)$ by*

$$a \prec' b \iff (\forall s)[a, b \in \text{range}(f_s) \implies f_s^{-1}(a) \prec f_s^{-1}(b)]$$

is computable and gives a tree structure on ω).

Proof. The construction makes it clear that $\text{range}(f_s) \subseteq \text{range}(f_{s+1})$ for all s. Now fix $a, b \in \text{range}(f_s)$. If $f_s^{-1}(a) \neq f_{s+1}^{-1}(a)$, then $f_s^{-1}(a)$ must lie on a successor tree $S_{e,s}$ such that $w_{e,s} \neq w_{e,s+1}$. Hence $f_{s+1}(g(f_s^{-1}(a))) = f_s(f_s^{-1}(a)) = a$, and $f_{s+1}^{-1}(a) = g(f_s^{-1}(a))$, where g is the upward embedding of $S_{e,s}$ into $S_{e,s+1}$ used in the construction. We consider four cases:

Case 1. Suppose $f_s^{-1}(b) \in S_{e,s}$ as well. Then also $f_{s+1}^{-1}(b) = g(f_s^{-1}(b))$, and since g is an embedding, we have

$$f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b) \iff f_s^{-1}(a) \prec f_s^{-1}(b).$$

Case 2. Suppose $f_s^{-1}(b) \in T[x_0] - S_{e,s} - \{x_0\}$. Then $f_s^{-1}(b) \perp f_s^{-1}(a)$. By Part 2 of Condition 3.4, we know $f_{s+1}^{-1}(b) \in T[x_0] - S_{e,s+1} - \{x_0\}$, so also $f_{s+1}^{-1}(b) \perp f_{s+1}^{-1}(a)$.

Case 3. Suppose $f_s^{-1}(b) \preceq x_0$. Then $f_s^{-1}(b) \prec f_s^{-1}(a)$, and $f_{s+1}^{-1}(b) = f_s^{-1}(b) \preceq x_0 \prec f_{s+1}^{-1}(a)$.

Case 4. If $f_s^{-1}(b) \perp x_0$, then $f_s^{-1}(b) \perp f_s^{-1}(a)$, and also $f_{s+1}^{-1}(b) = f_s^{-1}(b) \perp x_0 \prec f_{s+1}^{-1}(a)$, so $f_{s+1}^{-1}(b) \perp f_{s+1}^{-1}(a)$.

A similar analysis applies if $f_s^{-1}(a) = f_{s+1}^{-1}(a)$ and $f_s^{-1}(b) \neq f_{s+1}^{-1}(b)$. ∎

Lemma 3.8 *The tree $(T', \prec')$ is a computable tree isomorphic to T.*

Proof. We defined every f_s to be a 1-1 map, with $\text{range}(f_s) \subseteq \text{range}(f_{s+1})$. By Lemma 3.6, then, f is also 1-1.

The range of f is ω since at each of the (infinitely many) stages at which we needed a new element for the range of f_s, we took the smallest one available. If $f_{s+1}^{-1}(y) \neq f_s^{-1}(y)$ for some s, then $y = f_s(x)$ for some x on some $S_{e,s}$ which was redefined at stage $s+1$, and $f_{s+1}^{-1}(y) \in S_{e,s+1}$. But $S_{e,s}$ can only be redefined finitely often, since $\text{level}_T(\varphi_e(f(w_e))) < \omega$, so eventually $f_s^{-1}(y)$ will stabilize, forcing $y \in \text{range}(f)$.

Moreover, $\text{dom}(f) = \bigcup_s D_s = T$, so f is a bijection from T to T'. Since the partial order $\prec'$ on T' is defined by lifting $\prec$ from T via f, we know that f is an isomorphism. Computability of $\prec'$ follows from Lemma 3.7: given $a, b \in T'$, find a stage s such that $a, b \in \text{range}(f_s)$. Then $a \prec' b \iff f_s^{-1}(a) \prec f_s^{-1}(b)$. ∎

Lemma 3.9 *For every e, either $\varphi_e(f(w_e))$ diverges or*

$$\text{level}_{T'}(f(w_e)) \neq \text{level}_T(\varphi_e(f(w_e))).$$

Thus requirement $\mathcal{R}_e$ is satisfied by the element $f(w_e)$.

Proof. Let s_0 be a stage such that for all $s \geq s_0$, $w_{e,s} = w_e$ and $f_s(w_{e,s}) = f(w_e)$. Since $w_{e,s}$ is never redefined after stage s_0, we know that either $\varphi_e(f(w_e))$ diverges, or $\text{level}_{D_s}(\varphi_e(f(w_e))) \neq \text{level}_{D_s}(w_e)$ for all $s \geq s_0$. But since $\bigcup_s D_s = T$, the latter of these implies that $\text{level}_T(\varphi_e(f(w_e))) \neq \text{level}_T(w_e)$. Now $\text{level}_T(w_e) = \text{level}_{T'}(f(w_e))$ since f is an isomorphism, so φ_e maps the element $f(w_e)$ of T' to an element at a different level in T. Thus $\mathcal{R}_e$ is satisfied, and φ_e is not an isomorphism from T' to T. ■

This completes the proof of Proposition 3.2.

4 Trees of Height ω

4.1 Main Theorem

We now prove the desired result for trees of height ω.

Theorem 4.1 *No tree of height ω is computably categorical.*

The theorem will be proved in subsection 4.5, after we have established the necessary five propositions, covering five different types of tree. We use the notions of an extendible node and a side tree to define these cases. Recall (from page 5) that a node $x \in T$ is *extendible* if there exists an infinite path through T containing x. The set of all extendible nodes of T, if nonempty, forms a subtree of T, denoted by T_{ext}. T_{ext} need not be computable, even though T is.

The *side tree* above a node x is denoted $S[x]$, and is a subtree of $T[x]$.

$$S[x] = \{y \in T[x] : (\forall z \in T)[x \prec z \preceq y \implies z \notin T_{\text{ext}}]\}$$

(x itself may or may not be extendible.) Equivalently, consider the extendible immediate successors $x_1, x_2, \ldots$ of x. The side tree $S[x]$ is precisely $T[x] - \bigcup_i T[x_i]$. Thus x itself is the only node of $S[x]$ which can be extendible in T, and $S[x]$ contains no infinite paths, although it can have height ω if it is infinite-branching. $S[x]$ is not necessarily computable.

4.2 Three Cases Using Proposition 3.2

Proposition 4.2 *Let T be a computable tree of height ω, and suppose further that T has height ω above some nonextendible node y_0. Then T is not computably categorical.*

Proof. Let T and y_0 be as in the proposition. We claim there exists an $x_0 \in T$ with ω-many immediate successors, such that $\text{ht}_{x_0}(T) = \omega$ and T has finite height above every $x \succ x_0$. Indeed, consider the subtree

$$S = \{x \in T : \text{ht}_x(T) = \omega \;\&\; x \text{ is nonextendible} \;\&\; x \not\prec y_0\}.$$

S contains a $\prec$-least element (either y_0 or some predecessor of y_0), so S is indeed a subtree. However, S contains no infinite paths, so it must contain terminal nodes, all of which will lie above y_0. We take x_0 to be one of these.

(x_0 is terminal in S, that is; T will have height ω above x_0.) Therefore, T has finite height above every $x \succ x_0$, and moreover, this x_0 must be an ω-branch point, since otherwise one of its immediate successors in T would also be in S. Let $x_1, x_2, \ldots$ be the immediate successors of x_0 in T. Then $\sup_i \mathrm{ht}(T[x_i]) = \omega$, because $\mathrm{ht}_{x_0}(T) = \omega$. But $\mathrm{ht}(T[x_i]) < \omega$ for all $i \geq 1$, since otherwise x_i would lie in S. Therefore we must have $\limsup_i \mathrm{ht}(T[x_i]) = \omega$, and so Proposition 3.2 applies to T and T is not computably categorical. ∎

Proposition 4.3 *Suppose that the computable tree T of height ω contains an extendible node x_0 such that the side tree $S[x_0]$ has height ω. Then T is not computably categorical.*

Proof. If x_0 has an immediate successor in $S[x_0]$ above which T has height ω, then we apply Proposition 4.2 to this node. If all immediate successors of x_0 in $S[x_0]$ have finite height, then there must be infinitely many of them, say $x_1, x_2, \ldots$. Then $\limsup_{i \geq 1} \mathrm{ht}(T[x_i]) = \omega$, because $\sup_{i \geq 1} \mathrm{ht}(T[x_i]) = \omega$. Moreover, any immediate successor of x_0 in T either lies in $S[x_0]$ or is extendible. Hence Proposition 3.2 applies to x_0 itself. ∎

Proposition 4.4 *Suppose that in the computable tree T of height ω, there is a node $x_0 \in T_{ext}$ with infinitely many immediate successors in T_{ext}. Then T is not computably categorical.*

Proof. $\mathrm{ht}(T[y]) = \omega$ for every immediate successor y of x_0 in T_{ext}, so Proposition 3.2 applies to x_0. ∎

4.3 An Isolated Path

Proposition 4.5 *Let T be a computable tree of height ω. Suppose there is a node $x_0 \in T$ which is* uniquely extendible, *i.e. which lies on exactly one infinite path γ through T. If all side trees at nodes on γ above x_0 have finite height, then T is not computably categorical.*

Proof. Let x_0 be a uniquely extendible node on an infinite path γ through T, such that all side trees at nodes on γ above x_0 have finite height.

Let $x_0 \prec x_1 \prec x_2 \prec \ldots$ be all the nodes of γ above x_0. We apply Corollary 2.4 to the set of side trees $S[x_i]$ above nodes of γ, yielding an n such that for every $i \geq n$ and every finite subtree $S \subseteq S[x_i]$, there is some $j > i$ for which

S embeds into $S[x_j]$. Our diagonalization argument will take place entirely above x_n. (Notice that the sequence $\langle x_i \rangle_{i \in \omega}$ cannot necessarily be computed, and that the choice of n from Corollary 2.4 is nonuniform.)

We define $T_s = \{r, x_0, x_1, \ldots x_n\} \cup \{0, 1, \ldots s\}$, a tree under $\prec$. (As before, r represents the root of T.) We computably approximate the sequence $\langle x_i \rangle_{i \in \omega}$. For each s, let

$$\{x_n = x_{n,s} \prec x_{n+1,s} \prec \cdots \prec x_{l_s,s}\}$$

be the chain of maximal length in $T_s[x_n]$. (If there is more than one such chain, take the first such in the dictionary order derived from $<$.) Since all side trees have finite height, clearly $x_{i,s} \to x_i$ for each i. Indeed, $x_{i,s} = x_i$ for all s such that $\{x_n, \ldots x_m\} \subseteq T_s$, where $m = \max_{j<i}(j + \mathrm{ht}(S[x_j]))$. (However, $\mathrm{ht}(S[x_j])$ need not be computable in j.)

