# Low$_5$ Boolean Subalgebras and Computable Copies

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**Abstract**

It is known that the spectrum of a Boolean algebra cannot contain a low$_4$ degree unless it also contains the degree $\mathbf{0}$; it remains open whether the same holds for low$_5$ degrees. We address the question differently, by considering Boolean subalgebras of the computable atomless Boolean algebra $\mathcal{B}$. For such subalgebras $\mathcal{A}$, we show that it is possible for the spectrum of the unary relation $\mathcal{A}$ on $\mathcal{B}$ to contain a low$_5$ degree without containing $\mathbf{0}$.

## 1 Introduction

The question of coding low$_n$ sets into Boolean algebras is well-known and much studied. The initial result of Downey and Jockusch in [2] showed that every low Boolean algebra has a computable copy, and asked whether the same held more generally for low$_n$ Boolean algebras. Thurber gave an affirmative answer for the $n = 2$ case in [12], and Knight and Stob extended it to the cases $n = 3$ and $n = 4$ in [9]. Currently that remains the state of our knowledge, despite the efforts of many researchers, and recent work by Harris and Montalbán in [7], building on [8], has suggested that the $n = 5$

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case is not just more difficult, but actually substantially different from the previous ones.

The question can readily be adapted to the concept of the spectrum of a Boolean algebra. Recall that for any countable structure $\mathcal{S}$, the *spectrum of* $\mathcal{S}$ is the set of all Turing degrees of structures with domain $\omega$ which are isomorphic to $\mathcal{S}$:

$$\mathrm{Spec}(\mathcal{S}) = \{\deg(\mathfrak{M}) : \mathfrak{M} \cong \mathcal{S} \ \&\ \mathrm{dom}(\mathfrak{M}) = \omega\}.$$

So the existing results say that if the spectrum of a Boolean algebra contains any low$_4$ Turing degree, then it must also contain the degree $\mathbf{0}$.

In [6], Harizanov and the author drew connections from the spectra of certain types of structures to the spectra of relations on computable universal models of the same theory. For example, they showed that every nontrivial countable graph $\mathcal{S}$ can be embedded into the computable random graph $\mathcal{G}$ so that the spectrum of $\mathcal{S}$ (as a structure, i.e. as defined above) is precisely the spectrum of the image $R$ of $\mathcal{S}$ as a relation on $\mathcal{G}$. By definition, this latter spectrum is:

$$\mathrm{DgSp}_{\mathcal{G}}(R) = \{\deg(Q) : (\mathcal{G}, R) \cong (\mathcal{H}, Q) \text{ for some computable } \mathcal{H} \cong \mathcal{G}\}.$$

Of course, a unary relation $R$ on $\mathcal{G}$ forms a graph in its own right, under the restriction of the edge relation from $\mathcal{G}$, with degree computable from the degree of $R$. Indeed, for every unary relation $R$ on $\mathcal{G}$, there exists a countable graph $\mathcal{S}$ with $\mathrm{Spec}(\mathcal{S}) = \mathrm{DgSp}_{\mathcal{G}}(R)$, although cases exist in which $\mathcal{S}$ cannot be taken simply to be the restriction to $R$ of the edge relation from $\mathcal{G}$.

Harizanov and the author also studied these questions for linear orders. In that theory, the natural universal structure is the computable dense linear order $\mathcal{L}$ (with or without end points), and once again, the spectrum of an arbitrary countable linear order $\mathcal{S}$ can always be realized as the spectrum of its image within $\mathcal{L}$ under some embedding. In their terminology, $\mathcal{L}$ is *spectrally universal* for the class of all countable linear orders, just as $\mathcal{G}$ was spectrally universal for the class of all countable graphs. They asked whether the converse holds for linear orders as it did for graphs: when $R$ is a unary relation on $\mathcal{L}$, must $\mathrm{DgSp}_{\mathcal{L}}(R)$ be the spectrum of some linear order? Recognizing the close connections between linear orders and Boolean algebras, they also asked whether the computable atomless Boolean algebra $\mathcal{B}$ is spectrally universal for the class of countable Boolean algebras. In [3], the answer to the first question was shown to be negative, but in [1], Csima, Harizanov, Montalbán, and the author demonstrated that $\mathcal{B}$ is indeed spectrally universal for countable Boolean algebras.

The approach used in this article was inspired by work in [3], by Frolov, Harizanov, Kalimullin, Kudinov, and the author, on spectra of linear orders. It remains unknown whether there exists a linear order whose spectrum consists of precisely the non-low degrees, but in Theorem 4.8 of that work, it is shown that there is a relation $R$ on the computable dense linear order $\mathcal{L}$ with exactly that spectrum (as a relation on $\mathcal{L}$). The construction of that relation $R$ involved the notion of a *doubly dense* interval in $\mathcal{L}$: an interval in which both $R$ and its complement are dense. With one jump, an $R'$-oracle could distinguish such intervals from other intervals in which all elements of $\mathcal{L}$ lie in $R$. This made possible more coding than could have been accomplished by the same methods in an actual linear order, allowing $R$ to contain information so that its jump could enumerate a family of finite sets which is not $\mathbf{0}'$-enumerable. Moreover, any set $C$ whose jump can enumerate the same family is capable of computing a relation $S$ on $\mathcal{L}$ with $(\mathcal{L}, S) \cong (\mathcal{L}, R)$; this gave the desired result for the spectrum of the relation $R$. Of course, any unary relation on $\mathcal{L}$ is a linear order in its own right, under the restriction of the order $\prec$ of $\mathcal{L}$, but the spectrum of the linear order $(R, \prec)$ (as a structure) turned out not to exclude all low degrees. Without the ambient structure $\mathcal{L}$ present, the intervals in $R$ which used to be doubly dense could no longer be distinguished from intervals containing only elements of $R$, and the coding disintegrated.

In the next section, we use a similar approach to Boolean algebras. The natural analogue of the computable dense linear order is the computable atomless Boolean algebra $\mathcal{B}$. It was already known from work by Harizanov and the author that a unary relation on $\mathcal{B}$ could contain a low degree without containing the degree $\mathbf{0}$; see [6, Corollary 4.2]. There a follow-up question, attributed to Montalbán, asked whether such a relation could be a Boolean algebra in its own right (specifically, a Boolean subalgebra of $\mathcal{B}$). We do not have an answer to that precise question, but our main theorem in this paper, Theorem 2.4, shows that the answer to the same question about $\text{low}_5$ degrees is positive. Of course, $\text{low}_5$ is precisely the point at which the question for Boolean algebras as structures becomes open. So our result does not solve the question posed in [6] of whether the spectrum of a Boolean subalgebra of $\mathcal{B}$ (as a relation) is always the spectrum of some Boolean algebra (as a structure). However, we do expect our work to shed further light on the widespread investigations into the possibility of a $\text{low}_5$ Boolean algebra having no computable copy.

For details and definitions from computable model theory, [5] is a useful source, while more basic background on computability theory is available in many books, including [11].

# 2 Boolean Subalgebras of $\mathcal{B}$

The concept of double density, applied in [3] to linear orders, requires some refinement for Boolean algebras. (Thanks are due to the anonymous referee who pointed this out.) Our first definitions here use the language of a Boolean algebra with a subalgebra: functions $\vee$ and $\wedge$, constants 0 and 1, and a unary relation symbol $\mathcal{A}$ denoting the subalgebra. A *Boolean structure* $(\mathcal{B}, \mathcal{A})$ will be a model of the axioms for a Boolean algebra with Boolean subalgebra: the Boolean algebra axioms for the structure $\mathcal{B}$ itself, and similar axioms saying that $\mathcal{A}$ forms a Boolean algebra containing the same constants 0 and 1. For models of these axioms, $(\mathcal{B}', \mathcal{A}')$ is a substructure of $(\mathcal{B}, \mathcal{A})$ iff $\mathcal{A}' = \mathcal{A} \cap \mathcal{B}'$ (and the functions and constants in $\mathcal{B}$ restrict to those in $\mathcal{B}'$, of course). For brevity, we call such a $(\mathcal{B}', \mathcal{A}')$ a *Boolean substructure* of $(\mathcal{B}, \mathcal{A})$, and $(\mathcal{B}, \mathcal{A})$ a *Boolean extension* of $(\mathcal{B}', \mathcal{A}')$. Since the axiomatization is finite, we may check effectively whether a given finite structure in this language is a Boolean structure.