The requirements $\mathcal{R}_e$ are the same as in Proposition 3.2:

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T') \; [\mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x))].$$

This time, however, we will say that $\mathcal{R}_e$ is *satisfied at stage s* only if the witness node $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))$ converges and lies at a level of T_s different from $\mathrm{level}_{T_s}(w_{e,s})$.

Instead of simply freezing nodes, as in the proof of Proposition 3.2, we must freeze them with priority e. Thus, at each stage s, we define *envelopes* $E_{e,s}$ for each e, to provide negative restraints on redefining the isomorphism f on elements of $E_{e,s}$. If x lies in the envelope $E_{e,s}$, then $f_{s+1}(x) \neq f_s(x)$ only if necessary for the sake of a requirement $\mathcal{R}_i$ with $i \leq e$. Thus the envelopes will ensure that the functions f_s converge to a limit f with range ω.

Construction: f_0 is the identity map on T_0, and the witness nodes $w_{e,0}$ are undefined for all e. We define $E_{e,0} = \emptyset$ for all e.

At stage $s+1$, we search for the least $e \leq s+1$ such that one of the following holds:

1. $w_{e,s}$ is undefined.

2. For each i with $n \leq i \leq l_{s+1}$, the following holds:

$$x_{i,s+1} \preceq w_{e,s} \implies x_{i,s+1} \preceq w_{e-1,s}.$$

3. $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and

$$\mathrm{level}_{D_s}(w_{e,s}) = \mathrm{level}_{D_s}(\varphi_e(f_s(w_{e,s}))).$$

(Such an e must exist, because $w_{s+1,s}$ is undefined.) Let $w_{i,s+1} = w_{i,s}$ and

$$E_{i,s+1} = \{i \in D_{s+1} : (\exists z \in E_{i,s})\ y \preceq z\}$$

for all $i < e$, and let $w_{j,s+1}$ be undefined and $E_{j,s+1} = \emptyset$ for all $j > e$.

If case (1) holds for e, we let $w_{e,s+1}$ to be the $<$-least node in $D_s[x_n]$ with $\text{level}_{D_s[x_n]}(w_{e,s+1}) \geq e$ which does not lie in any $E_{i,s}$ with $i < e$ and such that

$$(\exists j)[x_{j,s} \preceq w_{e,s+1} \ \&\ x_{j,s} \not\preceq w_{e-1,s+1}].$$

We define $E_{e,s+1} = D_{s+1} = D_s \cup T_{s+1}$. (If no such node exists, then $w_{e,s+1}$ remains undefined, with $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$.)

If case (2) holds, we let $w_{e,s+1}$ diverge with $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$. (This is the case where $w_{e-1,s}$ and $w_{e,s}$ appear to lie in the same side tree along γ, in which case we cannot embed one upwards without disturbing the other.)

Otherwise, case (3) holds. We search for the least $t \geq \max(D_s)$ satisfying either of the following two conditions. Let $m_t = \max\{k : x_{k,t} \preceq w_{e,s}\}$ for each t.

Condition 4.6 *There exists $i < e$ such that $x_{m_t,t} \preceq w_{i,t}$.*

Condition 4.7 *There exists an embedding g of $D_s[x_{m_t,t}]$ into $T_t[x_{m_t,t}]$ with*

$$\text{level}_{T_t}(g(w_{e,s})) > \text{level}_{D_s}(w_{e,s}).$$

If Condition 4.6 holds for t, then we make $w_{e,s+1}$ undefined, and set $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$.

Otherwise, we use the embedding g given by Condition 4.7 to satisfy requirement $\mathcal{R}_e$. Let $w_{e,s+1} = g(w_{e,s})$, and for all $y \in D_s[x_{m_t,t}]$, define $f_{s+1}(g(y)) = f_s(y)$. For those $y \in D_s[x_{m_t,t}] - \text{range}(g)$, take $f_{s+1}(y)$ to be the least element of ω that is not yet in $\text{range}(f_{s+1})$ nor in $\text{range}(f_s)$. Let $D_{s+1} = D_s \cup T_t$, and let the envelope $E_{e,s+1} = D_{s+1}$.

(For the sake of clarity, we note that if $x_{m_t,t}$ does not lie in D_s, then

$$D_s[x_{m_t,t}] = \{y \in D_s : x_{m_t,t} \preceq y\}.$$

We do have $w_{e,s} \in D_s[x_{m_t,t}]$ by definition of m_t. If $D_s[x_{m_t,t}]$ does not have a single root, then we consider each minimal element in it to have level 0.)

In all three cases, we then define $f_{s+1}(y) = f_s(y)$ for those $y \in D_s$ on which f_{s+1} is not yet defined. Also, for each $y \in D_{s+1} - D_s$ on which f_{s+1} is not yet defined, choose the least element of ω which is not yet in range(f_{s+1}) to be $f_{s+1}(y)$. This completes the construction.

(The idea of the construction is that each witness element $w_{e,s}$ lies in the side tree above some x_i. When we need to satisfy $\mathcal{R}_e$, we do so by embedding the side tree containing $w_{e,s}$ into another side tree at a higher level. We define f_{s+1} so that $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. Since $\text{level}_T(\varphi_e(f_s(w_{e,s})))$ is finite, we will only have to repeat this process finitely often before reaching a stage s such that $f_s(w_{e,s})$ will satisfy $\mathcal{R}_e$ permanently.)

We first must prove that at each stage s at which we search for a t, we eventually find one. This requires a lemma guaranteeing our ability to embed trees upwards in $T[x_n]$.

Lemma 4.8 *For every $x_i \succeq x_n$ and every t, there is an embedding g of the tree $T_t[x_i]$ into $T[x_{i+1}]$.*

Proof. By the choice of n and Corollary 2.4, we know that every finite subtree of every $S[x_j]$ with $j \geq n$ embeds into some $S[x_k]$ with $k > j$. By induction, then, every finite subtree of every such $S[x_j]$ embeds into infinitely many $S[x_k]$ with $k > j$. Since there are only finitely many side trees $S[x_{j_0}], \ldots S[x_{j_n}]$ which intersect the finite tree T_t, we can embed $S[x_{j_0}] \cap T_t$ into some $S[x_{k_0}]$, then embed $S[x_{j_1}] \cap T_t$ into some $S[x_{k_1}]$ with $k_1 > k_0$, and so on. The union of these embeddings is the desired embedding g. ■

Lemma 4.9 *Fix any stage s, and take the corresponding e chosen in the construction. Then there exists a t for which Condition 4.7 holds.*

Proof. Since each sequence $\langle x_{i,t} \rangle_{t \in \omega}$ converges to x_i, we know that m_t converges to a limit m as $t \to \infty$. Thus $w_{e,s} \in S[x_m]$, and $m \geq n$. Moreover, there exists t such that $D_s \subseteq T_t$. By Lemma 4.8, there is an embedding $g : T_t[x_m] \to T[x_{m+1}]$, and then

$$\text{level}_{D_s}(w_{e,s}) \leq \text{level}_{T_t}(w_{e,s}) < \text{level}_T(g(w_{e,s}))$$

since $\text{level}_{T_t}(x) < \text{level}_T(g(x))$ for every $x \in T_t[x_m]$. ■

Lemma 4.10 *For every e, the sequence $\langle w_{e,s} \rangle_{s \in \omega}$ converges to a limit w_e, the sequence $\langle f_s(w_e) \rangle_{s \in \omega}$ converges to a limit $f(w_e)$, and either $\varphi_e(f(w_e))\uparrow$ or $\text{level}_T(\varphi_e(f(w_e))) \neq \text{level}_T(w_e)$. (Since $\text{level}_T(w_e) = \text{level}_{T'}(f(w_e))$, this satisfies $\mathcal{R}_e$.)*

Proof. Assume by induction that there exists a stage s_0 such that for all $s \geq s_0$ and all $i < e$, the hypotheses of the theorem hold: $w_{i,s} = w_i$, $f_s(w_i) = f(w_i)$, and either $\varphi_i(f(w_i))\uparrow$ or $\mathcal{R}_i$ is satisfied by $f(w_i)$ at stage s. Moreover, assume $x_{k,s} = x_k$ for every $k \leq j+1$ and every $s \geq s_0$, where j is maximal with $x_j \preceq w_{e-1}$, Then $m_s \geq j+1$ for every $s \geq s_0$, so $x_{m_s,s} \not\preceq w_i$ for all $i < e$ and $s \geq s_0$. If w_{e,s_0} is undefined, then at the first stage s after s_0 at which $\mathrm{ht}(D_s) > \mathrm{ht}(E_{e-1,s}) + \mathrm{level}_T(x_n)$, $w_{e,s}$ will be defined. Moreover, it will never again become undefined, since Condition 4.6 will never again be satisfied and case (2) will never apply.

Now if there is no stage $s \geq s_0$ such that $\varphi_{e,s}(w_{e,s})\downarrow$ and

$$\mathrm{level}_{D_s}(w_{e,s}) = \mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))),$$

then neither $w_{e,s}$ nor $f_s(w_{e,s})$ will ever be redefined after s_0. Then $\mathcal{R}_e$ will be satisfied by $w_e = \lim_s w_{e,s}$, because $\mathrm{level}_T(\varphi_e(f_s(w_{e,s})))$ is finite. Thus the lemma will be satisfied for e.

If there are stages $s \geq s_0$ where $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and $\mathrm{level}_{D_s}(w_{e,s}) = \mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s})))$, then Condition 4.6 will not hold at those stages, so at each such s we will find a t satisfying Condition 4.7 and follow the corresponding instructions for that t. Thus, $w_{e,s+1}$ will be redefined, but with $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. Moreover, by our choice of g, we will have

$$\mathrm{level}_{D_{s+1}}(w_{e,s+1}) > \mathrm{level}_{D_s}(w_{e,s}).$$

Now $\mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s})))$ may increase as s increases, but only finitely often, since $f_s(w_{e,s})$ is constant after s_0 and $\mathrm{level}_T(\varphi_e(f_s(w_{e,s}))) < \omega$. Therefore, we eventually reach a stage s_1 with

$$\mathrm{level}_{D_{s_1}}(\varphi_e(f_{s_1}(w_{e,s_1}))) = \mathrm{level}_T(\varphi_e(f_{s_1}(w_{e,s_1}))),$$

and for all $s > s_1 + 1$, $\mathcal{R}_e$ will be satisfied by $w_{e,s}$. Therefore $w_{e,s}$ will never again be redefined, and $\mathcal{R}_e$ will be satisfied by $w_e = \lim_s w_{e,s}$. ∎

Lemma 4.11 *For every $x \in T$, the sequence $\langle f_s(x)\rangle_{s\in\omega}$ converges to a limit. The limit function $f = \lim_s f_s$ has range ω.*

Proof. Fix x. The construction ensures that $x \in T_x \subseteq D_x \subseteq D_s = \mathrm{dom}(f_s)$ for all $s \geq x$. If $x_n \not\preceq x$, then $f_s(x) = f_{s+1}(x)$ for all s for which $f_s(x)$ is defined.