**Definition 2.1** Let $\mathcal{B}$ be the countable atomless Boolean algebra, with a Boolean subalgebra $\mathcal{A}$, and fix a nonzero element $x \in \mathcal{A}$. Write $\mathcal{B}_x = \{y \in \mathcal{B} : y \subseteq x\}$, and let $\mathcal{A}_x = \mathcal{B}_x \cap \mathcal{A}$. Then $(\mathcal{B}_x, \mathcal{A}_x)$ forms a Boolean structure in its own right, with greatest element $x$. We say that $\mathcal{A}$ is *doubly dense* within $x$ if, for every finite Boolean substructure $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0) \subseteq (\mathcal{B}_x, \mathcal{A}_x)$, and for every finite Boolean structure $(\mathcal{D}_0, \mathcal{C}_0)$ with Boolean substructure $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0)$, there exists a Boolean substructure $(\mathcal{D}_0', \mathcal{A} \cap \mathcal{D}_0') \subseteq (\mathcal{B}_x, \mathcal{A}_x)$ with an isomorphism $f : (\mathcal{D}_0, \mathcal{C}_0) \to (\mathcal{D}_0', \mathcal{A} \cap \mathcal{D}_0')$ which restricts to the identity on $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0)$.

Since we can check effectively whether any finite $(\mathcal{D}_0, \mathcal{C}_0)$ really is a Boolean extension of a given Boolean substructure $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0)$ of $(\mathcal{B}_x, \mathcal{A}_x)$, the property of being doubly dense is $\Pi_2^{\mathcal{A}}$.

To summarize this definition: every finite Boolean substructure of $(\mathcal{B}_x, \mathcal{A}_x)$ can be extended in every possible finite way within $(\mathcal{B}_x, \mathcal{A}_x)$. Moreover, the extension is an extension of Boolean structures, i.e. respecting $\mathcal{A}_x$. This is exactly the property we need for Lemma 2.3 below, which essentially says that the property of double density within an interval characterizes $\mathcal{A}$ in that interval up to isomorphism.

We will need a fixed computable copy of the atomless Boolean algebra. For this purpose, we fix a computable presentation $\mathcal{B}$ of **Intalg**($\mathbb{Q}$), the Boolean algebra of finite unions of left-closed, right-open intervals within the standard linear order $\prec$ on the rationals $\mathbb{Q}$. (Such an interval is allowed to be of the form $(-\infty, q)$ or $[q, +\infty)$.) Notice that since the domain of $\mathcal{B}$ officially is $\omega$, we will use the relation $<$ to denote the usual ordering of its elements, viewed as elements of $\omega$. The less-than relation in the Boolean algebra $\mathcal{B}$ is denoted by $\subseteq$ (and its strict version by $\subsetneq$), since we are thinking of elements of $\mathcal{B}$ as denoting intervals in $\mathbb{Q}$. As mentioned above, we also use $\prec$ to denote the computable linear ordering of $\mathbb{Q}$. Thus, for instance, $[q, r) \subseteq [q', r')$ iff $q' \preceq q$ and $r \preceq r'$. From here on, we switch back to the language of Boolean algebras, with no additional unary relation symbol for a Boolean subalgebra. Of course, we are still studying Boolean subalgebras! However, $\mathcal{B}$ is simply a computable atomless Boolean algebra. This means that the operations of meet $\cap$ and join $\cup$ are computable, and hence so are complementation and the containment relations $\subseteq$ and $\subsetneq$, but our subalgebras, not being in the language, are not required to be computable, and in general will not be.

The next lemma will be used in Theorem 2.4 every time we say "make $\mathcal{A}_s$ doubly dense below $x$ in $\mathcal{B}$." This really means to extend $\mathcal{A}_s$ (which will have finite intersection with $\mathcal{B}_x$) as in Lemma 2.2, and then to take $\mathcal{A}_{s+1}$ to be the Boolean subalgebra of $\mathcal{B}$ generated by $\mathcal{A}_s$ and the $\mathcal{A}$ from the lemma. If a finite set $S$ of elements of $\mathcal{B}$ has already been excluded from $\mathcal{A}$ by stage $s$, the lemma allows us to maintain this exclusion, by letting $S_0$ be the set of elements of $\mathcal{B}_x$ whose inclusion in $\mathcal{A}$ would force some element of $S$ to enter $\mathcal{A}$.

**Lemma 2.2** *For every $x \in \mathcal{B}$, every finite Boolean subalgebra $\mathcal{A}_0$ of $\mathcal{B}_x = \{y \in \mathcal{B} : y \subseteq x\}$, and every finite subset $S_0 \subseteq (\mathcal{B}_x - \mathcal{A}_0)$, there exists a Boolean subalgebra $\mathcal{A}$ of $\mathcal{B}_x$, extending $\mathcal{A}_0$ and disjoint from $S_0$, which is doubly dense below $x$ in $\mathcal{B}_x$. Moreover, $\mathcal{A}$ may be computed uniformly in $x$ and in a strong index for $\mathcal{A}_0$.*

*Proof.* This is a simple matter of writing down the requirements from Definition 2.1, with one requirement for each finite Boolean subalgebra of the $\mathcal{A}$

we build and for each finite Boolean extension $(\mathcal{D}_0, \mathcal{C}_0)$ thereof. The initial Boolean substructure $(\mathcal{B}_0, \mathcal{A}_0)$ uses the $\mathcal{A}_0$ given, with $\mathcal{B}_0$ generated by $\mathcal{A}_0$ and the elements of $S_0$, so that $S_0 \subseteq \mathcal{B}_0 - \mathcal{A}_0$. Each requirement says that some copy of a finite Boolean structure $(\mathcal{D}_0, \mathcal{C}_0)$ must appear as a Boolean substructure of $(\mathcal{B}_x, \mathcal{A})$, extending the $(\mathcal{B}_x, \mathcal{A}_s)$ we have built so far, and there are always infinitely many elements available in $\mathcal{B}_x$ of each type over $(\mathcal{B}_s, \mathcal{A}_s)$. At stage $s+1$ we make sure that only elements $> s$ are adjoined to $\mathcal{A}_s$. Thus our $\mathcal{A}$ will be not only computably enumerable, but computable.

Later on, when building subalgebras which may turn out to be doubly dense, we will refer to the above requirements informally as the *double-density requirements.* ■

**Lemma 2.3** *For any $x, x' \in \mathcal{B}$ and Boolean subalgebras $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{B}$, suppose that $\mathcal{A}$ is doubly dense within $x$ and $\mathcal{A}'$ doubly dense within $x'$. As above, consider $(\mathcal{B}_x, \mathcal{A}_x)$ as a Boolean structure in its own right, with Boolean subalgebra $\mathcal{A}_x = \mathcal{B}_x \cap \mathcal{A}$, and likewise $(\mathcal{B}_{x'}, \mathcal{A}'_{x'})$ with $\mathcal{A}'_{x'} = \mathcal{B}_{x'} \cap \mathcal{A}'$. Then there is an isomorphism $f$ of Boolean structures from $(\mathcal{B}_x, \mathcal{A}_x)$ onto $(\mathcal{B}_{x'}, \mathcal{A}'_{x'})$, computable in the join of deg$(\mathcal{A})$ and deg$(\mathcal{A}')$.*