Assume, therefore, that $x_n \prec x$. Let $k = \max\{i : x_i \preceq x\}$, so $x \in S[x_k]$. Let s_0 be a stage such that for all $s \geq s_0$ and for all $i \leq k+1$, we have $x_{i,s} = x_i$. Also, let $h = \max\{i + \mathrm{ht}(S[x_i]) : i \leq k+1\}$. Then by Lemma 4.10, there exists a stage $s_1 \geq s_0$ such that for all $s \geq s_1$ and for all $i \leq h$, we have $w_{i,s} = w_i$.

Suppose $s \geq s_1$ is a stage such that $f_s \not\subseteq f_{s+1}$, and take the corresponding index e. Then Condition 4.7 is satisfied for some $t > s$, yielding an embedding $g : D_s[x_{m_t,t}] \to T_t[x_{m_t,t}]$. By the construction, $w_{e,s} \neq w_{e,s+1}$, so we must have $e > h$. This forces $\mathrm{level}_{D_s}(w_{e,s}) > h$, since each $w_{i+1,s}$ is at a level $> i$, so $w_{e,s} \notin \bigcup_{i \leq k} S[x_i]$ by choice of h. Hence $x_{k+1} \preceq w_{e,s}$, and $m_t \geq k+1$ by definition of m_t (and since $t \geq s_0$). But then $x_{k+1} = x_{k+1,t} \preceq x_{m_t,t}$. Since $x \in S[x_k]$, we have $x_{k+1} \not\preceq x$, so $x_{m_t,t} \not\preceq x$. Hence $x \notin D_s[x_{m_t,t}]$, and so $x \notin \mathrm{dom}(g)$. Therefore $f_{s+1}(x) = f_s(x)$ for all $s \geq s_1$. We define $f = \lim_s f_s$.

To see that $\mathrm{range}(f) = \omega$, let $y \in \omega$. We assume inductively that $\{0, 1, \ldots y-1\} \subseteq \mathrm{range}(f)$. Therefore, if $y \notin \mathrm{range}(f)$, there would exist a stage at which y would be the least available fresh element, and so there must be a stage s_0 and an $x \in T$ for which $f_{s_0}(x) = y$. Moreover, then $y \in \mathrm{range}(f_s)$ for all $s \geq s_0$.

If there exists some stage $s_1 > s_0$ at which $f_{s_1-1}(x) \neq f_{s_1}(x)$, say for the sake of a requirement $\mathcal{R}_e$, then there must be an x' such that $f_{s_1}(x') = y$. At each such s_1, we will have $x' \in E_{e,s_1}$. Indeed, by taking s_1 so large that all $\mathcal{R}_i$ with $i \leq e$ are satisfied at all stages $s \geq s_1$, we may assume that $x' \in E_{e,s}$ for all $s \geq s_1$. But then $f_s(x') = f_{s+1}(x')$, so $y = f(x') \in \mathrm{range}(f)$. ■

Thus f is a 1-1 Δ^0_2 map from T to ω, hence an isomorphism from T to the tree $(T', \prec')$, where $T' = \omega$ and $\prec'$ is just the ordering $\prec$, induced on T' from T' by f.

Lemma 4.12 *The maps f_s satisfy Condition 3.1. Thus $\prec'$ is computable.*

Proof. The construction ensures that $D_s \subseteq D_{s+1}$ for all s. For every $x \in D_s - D_s[x_n]$, we have $f_s(x) = f_{s+1}(x)$. Therefore, Condition 3.1 clearly holds if either $f_s^{-1}(a)$ or $f_s^{-1}(b)$ is not in $T[x_n]$. So take $x, y \in D_s[x_n]$, with $a = f_s(x)$, $b = f_s(y)$, and let $x' = f_{s+1}^{-1}(a)$ and $y' = f_{s+1}^{-1}(b)$. We have four cases, depending on whether or not $x = x'$ and $y = y'$.

The first case, where $x = x'$ and $y = y'$, is trivial. Also, if $x \neq x'$ and $y \neq y'$, then x and y must both lie in $D_s[x_{m_t,t}]$, for which we find an embedding g into some $T_t[x_{m_t,t}]$. In this case,

$$x \preceq y \iff g(x) \preceq g(y) \iff x' \preceq y'$$

since $g(x) = x'$ and $g(y) = y'$. Thus Condition 3.1 is satisfied in these two cases.

Suppose $x \neq x'$ and $y = y'$. Then $x \in D_s[x_{m_t,t}]$. If $y \in D_s[x_{m_t,t}]$ also, then $x' = g(x) \prec g(y) = y'$. If not, then either $y \prec x_{m_t,t}$ (in which case $y \prec x$ and $y \prec g(x) = x'$, since range$(g) \subset T[x_{m_t,t}]$) or $y \perp x_{m_t,t}$ (in which case $y \perp x$ and $y \perp g(x) = x'$, again because range$(g) \subset T[x_{m_t,t}]$).

The preceding paragraph shows that in the third case, not only

$$x \prec y \iff x' \prec y'$$

but also

$$x \perp y \iff x' \perp y'.$$

Hence by symmetry, the fourth case, with $x = x'$ and $y \neq y'$, is also satisfied.

■

Thus $(T', \prec')$ is a computable tree, isomorphic to T, which satisfies every requirement $\mathcal{R}_e$. Hence T is not computably categorical, proving Proposition 4.5. ■

4.4 No Isolated Paths

An extendible node which lies on more than one infinite path is called *multiply extendible*, as opposed to the uniquely extendible nodes of Subsection 4.3. We now consider the case of a tree in which every extendible node is multiply extendible. This implies that every extendible node lies on infinitely many infinite paths. (We also assume that the tree contains at least one extendible node!)

Proposition 4.13 *Let T be a computable tree of height ω such that T_{ext} is non-empty and finite-branching and every $x \in T_{ext}$ lies on infinitely many infinite paths through T. If all side trees in T have finite height, then T is not computably categorical.*

Proof. We use the same requirements $\mathcal{R}_e$ as in Propositions 3.2 and 4.5. The idea of this construction is that for each e, we devote an entire level l_e of T to satisfying $\mathcal{R}_e$. By the assumptions of the Proposition, we know that there exists at least one extendible node at level l_e, and at most finitely many of them. Also, there may exist any number of nonextendible nodes at level l_e.

Since we cannot tell the extendible nodes from the nonextendible ones at any stage s, we consider all the nodes at level $l_{e,s}$ at that stage, and denote them by $v^0_{e,s}, v^1_{e,s}, \dots v^{n_{e,s}}_{e,s}$.

Now since the Proposition assumes that the side tree above each extendible node has finite height, and since there exist only finitely many extendible nodes at levels $\leq l_e$, there must exist a number d_e such that every node x at level l_e with $\mathrm{ht}_x(T_s) \geq d_e$ must be extendible. We do not know d_e, but at each stage we focus on those nodes at level $l_{e,s}$ in $D_s \supseteq T_s$ above which D_s has maximal height. Thus, we will eventually be considering only extendible nodes and their successors. Above these nodes we look for upward embeddings to use to satisfy $\mathcal{R}_e$. Since every extendible node x lies on infinitely many infinite paths, and since T_{ext} is finite-branching, $T[x]$ must contain a subtree of type $2^{<\omega}$, and any finite tree can be embedded into $2^{<\omega}$ at arbitrarily high levels. Thus we can find upward embeddings of $D_s[x]$ above x whenever needed, as long as x is extendible.

(For trees defined using the infimum function, it is not immediate that the required embeddings exist. For instance, a tree with three nodes at level 1 does not embed into $2^{<\omega}$. Therefore, we need the following lemma.

Lemma 4.14 *Let T be a tree such that T_{ext} is nonempty and finite-branching and contains no isolated paths. Then all but finitely many nodes $x \in T_{ext}$ have the property that for every finite $S \subseteq T[x]$, there exists $y \succ x$ with $y \in T_{ext}$ such that S embeds into $T[y]$.*

Proof. Suppose the lemma failed, so the the set U of nodes where it fails (with the root r of T adjoined) forms an infinite subtree of T_{ext}. Since T_{ext} is finite-branching, König's Lemma provides an infinite path through U, which in turn yields an infinite path through T_{ext} containing infinitely many nodes $r \prec u_0 \prec u_1 \prec \cdots$ from U. Now for each i, some finite $S_i \subseteq T[u_i]$ embeds into no $T[y]$ with $u_i \prec y$. In particular, S_i does not embed into any $T[u_j]$ with $j > i$. Such a sequence would violate Lemma 2.3. ∎

It follows that, by considering only nodes at sufficiently high levels, we can guarantee the existence of $\wedge$-preserving embeddings. Thus Proposition 4.13 also holds for trees defined using the infimum.)

The notation is as in the previous proofs, except that there may be more than one potential witness for a given requirement $\mathcal{R}_e$ at a given stage s. We denote these witnesses by $w^0_{e,s}, w^1_{e,s}, \dots w^{n_{e,s}}_{e,s}$. Also, we will keep track of the original position of each of these witnesses. When $w^k_{e,s}$ is defined, we will set $v^k_{e,s} = w^k_{e,s}$, but as $w^k_{e,s}$ is embedded further up in the tree, $v^k_{e,s}$ stays fixed. The only stages at which $v^k_{e,s}$ will be redefined are those at which a requirement of higher priority receives attention and those at which $v^k_{e,s}$ acquires a new predecessor. For a given e and s, the elements $v^k_{e,s}$ will be at the same level for all k, and we will denote this level by $l_{e,s}$.

Let r be the root of T. We define $T_s = \{r\} \cup \{0, 1, \ldots s\}$, a tree under $\prec$. Again, we will define envelopes $E_{e,s}$, in order to ensure that $\text{range}(f) = \omega$.

The requirements $\mathcal{R}_e$ are as follows:

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T') \, [\text{level}_{T'}(x) \neq \text{level}_T(\varphi_e(x))].$$

$\mathcal{R}_e$ *receives attention at stage* s if some witness node $w^k_{e,s}$ is embedded upwards at stage s, if $w^0_{e,s}$ is newly defined at stage s, or if the height of the envelope $E_{e,s}$ increases at stage s. When this happens, all actions previously taken for the sake of requirements $\mathcal{R}_j$ with $j > e$ are injured. However, this will only occur finitely often for each e.