*Proof.* According to Definition 2.1, $x \in \mathcal{A}$ and $x' \in \mathcal{A}'$, so $(\mathcal{B}_x, \mathcal{A}_x)$ and $(\mathcal{B}_{x'}, \mathcal{A}'_{x'})$ are indeed Boolean structures. We build $f$ by a back-and-forth construction, starting with $f_0(0) = 0$ and $f_0(x) = x'$. Suppose that at stage $s$ we have built a finite partial isomorphism $f_s$, respecting $\mathcal{A}$ and $\mathcal{A}'$ as desired, whose domain $\mathcal{B}_s$ is a finite Boolean subalgebra of $\mathcal{B}_x$. Then $(\mathcal{B}_s, \mathcal{A} \cap \mathcal{B}_s)$ is a finite Boolean substructure of $(\mathcal{B}_x, \mathcal{A}_x)$, and Definition 2.1 is exactly the property we need to extend $f_s$ to the least element $y$ of $\mathcal{B}_x - \mathcal{B}_s$: just let $(\mathcal{D}_0, \mathcal{C}_0)$ be the finite Boolean substructure of $(\mathcal{B}_x, \mathcal{A}_x)$ generated by $\mathcal{B}_s$ and $y$, with $\mathcal{C}_0 = \mathcal{A} \cap \mathcal{D}_0$. Since $\mathcal{A}'$ is doubly dense within $x'$, there exists a finite Boolean substructure of $(\mathcal{B}_{x'}, \mathcal{A}'_{x'})$ isomorphic to $(\mathcal{D}_0, \mathcal{C}_0)$ via an isomorphism extending $f_s$. Since everything is finite, it is easy to find such a Boolean substructure and such an isomorphism, which we define to be $f_{s+1}$. The backwards direction is exactly the same, since $\mathcal{A}$ is likewise doubly dense within $x$. ■

We use the term $\mathcal{A}$-*atom* to mean an element $a \in \mathcal{A}$ which is an atom of $\mathcal{A}$. Of course, such an $a$ will still have densely many elements of $\mathcal{B}$ within it, but none of them except 0 and $a$ itself lies in $\mathcal{A}$.

**Theorem 2.4** *Let* $\boldsymbol{c}$ *be any Turing degree which is not low*$_4$*. Then there exists a Boolean subalgebra* $\mathcal{A}$ *of the computable atomless Boolean algebra* $\mathcal{B}$ *for which* $DgSp_{\mathcal{B}}(\mathcal{A})$ *contains* $\boldsymbol{c}$ *but does not contain* $\mathbf{0}$*. In particular, there exist low*$_5$ *degrees* $\boldsymbol{c}$ *for which this holds.*

*Proof.* First we describe a straightforward presentation of $\mathcal{A}$, and explain why the degree $\mathbf{0}$ cannot lie in its spectrum. Then we will construct a Boolean subalgebra $\mathcal{D}$ of $\mathcal{B}$ such that $(\mathcal{B}, \mathcal{A}) \cong (\mathcal{B}, \mathcal{D})$ and $\deg(\mathcal{D}) = \boldsymbol{c}$. These two results suffice to establish the theorem.

In building $\mathcal{A}$, we will often say of an interval $[m, m+1)$ (with $m \in \omega$) that we make $\mathcal{A}$ *converge to the right* in $[m, m+1)$. This means that all of the following intervals are enumerated into $\mathcal{A}$:

$$\left[m, m+\frac{1}{2}\right), \ \left[m+\frac{1}{2}, m+\frac{2}{3}\right), \ \left[m+\frac{2}{3}, m+\frac{3}{4}\right), \ \left[m+\frac{3}{4}, m+\frac{4}{5}\right), \ldots$$

Notice that this does *not* force $[m, m+1)$ to lie in $\mathcal{A}$, although of course every finite union of these intervals and of their complements (in $\mathcal{B}$) must enter the Boolean subalgebra $\mathcal{A}$. We will specify whether the interval $[m, m+1)$ belongs to $\mathcal{A}$ or not. (Usually it will not.)

Now we describe the Boolean subalgebra $\mathcal{A}$ of $\mathcal{B}$. First, the interval $(-\infty, 0)$ is in $\mathcal{A}$, and $\mathcal{A}$ is doubly dense inside $(-\infty, 0)$. Next, fix any set $C \in \boldsymbol{c}$, and let $\{n_0 < n_1 < n_2 < \cdots\} = C^{(4)}$, the fourth jump of $C$. We put the interval $[0, 2^{n_0})$ into $\mathcal{A}$. Then, as described above, we make $\mathcal{A}$ converge to the right in each of the intervals

$$[0,1), [1,2), \ldots, [2^{n_0}-1, 2^{n_0}).$$

We do *not* put any of these intervals themselves into $\mathcal{A}$ (unless $n_0 = 0$, in which case $[0,1)$ lies in $\mathcal{A}$). Write $q_0 = 0$ and $r_0 = 2^{n_0}$.

Next we put the interval $[2^{n_0}, 2^{n_0}+1)$ into $\mathcal{A}$, and make $\mathcal{A}$ doubly dense there. Then we define

$$q_1 = 2^{n_0}+1, \qquad r_1 = 2^{n_0}+1+2^{n_1},$$

put the interval $[q_1, r_1)$ into $\mathcal{A}$, and repeat the process, making $\mathcal{A}$ converge to the right in each interval

$$[q_1, q_1+1), [q_1+1, q_1+2), \ldots, [r_1-1, r_1).$$

Next $[r_1, r_1+1)$ enters $\mathcal{A}$ and we make $\mathcal{A}$ doubly dense there. Then we set $q_2 = r_1 + 1$ and $r_2 = q_2 + 2^{n_2}$, and put $[q_2, r_2)$ into $\mathcal{A}$, with $\mathcal{A}$ converging to the right in every unit integer interval inside there. We continue this way forever.

Having defined $\mathcal{A}$ on these unit intervals which together form a partition of $\mathcal{B}$, we declare the entire set $\mathcal{A}$ to be the Boolean subalgebra of $\mathcal{B}$ generated by those subelements which lie within single intervals and have already been placed in $\mathcal{A}$. The main point is that when defining $\mathcal{A}$, if $b_0, \ldots, b_n \notin \mathcal{A}$ and these elements come from $(n+1)$ distinct unit intervals, then one can choose whether their union should lie in $\mathcal{A}$ or not, and we always choose not to place it in $\mathcal{A}$. (In contrast, within a single doubly dense interval, two elements not in $\mathcal{A}$ may have their union in $\mathcal{A}$; double density forces this to happen sometimes. Likewise, each individual interval $[q_i, q_i+1), \ldots, [r_i-1, r_i)$ does not lie in $\mathcal{A}$, yet their union does.)

To summarize, we have an infinite ascending sequence

$$q_0 \prec r_0 \prec q_1 \prec r_1 \prec q_2 \prec \cdots$$

of rationals, with $r_i = q_i + 2^{n_i}$ and $q_{i+1} = r_i + 1$ for every $i \in \omega$. $\mathcal{A}$ is doubly dense within each interval $[r_i, q_{i+1})$, as well as in $(-\infty, q_0)$. Each interval $[q_i, r_i)$ lies in $\mathcal{A}$ and consists of $2^{n_i}$ adjacent unit intervals, none of which lies in $\mathcal{A}$ (unless $n_i = 0$), but within each of which $\mathcal{A}$ converges to the right. Moreover, each of these unit intervals $[m, m+1)$ may be seen as the disjoint union of the intervals

$$\left[m, m+\frac{1}{2}\right) \sqcup \left[m+\frac{1}{2}, m+\frac{2}{3}\right) \sqcup \left[m+\frac{2}{3}, m+\frac{3}{4}\right) \sqcup \cdots,$$

each of which is an $\mathcal{A}$-atom. (This is clear because no step in our description of $\mathcal{A}$ ever put into $\mathcal{A}$ any interval with an endpoint in the interior $(m + \frac{k}{k+1}, m + \frac{k+1}{k+2})$ of any of these intervals.)

Of course, if $\mathcal{A}$ converges to the right in $[m, m+1)$, then there is a Boolean-algebra isomorphism, respecting $\mathcal{A}$, from this interval onto any $[m+\frac{k}{k+1}, m+1)$. There is also such an isomorphism from $[m, m+1)$ onto the union of this interval with any finite number of $\mathcal{A}$-atoms in $\mathcal{B}$. The portion of the interval of interest to us is the right end, where the "convergence" occurs. Each interval $[m, m+1)$ with $q_i \leq m < r_i$, for any $i$, is called a *single $\mathcal{A}$-supremum*, as is any other interval of $\mathcal{B}$ which is isomorphic over $\mathcal{A}$ to one of these, according to the following definition.