Construction: f_0 is the identity map on T_0, and the witness nodes $w^k_{e,0}$ and their original positions $v^k_{e,0}$ are undefined for all e and k. Also undefined are $n_{e,0}$ and $l_{e,0}$ for all e, and all $E_{e,0}$ are empty.

At stage $s + 1$, we execute the following steps for each $e \leq s$, starting with $e = 0$. If a requirement $\mathcal{R}_e$ receives attention, then we do not execute the steps for any $j > e$.

1. If $w^0_{e,s}$ is undefined, and there exists an element x of D_s with

$$\text{level}_{D_s}(x) > \max \bigcup_{i<e} \{\text{level}_{D_s}(y) : y \in E_{i,s}\},$$

   then let $l_{e,s+1}$ be its level, and let $w^0_{e,s+1}, \ldots w^{n_{e,s+1}}_{e,s+1}$ be all the elements of D_s at level $l_{e,s+1}$. Let $v^k_{e,s+1} = w^k_{e,s+1}$ for each k. Requirement $\mathcal{R}_e$ has now received attention. Let $D_{s+1} = D_s \cup T_{s+1}$, and set $E_{e,s+1} = D_{s+1}$. For each $j > e$ we set

$$E_{j,s+1} = \{y \in D_s : (\exists z \in E_{j,s}) \; y \preceq z\}.$$

2. If $w^0_{e,s}$ is undefined, and there does not exist any element x at a sufficiently high level to satisfy condition (1), then let $w_{e,s+1}\uparrow$ also, and set

$$E_{e,s+1} = \{y \in D_s : (\exists z \in E_{e,s}) \; y \preceq z\}.$$

   Then $\mathcal{R}_e$ has not received attention at this stage.

3. Otherwise, $w^0_{e,s}, \ldots w^{n_{e,s}}_{e,s}$ are defined, as are the corresponding $v^k_{e,s}$. Find the least stage $t \geq \max(D_s)$ such that one of the following holds:

   (a) There exists $m \leq n_{e,s}$ and an embedding $g : D_s[v^m_{e,s}] \to T_t[v^m_{e,s}]$ such that
   $$\mathrm{level}_{T_t}(g(w^m_{e,s})) \geq \mathrm{level}_{D_s}(w^m_{e,s}) + s.$$

   (b) There exists $x \in T_t$ with $\mathrm{level}_{T_t}(x) = l_{e,s}$ and $\mathrm{ht}_x(T_t) \geq s$, such that either $x \notin D_s$ or $\mathrm{level}_{D_s}(x) < l_{e,s}$.

   If (b) holds and (a) fails at stage t, let $w^k_{e,s+1} = w^k_{e,s}$ for all $k \leq n_{e,s}$, and let $l_{e,s+1} = l_{e,s}$. For each k, if $\mathrm{level}_{D_s}(v^k_{e,s}) = l_{e,s}$, let $v^k_{e,s+1} = v^k_{e,s}$; otherwise let $v^k_{e,s+1}$ be the predecessor of $v^k_{e,s}$ at level $l_{e,s}$ in D_s. If there exist elements $x \in D_s$ with $\mathrm{level}_{D_s}(x) = l_{e,s}$ such that $x \notin \{v^0_{e,s+1}, \ldots v^{n_{e,s}}_{e,s+1}\}$, then define those x's to be $w^{1+n_{e,s}}_{e,s+1}, w^{2+n_{e,s}}_{e,s+1}, \ldots$, with $v^k_{e,s+1} = w^k_{e,s+1}$ for each, and define $n_{e,s+1}$ to be the greatest superscript required. (If there are no such x, then $n_{e,s+1} = n_{e,s}$.) Define
   $$E_{e,s+1} = \{y \in D_s : (\exists z \in E_{e,s})\ [y \preceq z]\}.$$

   If $l_{e+1,s}\downarrow$ and $\mathrm{ht}(E_{e,s+1}) \geq l_{e+1,s}$, then we say that $\mathcal{R}_e$ has received attention at stage $s+1$, and for each $j > e$ we set
   $$E_{j,s+1} = \{y \in D_s : (\exists z \in E_{j,s})\ [y \preceq z]\}.$$

   Otherwise $\mathcal{R}_e$ has not received attention.

   If (a) holds at stage t, let m be the least index for which it holds, and let g be the corresponding embedding. If $\varphi_{e,s}(f_s(w^m_{e,s}))\uparrow$, or if
   $$\mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w^m_{e,s}))) \neq \mathrm{level}_{D_s}(w^m_{e,s}),$$
   then we proceed exactly as in the preceding paragraph. Otherwise, $\mathcal{R}_e$ receives attention as follows. For every node $y \in D_s[v^m_{e,s}]$, define $f_{s+1}(g(y)) = f_s(y)$ and define $f_{s+1}(y)$ vto be the least element of ω which is not already in $\mathrm{range}(f_{s+1}) \cup \mathrm{range}(f_s)$. Let $w^m_{e,s+1} = g(w^m_{e,s})$. We define $l_{e,s+1} = l_{e,s}$. For each k, let $v^k_{e,s+1}$ be that predecessor of $v^k_{e,s}$ at level $l_{e,s}$ in D_s. (Quite possibly, this will be $v^k_{e,s}$ itself.) Also, if there are any $x \in D_s$ at level $l_{e,s}$ which are not in $\{v^k_{e,s+1} : k \leq n_{e,s}\}$, then define those x's to be $w^{1+n_{e,s}}_{e,s+1}, w^{2+n_{e,s}}_{e,s+1}, \ldots$, with $v^k_{e,s+1} = w^k_{e,s+1}$ for each, and define $n_{e,s+1}$ to be the greatest superscript required. (If there are no such x, then $n_{e,s+1} = n_{e,s}$.) Finally, let $D_{s+1} = D_s \cup \mathrm{range}(g) \cup T_{s+1}$, and let $E_{e,s+1} = D_{s+1}$, with $E_{j,s+1} = \emptyset$ for all $j > e$.

4. If $\mathcal{R}_e$ has received attention at stage $s+1$, we make all $n_{j,s+1}$, $l_{j,s+1}$, $v^k_{j,s+1}$ and $w^k_{j,s+1}$ undefined for all $j > e$, and skip all steps for all those j. Otherwise we increment e by 1 and return to Step 1.

Once we have either given attention to a requirement or completed the steps with $e = s$, we define $f_{s+1}(y) = f_s(y)$ for those $y \in D_s$ on which f_{s+1} is not yet defined. Also, for each $y \in D_{s+1} - D_s$ on which f_{s+1} is not yet defined, choose the least element of ω which is not yet in range(f_{s+1}) to be $f_{s+1}(y)$. This completes the construction.

Lemma 4.15 *For each s and each $e \leq s$, either 3(a) or 3(b) must hold for some t.*

Proof. Suppose there exists an extendible node y among $\{v^0_{e,s}, \dots v^{n_{e,s}}_{e,s}\}$. Then by the assumption of the proposition, there is a copy of $2^{<\omega}$ embedded into $T[y]$, and any finite tree can be embedded into $2^{<\omega}$ with the root mapping to a node at an arbitrarily high level of $2^{<\omega}$. Thus 3(a) will eventually hold.

Otherwise, none of $v^0_{e,s}, \dots v^{n_{e,s}}_{e,s}$ is extendible. Now some node x on level $l_{e,s}$ of T must be extendible. If $x \in D_s$, then we must have $\text{level}_{D_s}(x) < l_{e,s}$, since no node at level $l_{e,s}$ in D_s is extendible. Otherwise $x \notin D_s$, and either way we will eventually reach a stage t at which 3(b) holds of x. ∎

Lemma 4.16 *For every e the following hold:*

- $\lim_s \text{ht}(E_{e,s})$ *exists and is finite.*
- *The sequence $\langle l_{e,s}\rangle_{s\in\omega}$ converges to some $l_e \in \omega$.*
- *For every $k \in \omega$, either $\langle w^k_{e,s}\rangle_{s\in\omega}$ and $\langle v^k_{e,s}\rangle_{s\in\omega}$ converge to elements w^k_e and v^k_e in ω, or there exists a stage t such that $w^k_{e,s}\uparrow$ and $v^k_{e,s}\uparrow$ for all $s > t$.*
- *The requirement $\mathcal{R}_e$ receives attention at only finitely many stages, and is satisfied.*

Proof. We proceed by induction on e. Fix e, and assume s_0 is a stage satisfying all of the following conditions for every $s \geq s_0$ and every $i < e$:

1. $\mathcal{R}_i$ does not receive attention at stage s;
2. $l_{i,s} = l_i$;

3. Every $v \in T_{\text{ext}}$ with $\text{level}_T(v) = l_e$ satisfies $\text{level}_{T_s}(v) = l_e$, and hence is of the form $v^k_{e,s}$ for some k;

4. $v^k_{i,s} = v^k_i$ and $w^k_{i,s} = w^k_i$ for all k such that $v^k_{i,s} \in T_{\text{ext}}$ (Notice that each level of T_{ext} is finite, since the proposition assumes that T_{ext} is finitely branching. Hence only finitely many $v^k_{i,s}$ lie in T_{ext}.);

5. $\text{ht}(T_s) > l_{e-1}$.

Condition 3 simply says that we have waited until all predecessors of each $v \in T_{\text{ext}}$ at level l_e have appeared in T_{s_0}. This is possible because T_{ext} is finite-branching. Notice that this condition implies the same condition for all $i \leq e$.

Now $l_{e,s}$ is never redefined in the construction, and it can only become undefined at stages at which some $\mathcal{R}_i$ with $i < e$ receives attention. Hence $l_{e,s} = l_{e,s_0+1}$ for all $s > s_0$, so $l_{e,s}$ converges to a limit $l_e = l_{e,s_0+1}$. Also, after stage s_0 in the construction, $v^k_{e,s}$ can only be redefined to be a predecessor of itself, and that only when it has acquired a new predecessor. But by Condition 3, each $v^k_{e,s}$ acquires no new predecessors in T after stage s_0, so each sequence $\langle v^k_{e,s} \rangle_{s\in\omega}$ converges to a limit $v^k_e = v^k_{e,s_0}$.

Similarly, $w^k_{e,s}$ is never undefined after stage s_0, although it may be redefined at certain stages at which $\mathcal{R}_e$ receives attention. If $v^k_{e,s} \notin T_{\text{ext}}$, then $\text{ht}_{v^k_{e,s}}(T)$ is finite, and the corresponding $w^k_{e,s}$ can only be embedded finitely often by step 3(a), since each embedding (at a stage $s+1$) moves it up by at least s levels in D_s. Hence all those sequences $\langle w^k_{e,s} \rangle_{s\in\omega}$ converge.

For each of the finitely many k with $v^k_{e,s} \in T_{\text{ext}}$, it is possible for 3(a) to hold for k at infinitely many stages. However, we only actually apply the embedding g to redefine $w^k_{e,s}$ at stages $s+1$ such that $\varphi_{e,s}(f_s(w^k_{e,s}))\downarrow$ and $\text{level}_{D_s}(\varphi_{e,s}(f_s(w^k_{e,s}))) = \text{level}_{D_s}(w^k_{e,s})$. By the construction, we always have $f_{s+1}(w^k_{e,s+1}) = f_s(w^k_{e,s})$, even if $w^k_{e,s+1} \neq w^k_{e,s}$. At each stage $s+1$ at which $w^k_{e,s}$ is redefined, we have

$$\text{level}_{D_{s+1}}(w^k_{e,s+1}) \geq \text{level}_{D_s}(w^k_{e,s}) + s.$$

If this happens sufficiently often, then we must reach a stage s_1 at which $\text{level}_{D_{s_1}}(w^k_{e,s_1}) > \text{level}_T(\varphi_{e,s_1}(f_{s_1}(w_{e,s_1})))$, since T has height ω, and after stage s_1, we will never redefine $w^k_{e,s}$ again, even if 3(a) does apply. Hence each of these sequences $\langle w^k_{e,s} \rangle_{s\in\omega}$ does converge to a limit w^k_e.

Now there must be an element of T_{ext} on level l_e, and this element will be designated at some stage s as $v^k_{e,s}$ for some k. We note first that since all side trees are finite and T_{ext} is finitely-branching, there is a d such that every nonextendible node x at any level $\leq l_e$ satisfies $\text{ht}_x(T) < d$. (Also, assume d is sufficiently large that $l_{e,d} = l_e$.) Once we reach stages $s \geq d$, therefore, 3(a) will never again hold for any m with $v^m_{e,s}$ nonextendible, and 3(b) will not hold for any nonextendible x. Thus only the finitely many extendible nodes $v^k_{e,s}$ will satisfy either 3(a) or 3(b) at any subsequent stage. But every extendible node v at level l_e in T already satifies $\text{level}_{T_{s_0}}(v) = l_e$, by inductive hypothesis, so 3(b) will never hold again. By Lemma 4.15, there must exist an m, with v^m_e extendible, which satisfies 3(a) at infinitely many stages. (If there is more than one such, choose the least of them, just as we did at each stage of the construction.)

If $\varphi_e(f_s(w^m_{e,s}))\uparrow$ for the corresponding w^m_e, then $w^m_{e,s}$ is never redefined, and $f_{s+1}(w^m_e) = f_s(w^m_e)$ for all s, so $\varphi_e(f(w^m_e))\uparrow$, where $f = \lim_s f_s$ as defined below. Hence $\mathcal{R}_e$ is satisfied, since φ_e is not total. On the other hand, if $\varphi_e(f_s(w^m_{e,s}))\downarrow$, then for every stage s at which

$$\text{level}_{D_s}(\varphi_{e,s}(f_s(w^m_{e,s}))) = \text{level}_{D_s}(w^m_{e,s}),$$

either there will be a subsequent stage s' at which 3(a) applies to $v^m_{e,s'}$ and $\mathcal{R}_e$ receives attention and $w^m_{e,s'}$ is embedded at a greater level, or else

$$(\forall s' > s)[\text{level}_{D_{s'}}(w^m_{e,s'}) < \text{level}_{D_{s'}}(\varphi_{e,s'}(f_{s'}(w^m_{e,s'})))].$$

In the latter case, $w^m_{e,s'}$ will never again be redefined, leaving $\mathcal{R}_e$ satisfied by the witness $f(w^m_e)$. In the former case, we again have

$$\text{level}_{D_{s'+1}}(\varphi_e(f_{s'+1}(w^m_{e,s'+1}))) < \text{level}_{D_{s'}}(w^m_{e,s'}).$$

But

$$\text{level}_{D_s}(\varphi_e(f(w^m_e))) \leq \text{level}_T(\varphi_e(f(w^m_e))) < \omega,$$

so eventually we reach a stage s with $\text{level}_T(\varphi_e(f(w^m_e))) < \text{level}_{D_s}(w^m_{e,s})$. After this stage, $w^m_{e,s}$ is never redefined, leaving

$$\text{level}_T(\varphi_e(f(w^m_e))) < \text{level}_T(w^m_e) = \text{level}_{T'}(f(w^m_e)).$$

Thus requirement $\mathcal{R}_e$ is satisfied.

We note that since each sequence $\langle w^k_{e,s}\rangle_{s\in\omega}$ converges to w^k_e, none of them changes value more than finitely often. Moreover, the stage d designated above has the property that only finitely many elements $w^k_{e,s}$ are ever redefined after stage d, namely those corresponding to extendible v^k_e.

Moreover, since there are only finitely many stages s at which any of the elements $w^k_{e,s}$ is redefined, we eventually reach a stage s_1 after which none of them is ever redefined. Now E_{e,s_1} is finite. Let s_2 be a stage such that

$$(\forall y \in T)[(\exists z \in E_{e,s_1})[y \preceq z] \implies y \in T_{s_2}].$$

That is, every predecessor of each of the (finitely many) elements $x \in E_{e,s_1}$ appears in T_{s_2}. Then for all $s \geq s_2$, we have $E_{e,s} = E_{e,s_2}$. Hence $\lim_s \mathrm{ht}(E_{e,s}) = \mathrm{ht}(E_{e,s_2})$. Thus $\mathcal{R}_e$ only receives attention finitely often.

This completes the induction. ■

Lemma 4.17 *For each x, the sequence $\langle f_s(x)\rangle_{s\in\omega}$ converges. The limit function $f = \lim_s f_s$ has range ω.*

Proof. We need to show that both $\lim_s f_s(x)$ and $\lim_s f_s^{-1}(y)$ exist for all x and y in ω.

First of all, we have $x \in T_s \subseteq D_s$ for every $s \geq x$, so $f_s(x)\downarrow$ for all sufficiently large s. Also, by the construction, we have $\mathrm{range}(f_s) \subseteq \mathrm{range}(f_{s+1})$ for every s. Moreover, each time we need a new element for the range of f_{s+1}, we take the least available one, so clearly every $y \in \omega$ lies in $\mathrm{range}(f_s)$ for all sufficiently large s.

So suppose $f_s(x) \neq f_{s+1}(x)$ for some s. The only way this can occur in our construction is if 3(a) holds for some e and m, and we execute an upwards embedding g of $D_s[v^m_{e,s}]$ into $T[v^m_{e,s}]$ at stage $s+1$ in order to satisfy $\mathcal{R}_e$. If this happens, then $E_{e,s+1} = D_{s+1} \supseteq \mathrm{range}(g)$, so $x \in E_{e,s+1}$. Similarly, if $f_s^{-1}(y) \neq f_{s+1}^{-1}(y)$ for some s, then $f_{s+1}^{-1}(y) \in E_{e,s+1}$.

The only way we could then have $f_t(x) \neq f_{t+1}(x)$ or $f_t^{-1}(y) \neq f_{t+1}^{-1}(y)$ for any $t > s$ is if some $\mathcal{R}_i$ with $i \leq e$ receives attention at stage $t+1$. This could happen for the following reasons:

Case 1: Step 3(a) applies to $\mathcal{R}_i$ for some $i \leq e$, and we execute the corresponding upward embedding g. In this case, $E_{i,t+1} = D_{t+1}$, so $x \in E_{i,t+1}$ and $f_{s+1}^{-1}(y) = g(x) \in E_{i,t+1}$.

Case 2: $w^0_{i,t}\uparrow$ and $w^0_{i,t+1}\downarrow$, for some $i \leq e$. However, although $\mathcal{R}_i$ does receive attention in this case, the construction leaves $E_{e,t} \subseteq B_{e,t+1}$. Hence $x \in E_{e,t+1}$, and $f_{t+1}(x) = f_t(x)$. Similarly, $f_{s+1}^{-1}(y) = f_s^{-1}(y) \in E_{e,t+1}$.

Case 3: $\mathrm{ht}(E_{i,t+1}) > l_{i+1,t}$ for some $i < e$. Again, the construction leaves $E_{e,t} \subseteq E_{e,t+1}$, so $x \in E_{e,t+1}$ and $f_{t+1}(x) = f_t(x)$ and $f_{s+1}^{-1}(y) = f_s^{-1}(y) \in E_{e,t+1}$.

Thus, for every $t > s$, we have both x and $f_t^{-1}(y)$ in $Ev_{i,t}$ for some $i \leq e$. Therefore, $f_{t+1}(x) \neq f_t(x)$ and $f_{t+1}^{-1}(y) \neq f_t^{-1}(y)$ each can occur only for the sake of an upwards embedding on behalf of some $\mathcal{R}_i$ with $i \leq e$. By Lemma 4.16, this can only occur finitely often. Hence the sequences $\langle f_s(x)\rangle_{s\in\omega}$ and $\langle f_s^{-1}(y)\rangle_{s\in\omega}$ both converge, making $f = \lim_s f_s$ a Δ_2^0-bijection from ω to ω. ∎

As usual, we lift the partial order $\prec$ from T to an order $\prec'$ on T', making f an isomorphism from T to T'.

Lemma 4.18 *The functions f_s satisfy Condition 3.1. Hence $\prec'$ is computable.*

Proof. We have already seen that $\text{range}(f_s) \subseteq \text{range}(f_{s+1})$. Take $a, b \in \text{range}(f_s)$. The only way for $f_{s+1}^{-1}(b) \neq f_s^{-1}(b)$ is if $f_s^{-1}(b)$ lies in some subtree $D_s[v_{e,s}^m]$ which is embedded upward via some g as part of Step 3(a) for some e at stage $s+1$. If $f_s^{-1}(a)$ is also embedded upward at stage $s+1$, then since g is a homomorphism of trees, we have:

$$f_s^{-1}(a) \prec f_s^{-1}(b) \iff g(f_s^{-1}(a)) \prec g(f_s^{-1}(b)) \iff f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b).$$

Otherwise, $f_s^{-1}(a) \notin D_s[v_{e,s}^m]$. In this case:

$$\begin{aligned} f_s^{-1}(a) \prec f_s^{-1}(b) &\iff f_s^{-1}(a) \prec v_{e,s}^m \\ &\iff f_{s+1}^{-1}(a) \prec v_{e,s}^m \\ &\iff f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b). \end{aligned}$$

The case $f_{s+1}^{-1}(b) = f_s^{-1}(b)$ is simpler, since this implies $f_s^{-1}(b) \notin D_s[v_{e,s}^m]$. Thus, if $f_s^{-1}(a) \prec f_s^{-1}(b)$, we know that $f_s^{-1}(a) = f_{s+1}^{-1}(a)$, so $f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b)$ and conversely as well. ∎

Thus $(T', \prec')$ is a computable tree, isomorphic to T via f, yet not computably isomorphic to T, since every requirement $\mathcal{R}_e$ is satisfied. Therefore, T is not computably categorical. This completes the proof of Proposition 4.13. ∎

4.5 Proof of the Theorem

Proof of Theorem 4.1. We need only confirm that the preceding propositions cover all possible cases. First, if T contains no extendible nodes, then Proposition 4.2 applies to the root of T, since $\mathrm{ht}(T) = \omega$. If T_{ext} is nonempty and infinite-branching, then Proposition 4.4 covers this case. If T_{ext} is nonempty and finite-branching, then we ask whether there exist side trees of height ω. If so, then Proposition 4.3 gives the result. Otherwise, every side tree has finite height. If every extendible node lies on infinitely many infinite paths, we apply Proposition 4.13. If there exists a node $x \in T_{\mathrm{ext}}$ which lies on only finitely many infinite paths through T, then by following those finitely many infinite paths upwards until they all diverge, we find a node $x_0 \in T_{\mathrm{ext}}$ which fits Proposition 4.5. ■

5 Trees of Height $> \omega$

Having established that no tree of height ω is computably categorical, we now prove the same result for trees of height $> \omega$. Recall that for trees T of height ω, we considered two cases in which T contains an infinite path, and used guessing procedures to find that (or those) paths. Now the existence of a node x_ω at level ω simplifies matters considerably, since the predecessors of x_ω form a computable infinite chain in T. (Technically, this chain is not a path, since it is not a maximal chain, but it is still perfectly useful for our purposes.) We will appeal again to Kruskal's Theorem to guarantee the existence of the necessary embeddings upwards along this chain, and use them to satisfy the requirements.

On the other hand, having $\mathrm{ht}(T) > \omega$ creates a different set of problems. Previously, with every node in T sitting at a finite level, we knew that each requirement would only require finitely many upwards embeddings in order to be satisfied. Now, it is possible that the node $\varphi_{e,s}(f_s(w_{e,s}))$ lies at an infinite level in T, in which case we might have to redefine $f_s(w_{e,s})$ to lie at higher levels in T' infinitely often, thereby injuring the lower-priority requirements infinitely many times. (Also, this would prevent $f_s(w_e)$ from converging, ruining the isomorphism from T to T'.) We avoid this difficulty by watching for predecessors of $\varphi_{e,s}(f_s(w_{e,s}))$ and using their preimages (under φ_e) as new witness nodes. Eventually we will find such a predecessor sitting at a finite level of T, and for this one we will only need finitely many upwards embeddings.

Of course, the preimage under φ_e of a predecessor of $\varphi_{e,s}(f_s(w_{e,s}))$ will not necessarily be a predecessor of $f_s(w_{e,s}))$ in T'. However, if indeed it is not a predecessor, then clearly φ_e was not an isomorphism. We can check effectively whether or not this is the case, and if it is not a predecessor, then $\mathcal{R}_e$ is automatically satisfied.

Theorem 5.1 *No computable tree of infinite height is computably categorical.*

Proof. Theorem 4.1 covers the case of a tree of height ω, so assume that T is a tree under $\prec$ with $\mathrm{ht}(T) > \omega$. Then T contains a node x_ω at level ω. The set S of predecessors of x_ω is a computable set, ordered in order type ω.

Each of our requirements $\mathcal{R}_e$ guarantees that φ_e is not an isomorphism from T' to T, just as before, but the exact statement is slightly different:

$\mathcal{R}_e$: φ_e bijective $\Longrightarrow$ either $(\exists x \in T')$ $[\text{level}_{T'}(x) \neq \text{level}_T(\varphi_e(x))]$ or $(\exists x, y \in T')[x \not\prec' y$ and $\varphi_e(x) \prec \varphi_e(y)]$.

If the second clause of the conclusion applies, or if there exists an s for which $\varphi_{e,s}$ is not one-to-one, we will say that $\mathcal{R}_e$ is *finitely satisfied*, since each of these facts will become evident at some finite stage of the construction. In contrast, we can never be sure at any finite stage whether or not we have permanently satisfied the first clause of the conclusion, or whether φ_e is total or onto.

Let $r = x_0 \prec x_1 \prec \cdots$ be all the predecessors of x_ω in T. We apply Corollary 2.4 to the collection of trees $\{S_i : i \in \omega\}$, where

$$S_i = T[x_i] - T[x_{i+1}].$$

(Thus the tree S_i has root x_i and contains those nodes lying above x_i but not above x_{i+1}.) Clearly S_i is computable. In the construction below, we will write $S_{i,s}$ for $S_i \cap D_s$. Let n be the number given by the corollary, such that every finite subtree of every S_i with $i \geq n$ embeds into some S_j with $j > i$.

Construction: f_0 is the identity map on $T_0 = \{x_i : i \leq n\} \cup \{x_\omega\}$, and the witness nodes $w_{e,0}$ and their traces $v_{e,0}$ are undefined for all e. For each s we define $T_{s+1} = T_s \cup \{s\}$.

At stage $s + 1$, we say that a requirement $\mathcal{R}_e$ $(e \leq s)$ is *finitely satisfied* if there exist distinct numbers $x \leq s$ and $y \leq s$ in the domain of $\varphi_{e,s}$ such that $\varphi_{e,s}(x) = \varphi_{e,s}(y)$, or such that $x \prec' y$ and $\varphi_{e,s}(x) \not\prec \varphi_{e,s}(y)$, or such that $x \not\prec' y$ and $\varphi_{e,s}(x) \prec \varphi_{e,s}(y)$. (In any of these three cases, we know right away that φ_e is not an isomorphism.) Search for the least $e \leq s + 1$ such that $\mathcal{R}_e$ is not yet finitely satisfied and one of the following cases holds:

1. $w_{e,s}$ is undefined; or

2. $w_{e,s}$ is defined and $\text{level}_{D_s}(w_{e,s}) \leq \text{level}_{D_s}(v_{e,s}) + 1$ and $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and
$$\text{level}_{D_s}(w_{e,s}) = \text{level}_{D_s}(\varphi_e(f_s(w_{e,s})));$$
or

3. $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and there exist nodes $w \in D_s$ and $w' \in \text{range}(f_s)$ such that $w \prec \varphi_{e,s}(f_s(w_{e,s}))$ and $\varphi_{e,s}(w')\downarrow= w$ and $\text{level}_{D_s}(w) = 1 + \text{level}_{D_s}(v_{e,s})$.

(Such an e must exist, because $w_{s+1,s}$ is undefined.) We say that $\mathcal{R}_e$ *receives attention* at this stage. For all $i < e$, let $w_{i,s+1} = w_{i,s}$ and $v_{i,s+1} = v_{i,s}$. For all $j > e$, let $w_{j,s+1}$ and $v_{j,s+1}$ be undefined. We proceed according to which of the three cases above held.

1. If $w_{e,s}$ is undefined,, we search for the $\prec$-least node w in $D_s[x_n]$ satisying the following conditions:

   - $w \prec x_\omega$;
   - $w \not\preceq w_{i,s}$ for every $i < e$ such that $\mathcal{R}_i$ is not yet finitely satisfied;
   - for every $x < e$, either $w \not\preceq x$ or $x_\omega \preceq x$; and
   - for every $y < e$, there exists $x \in D_s$ such that $f_s(x) = y$ and either $w \not\preceq x$ or $x_\omega \preceq x$.

   Let $v_{e,s+1} = w_{e,s+1} = w$. (If there is no such w, then leave $w_{e,s+1}$ undefined.) Let $D_{s+1} = D_s \cup T_{s+1}$.

2. If $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and

   $$\text{level}_{D_s}(w_{e,s}) = \text{level}_{D_s}(\varphi_e(f_s(w_{e,s}))),$$

   then search for the least $t > s$ such that there exists $x < t$ with $v_{e,s} \prec x \prec x_\omega$ for which $D_s[v_{e,s}]$ embeds into $T_t[x]$ via an embedding g such that $g(v_{e,s}) = x$ and $g(x_\omega) = x_\omega$. Since $v_{e,s} \prec x$, clearly

   $$\text{level}_{T_t}(g(w_{e,s})) > \text{level}_{D_s}(w_{e,s}).$$

   Fix this t, x, and g.

   We use the embedding g to satisfy (for the time being) the first clause of $\mathcal{R}_e$. Let $D_{s+1} = D_s \cup T_t$, $v_{e,s+1} = v_{e,s}$, and $w_{e,s+1} = g(w_{e,s})$, and for all $y \in D_s[v_{e,s}] - D_s[x_\omega]$, define $f_{s+1}(g(y)) = f_s(y)$. For those $y \in D_{s+1}[v_{e,s}] - D_s[x_\omega] - \text{range}(g)$, take $f_{s+1}(y)$ to be the least element of ω that is not yet in range(f_{s+1}) nor in range(f_s). Notice that although we have temporarily fulfilled $\mathcal{R}_e$, we do *not* state that $\mathcal{R}_e$ is satisfied, since possibly $\text{level}_T(\varphi_e(f_s(w_{e,s+1}))) > \text{level}_{T_t}(\varphi_e(f_s(w_{e,s+1})))$. We will continue to scrutinize $\mathcal{R}_e$ at subsequent stages.

3. Otherwise, we have the nodes $w \in D_s$ and $w' \in \text{range}(f_s)$ given in Case (3). Let $D_{s+1} = D_s \cup T_{s+1}$. Since $\mathcal{R}_e$ is not finitely satisfied, we must have $f_s^{-1}(w') \prec w_{e,s}$. Define $w_{e,s+1} = f_s^{-1}(w')$, and let $v_{e,s+1} = v_{e,s}$.

In all three cases, we then define $f_{s+1}(y) = f_s(y)$ for those $y \in D_s$ on which f_{s+1} is not yet defined. Also, for each $y \in D_{s+1} - D_s$ on which f_{s+1} is not yet defined, choose the least element of ω which is not yet in range(f_{s+1}) to be $f_{s+1}(y)$. For each e such that $w_{e,s+1}$ is defined, let $v_{e,s+1} = w_{e,s+1} \wedge x_\omega$. This completes the construction.

(This construction is most comparable to that of Proposition 4.5, in which we assumed that T contained an isolated infinite path. Here the path may not be isolated, but the node x_ω allows us to identify it anyway. The twist which we must add appears in Case (3) of the construction, in which we ensure that the $\lim_s w_{e,s}$ will lie at a finite level, or else that φ_e fails to preserve the relation $\prec'$.)

We first must prove that at each stage s at which Case (2) applies, we do eventually find an embedding, This requires a lemma guaranteeing our ability to embed trees upwards in $T[x_n]$.

Lemma 5.2 *For every $x_i \succeq x_n$ and every t, there is an embedding g of the tree $D_s[x_i]$ into $T[x_{i+1}]$ with $g(x_\omega) = x_\omega$.*

Proof. By the choice of n and Corollary 2.4, we know that every finite subtree of every S_j with $j \geq n$ embeds into some S_k with $k > j$. By induction, then, every finite subtree of every such S_j embeds into infinitely many S_k with $k > j$. We may also assume that in each such embedding, x_j is mapped to x_k. Since there are only finitely many side trees $S_{j_0}, \ldots S_{j_n}$ which intersect the finite tree D_s, we can embed $S_{j_0} \cap D_s$ into some S_{k_0}, then embed $S_{j_1} \cap D_s$ into some S_{k_1} with $k_1 > k_0$, and so on. The union of these embeddings with the identity map on $D_s[x_\omega]$ respects the order $\prec$ (since each x_{j_i} is mapped to some other predecessor x_k of x_ω), and is the desired embedding g. ∎

Having thus guaranteed that every stage will eventually terminate, we turn to the question of convergence.

Lemma 5.3 *For every e, either $\mathcal{R}_e$ is finitely satisfied at some stage s, or else:*

- *the sequence $\langle v_{e,s} \rangle_{s\in\omega}$ converges to a limit $v_e \prec x_\omega$, and $v_i \prec v_e$ for every $i < e$ such that $\mathcal{R}_i$ is not finitely satisfied;*
- *the sequence $\langle w_{e,s} \rangle_{s\in\omega}$ converges to a limit w_e with $v_e \prec w_e$; and*
- *the sequence $\langle f_s(w_e) \rangle_{s\in\omega}$ converges to a limit $f(w_e)$.*

Proof. We proceed by induction on e. Suppose that $\mathcal{R}_e$ is never finitely satisfied, and let s_0 be a stage so large that for all $s \geq s_0$ and all $i < e$, the hypotheses of the theorem hold. (In particular, assume that $v_{i,s} = v_i$, $w_{i,s} = w_i$, and $f_s(w_i) = f(w_i)$ for all $s \geq s_0$.) Since x_ω has infinitely many predecessors, there must exist a stage $s_1 > s_0$ at which v_{e,s_1} and w_{e,s_1} are defined. Moreover, they will never again become undefined, since no $\mathcal{R}_i$ with $i < e$ will ever again receive attention. Indeed, $v_{e,s} = v_{e,s_1}$ for every $s \geq s_1$, since Cases (2) and (3) both define $v_{e,s+1} = v_{e,s}$, so we may write $v_e = v_{e,s_1}$. We may also assume that $\varphi_{e,s_1}(f_{s_1}(w_{e,s_1})$ converges, since if there is no such s_1, then $\mathcal{R}_e$ will never again receive attention and the theorem will be satisfied.)

Notice that $v_{e,s+1}$ becomes undefined at any stage at which $v_{i,s+1} \neq v_{i,s}$ for some $i < e$. Now if s is the last stage at which $v_{e,s}$ is undefined, then $v_{e,s+1} = w_{e,s+1} \not\preceq w_{i,s}$ for every $i < e$. Hence $v_{e,s+1} \not\preceq v_{i,s+1}$ for any such i. However, $v_{e,s+1} \prec x_\omega$, forcing $v_{i,s+1} \prec v_{e,s+1}$ for each such i. The construction never allows $v_{e,t+1}\!\downarrow \neq v_{e,t}\!\downarrow$, so we must have $v_i \prec v_e$, as the theorem demands.

Notice that at any stage $s+1 > s_1$ at which $\mathcal{R}_e$ satisfies Case (2), it will receive attention and the resulting embedding will guarantee $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. Hence $\varphi_e(f_{s+1}(w_{e,s+1})) = \varphi_e(f_s(w_{e,s}))$ for all such s. Also, at any stage at which $\mathcal{R}_e$ satisfies Case (3), either $\mathcal{R}_e$ will be finitely satisfied or

$$\varphi_e(f_{s+1}(w_{e,s+1})) = \varphi_e(w')\!\downarrow = w \prec \varphi_e(f_s(w_{e,s}))$$

where w and w' are as given in Case (3).

Thus, $\varphi_e(f_{s+1}(w_{e,s+1})) \preceq \varphi_e(f_s(w_{e,s}))$ in T for every $s+1 > s_1$. Since T is a tree, the infinite nonincreasing sequence $\langle \varphi_e(f_s(w_{e,s})) \rangle_{s \geq s_1}$ must converge to a limit, so there exists a stage s_2 after which this sequence is constant. The construction then makes it clear that Case (3) will never apply after stage s_2. This implies that no more nodes w are found which satisfy the hypotheses of Case (3). Therefore, either there is no predecessor w of $\varphi_e(f_{s_2}(w_{e,s_2}))$ in T with $\mathrm{level}_T(w) = 1 + \mathrm{level}_T(v_e)$, in which case $\varphi_e(f_{s_2}(w_{e,s_2}))$ must lie at a finite level of T, or else any such predecessor does not lie in the range of φ_e, in which case φ_e is not bijective. (Recall that we are assuming here that $\mathcal{R}_e$ is not finitely satisfied.) In the latter case, neither Case (2) nor Case (3) will ever again apply to $\mathcal{R}_e$, so $w_{e,s} = w_{e,s_2}$ for every $s \geq s_2$.

In the former case, where $\varphi_e(f_{s_2}(w_{e,s_2}))$ lies at a finite level of T, we know that Case (3) will never again apply, so Case (2) will only apply finitely many more times. (Each time Case (2) applies, we have $\mathrm{level}_{D_{s+1}}(w_{e,s+1}) >$

$\text{level}_{D_s}(w_{e,s})$, but Case (2) is impossible when $\text{level}_{D_s}(w_{e,s}) > 1 + \text{level}_T(v_e)$.) Hence we will reach a stage after which $\mathcal{R}_e$ never again receives attention, and thus $\langle w_{e,s}\rangle_{s\geq s_2}$ converges to a limit w_e. We also note that $v_e \preceq w_{e,s} \preceq w_{e,s+1}$ for every $s \geq s_2$, so that $v_e \preceq w_e$, as the theorem claims. Furthermore, we already saw that $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$ for every $s \geq s_1$, so clearly the sequence $\langle f_s(w_e)\rangle_{s\geq s_2}$ converges. This completes the proof of the lemma. ■

Lemma 5.4 *The functions f_s converge to a limit f which is bijective.*

Proof. Every x lies in D_s for all $s > x$, so $f_s(x)$ will be defined for all sufficiently large s. Also, each time a fresh element y was needed for the range of f_s, we chose the least y available. Once $f_s^{-1}(y)$ is defined, y will remain in range(f_s) at all subsequent stages, since the embeddings in Case (2) always preserve the range.

In Case (1) we ensured that $v_e \prec x$ only if $e \leq x$ or $x_\omega \preceq x$. In the latter case, $f_s(x)$ will never be redefined. In the former case, we may have $f_{s+1}(x) \neq f_s(x)$ at stages $s+1$ at which Case (2) applies to a requirement $\mathcal{R}_e$ with $e \leq x$. However, Lemma 5.3 shows that there are only finitely many such stages. Similarly, Case (1) and this lemma ensure that $f_{s+1}^{-1}(y)$ will only be redefined finitely often.

Thus $f = \lim_s f_s$ has domain and range ω. Injectivity follows from the injectivity of each f_s. ■

Lemma 5.5 *For every e, either $\mathcal{R}_e$ is finitely satisfied at some stage s, or $\varphi_e(f(w_e))\uparrow$ or $\text{level}_T(\varphi_e(f(w_e))) \neq \text{level}_T(w_e)$. (Since $\text{level}_T(w_e) = \text{level}_{T'}(f(w_e))$, this guarantees that $\mathcal{R}_e$ is satisfied.)*

Proof. Suppose $\varphi_e(f(w_e))$ converges. (We also continue to assume that $\mathcal{R}_e$ is not finitely satisfied, so that φ_e is one-to-one and maps $\prec'$ to $\prec$.) Let t_0 be so large that $\mathcal{R}_e$ never receives attention after stage t_0.

Let $l_e = \text{level}_T(v_e)$. If $\text{level}_T(w_e) > l_e+1$, then let u be the predecessor of w_e at level l_e+1 in T. Now since φ_e is assumed to be total, we know that there is some $w \in T$ such that $\varphi_e(f(u))\downarrow = w$. If $\text{level}_T(w) \neq \text{level}_T(u)$, then $\mathcal{R}_e$ holds, since $\text{level}_T(u) = \text{level}_{T'}(f(u))$. Otherwise, $\text{level}_T(w) = \text{level}_T(u) = l_e+1$, and Case (3) must apply to w at some stage, with $w' = f(u)$. (Notice that $w \prec \varphi_e(f(w_e))$ since by assumption φ_e maps the ordering $\prec'$ on T' to $\prec$ on T.) This contradicts our choice of t_0, completing the proof of the lemma. ■

Lemma 5.6 *The maps f_s satisfy Condition 3.1. Thus $\prec'$ is computable.*

Proof. The construction ensures that $D_s \subseteq D_{s+1}$ for all s. For every $x \in D_s - D_s[x_n]$, we have $f_s(x) = f_{s+1}(x)$. Therefore, Condition 3.1 clearly holds if either $f_s^{-1}(a)$ or $f_s^{-1}(b)$ is not in $T[x_n]$. So take $x, y \in D_s[x_n]$, with $a = f_s(x)$, $b = f_s(y)$, and let $x' = f_{s+1}^{-1}(a)$ and $y' = f_{s+1}^{-1}(b)$. We have four cases, depending on whether or not $x = x'$ and $y = y'$.

The first case, where $x = x'$ and $y = y'$, is trivial. Also, if $x \neq x'$ and $y \neq y'$, then x and y must both lie in $D_s[v_{e,s}] - D_s[x_\omega]$, where $\mathcal{R}_e$ receives attention in Case (2) at stage $s + 1$. In this case, we find an embedding g into some $T_t[x]$ with $x_n \prec x \prec x_\omega$. Thus

$$x \preceq y \iff g(x) \preceq g(y) \iff x' \preceq y'$$

since $g(x) = x'$ and $g(y) = y'$. Thus Condition 3.1 is satisfied in these two cases.

Suppose $x \neq x'$ and $y = y'$. Then $x \in D_s[v_{e,s}]$ as above. If $y \in D_s[v_{e,s}] - D_s[x_\omega]$, then $x' = g(x) \prec g(y) = y'$. If not, then either $y \prec v_{e,s}$ (in which case $y \prec x$ and $y \prec g(x) = x'$, since range$(g) \subset T[v_{e,s}]$) or $y \perp x_{m_t,t}$ (in which case $y \perp x$ and $y \perp g(x) = x'$, again because range$(g) \subset T[x_{m_t,t}]$), or $x_\omega \preceq y$. In this last case the consition is satisfied, since

$$x \prec y \iff x \prec x_\omega \iff g(x) \prec g(x_\omega) = x_\omega \iff x' \prec y = y'.$$

The preceding paragraph shows that in the third case, not only

$$x \prec y \iff x' \prec y'$$

but also

$$x \perp y \iff x' \perp y'.$$

Hence by symmetry, the fourth case, with $x = x'$ and $y \neq y'$, is also satisfied. ∎

Thus $(T', \prec')$ is a computable tree, isomorphic to T via the Δ_2 function f, and T' satisfies every requirement $\mathcal{R}_e$. Hence T is not computably categorical, proving Theorem 5.1. ∎

Intuitively it can be difficult to see where the action occurs in the proof of Theorem 5.1, particularly in Lemma 5.3, which is the heart of the proof.

Essentially the argument for convergence of $w_{e,s}$ comes down to the fact that each time $\mathcal{R}_e$ receives attention in Case (3), we generate another element in a descending sequence in T, and the definition of tree guarantees that this sequence must be finite. (In fact, we can say more: once the actual predecessors of $\varphi_e(f_s(w_{e,s}))$ at all levels $\leq l_e$ in T have appeared, and once φ_e^{-1} has converged on all of them, $\mathcal{R}_e$ will never again receive attention in Case (3).) Thereafter, any more upwards embeddings of $w_{e,s}$ would cause Case (3) to apply again, which we know cannot occur, so Case (2) must never again apply either. We conclude that $\mathcal{R}_e$ must be satisfied, becasue if φ_e really were an isomorphism, either (2) or (3) would apply at some subsequent stage. In the architecture of this proof, therefore, it is the well-ordering of the predecessors of each element of T which drives the result home. (One could write a similar proof for Proposition 4.5, but the one given was more straightforward and offered a better insight into the reasons why the Proposition held, perhaps at some cost in elegance.)

6 Effectively Infinite Dimension

Recall that the *computable dimension* of a computable structure is the number of computable isomorphism classes of computable copies of that structure. Theorem 4.1 shows that every computable tree of infinite height has computable dimension at least 2. A theorem of Goncharov from [7] states that if $\mathcal{A}$ is a computable structure which has two computable copies that are Δ^0_2-isomorphic but not computably isomorphic, then $\mathcal{A}$ has computable dimension ω. The isomorphisms which we constructed in our proofs are all Δ^0_2, so in fact the these trees all have computable dimension ω.

It is possible to strengthen this statement even further, by avoiding countably many isomorphism classes simultaneously. That is, given a uniformly presented list $\{T_i : i \in \omega\}$ of computable copies of T, one can construct another computable copy T' of T which is not computably isomorphic to any T_i. Thus, the computable dimension of T is *effectively infinite*; one might even call it *effectively uncountable*, for, although there are only countably many computable isomorphism classes, there is no effective enumeration of them.

Proposition 6.1 *Let T be a tree of infinite height, and let $\{T_i\}$ be a computable (finite or infinite) sequence of computable trees isomorphic to T. Then there exists a computable tree T' isomorphic to T, such that no T_i is computably isomorphic to T'.*

(The fact that the set $\{T_i\}$ is allowed to be infinite gives rise to the term *effectively uncountable*. If we could only prove this proposition for finite sets $\{T_i\}$, then we would only say that the dimension was effectively infinite.)

Proof. For trees of height ω, the construction proceeds exactly as in Propositions 3.2, 4.5, and 4.13, according to which of these propositions applies to T. We let T_0 play the role of T as a template for T', constructing T' to be isomorphic to T_0 via a Δ^0_2-isomorphism $f = \lim_s f_s$, with $D_s = \text{domain}(f_s) \subset T_0$. The only difference is that instead of checking whether $\text{level}_{D_s}(w_{e,s}) = \text{level}_{D_s}(\varphi_e(f_s(w_{e,s})))$ at each stage s, we have witness elements $w_{e,i,s} \in T_0$ to ensure that φ_e is not an isomorphism from T' to T_i, and we check at each stage s whether

$$\text{level}_{D_s}(w_{e,i,s}) = \text{level}_{T_{i,s}}(\varphi_e(f_s(w_{e,i,s}))).$$

If it is, then we proceed to embed $w_{e,i,s}$ further upwards in T_0, which pushes $f_s(w_{e,i,s})$ further up in T'. Eventually $\varphi_e(f_s(w_{e,i,s}))$ reaches its final level in T_i, and one last upwards embedding guarantees that φ_e is not an isomorphism from T' to T_i.

The case of a tree of height $> \omega$ requires similar modifications. At stage $s+1$ of that construction, we say that a requirement $\mathcal{R}_{e,i}$ is *finitely satisfied* if there exist distinct numbers $x \leq s$ and $y \leq s$ in the domain of $\varphi_{e,s}$ such that $\varphi_{e,s}(x) = \varphi_{e,s}(y)$, or such that $x \prec' y$ and $\varphi_{e,s}(x) \not\prec_i \varphi_{e,s}(y)$, or such that $x \not\prec' y$ and $\varphi_{e,s}(x) \prec_i \varphi_{e,s}(y)$. (Here $\prec_i$ denotes the partial order on the tree T_i, so each of these conditions ensures that φ_e is not an isomorphism from T' to T_i.) Then we search for the least pair $\langle e, i\rangle \leq s+1$ such that $\mathcal{R}_{e,i}$ is not yet finitely satisfied and one of the following cases holds:

1. $w_{e,i,s}$ is undefined; or

2. $\text{level}_{D_s}(w_{e,i,s}) \leq \text{level}_{D_s}(v_{e,i,s}) + 1$ and $\varphi_{e,s}(f_s(w_{e,i,s}))\downarrow$ and

$$\text{level}_{D_s}(w_{e,i,s}) = \text{level}_{T_{i,s}}(\varphi_e(f_s(w_{e,i,s})));$$

   or

3. $\varphi_{e,s}(f_s(w_{e,i,s}))\downarrow$ and there exist nodes $w \in T_{i,s}$ and $w' \in \text{range}(f_s)$ such that $w \prec_i \varphi_{e,s}(f_s(w_{e,i,s}))$ and $\varphi_{e,s}(w')\downarrow = w$ and $\text{level}_{T_{i,s}}(w) = 1 + \text{level}_{D_s}(v_{e,i,s})$.

Corresponding adjustments through the rest of the proof guarantee that each $\mathcal{R}_{e,i}$ is satisfied, so that φ_e is not an isomorphism from T' to T_i. ■

References

[1] C. J. Ash; Categoricity in Hyperarithmetical Degrees, *Annals of Pure and Applied Logic* **34** (1987), 1-14.

[2] J. N. Crossley, A. B. Manaster, & M. F. Moses; Recursive Categoricity and Recursive Stability, *Annals of Pure and Applied Logic* **31** (1986), 191-204.

[3] R. Downey; On Presentations of Algebraic Structures, in *Complexity, Logic, and Recursion Theory*, ed. A. Sorbi (New York: Dekker, 1997), 157-205.

[4] Y. L. Ershov & S. S. Goncharov; *Constructive Models* (New York: Kluwer Academic/Plenum Press, 2000).

[5] S. S. Goncharov; Autostability and Computable Families of Constructivizations, *Algebra and Logic* **14** (1975), 647-680 (Russian), 392-409 (English translation).

[6] S. S. Goncharov; Autostable Models and Algorithmic Dimensions, *Handbook of Recursive Mathematics*, vol. 1 (Amsterdam: Elsevier, 1998), 261-287.

[7] S. S. Goncharov; Nonequivalent Constructivizations, *Proc. Math. Inst. Sib. Branch Acad. Sci.* (Novosibirsk: Nauka, 1982).

[8] S. S. Goncharov; The Problem of the Number of Non-self-equivalent Constructivizations, *Algebra i Logika* **19** (1980), 621-639.

[9] S. S. Goncharov & V. D. Dzgoev; Autostability of Models, *Algebra and Logic* **19** (1980), 45-58 (Russian), 28-37 (English translation).

[10] B. Khoussainov & R. A. Shore; Computable Isomorphisms, Degree Spectra of Relations, and Scott Families, *Annals of Pure and Applied Logic* **93** (1998), 153-193.

[11] B. Khoussainov & R. A. Shore; Effective Model Theory: The Number of Models and their Complexity, *Models And Computability: Invited Papers from Logic Colloquium '97*, ed. S. B. Cooper & J. K. Truss, LMSLNS **259** (Cambridge: Cambridge University Press, 1999), 193-240.

[12] J.B. Kruskal; Well Quasi-Ordering, the Tree Theorem, and Vázsonyi's Conjecture, *Transactions of the American Mathematical Society* **95** (1960), 210-225.

[13] S. Lempp, C. McCoy, R. G. Miller, & R. Solomon; Computable Categoricity of Trees of Finite Height, to appear.

[14] C. St. J. A. Nash-Williams; On Well-quasi-ordering Finite Trees, *Proc. Cambridge Phil. Soc.* **59** (1963), 833-835.

[15] J. B. Remmel; Recursively Categorical Linear Orderings, *Proceedings of the American Mathematical Society* **83** (1981), 387-391.

[16] J. B. Remmel; Recursive Isomorphism Types of Recursive Boolean Algebras, *Journal of Symbolic Logic* **46** (1981), 572-594.

[17] S. G. Simpson; Nonprovability of Certain Combinatorial Properties of Finite Trees, *Harvey Friedman's Research on the Foundations of Mathematics*, ed. L. A. Harrington, M. D. Morley, A. Scedrov & S. G. Simpson (Amsterdam: North-Holland, 1985), 87-117.

[18] R. I. Soare; *Recursively Enumerable Sets and Degrees* (New York: Springer-Verlag, 1987).