**Definition 2.5** An element $x \in \mathcal{B}$ is an $\mathcal{A}$-*supremum* if $x$ is the least upper bound in $\mathcal{B}$ of an infinite set $\{a_i\}$ of $\mathcal{A}$-atoms. (That is, all $a_i \subseteq x$, and for every $y \subsetneq x$ in $\mathcal{B}$ there is an $i$ with $a_i \not\subseteq y$.) Such an element is a *single* $\mathcal{A}$-*supremum* if it is not the union of two disjoint $\mathcal{A}$-suprema. The union of $k$ pairwise-disjoint single $\mathcal{A}$-suprema will be called a $k$-*fold* $\mathcal{A}$-*supremum*.

**Lemma 2.6** *For the $\mathcal{A}$ defined above, if an element $x \in \mathcal{B}$ contains infinitely many $\mathcal{A}$-atoms, then $x$ contains some interval of the form $[m - \frac{1}{k}, m)$, with $m \notin \{q_i : i \in \omega\}$.*

*Proof.* Every $x \in \mathcal{B}$ is a finite union of intervals, so some interval $[q, r)$ within $x$ contains infinitely many $\mathcal{A}$-atoms. Now $\mathcal{A}$-atoms appear only within the intervals $[q_i, r_i)$, not in the doubly dense intervals of $\mathcal{B}$. If $r = +\infty$, then the conclusion is immediate, so assume $r \in \mathbb{Q}$. It follows that $[q, r)$ intersects only finitely many intervals $[q_i, r_i)$, and so must contain infinitely many $\mathcal{A}$-atoms from one particular $[q_j, r_j)$. But since $[q, r)$ is itself an interval, it must contain cofinitely many of those $\mathcal{A}$-atoms. ■

**Lemma 2.7** *Let $x$ be any element of $\mathcal{B}$, and define $\mathcal{A}$ as above. Then $x$ is a single $\mathcal{A}$-supremum iff all the following properties hold of $x$.*

1. *For all $y \subseteq x$, $\mathcal{A}$ is not doubly dense within $y$.*

2. *$x$ contains infinitely many $\mathcal{A}$-atoms.*

3. *For every $y \in \mathcal{B}$, either $x \cap y$ or $(x - y)$ is contained within a finite union of $\mathcal{A}$-atoms.*

4. *Every $\mathcal{A}$-atom either is contained within $x$ or does not intersect $x$.*

*Thus the property of being a single $\mathcal{A}$-supremum is $\Pi^{\mathcal{A}}_3$, and for any $k > 1$, the property of being a $k$-fold $\mathcal{A}$-supremum is $\Sigma^{\mathcal{A}}_4$, uniformly in $k$.*

*Proof.* First we note that all four properties described are indeed $\Pi^{\mathcal{A}}_3$. Since every nonzero element of $\mathcal{B}$ contains a subelement not in $\mathcal{A}$, Property (1) can be expressed by:

$$(\forall y \subseteq x)[y \neq 0 \implies \text{some double-density requirement fails in } \mathcal{B}_y].$$

Saying that a single double-density requirement fails is a $\Sigma^{\mathcal{A}}_2$ statement: there is a finite Boolean substructure of $\mathcal{B}_y$ and a Boolean extension of it which is not realized in $\mathcal{B}_y$. So Property (1) is $\Pi_3^{\mathcal{A}}$. Property (2) is most easily stated with the $\exists^\infty$ quantifier, which is equivalent to an $\forall\exists$ quantifier:

$$(\exists^\infty y)[0 \neq y \subseteq x \ \&\ y \in \mathcal{A} \ \&\ \forall z \subsetneq y(z \in \mathcal{A} \implies z = 0)].$$

Property (3) is given by:

$$(\forall y \subseteq x)(\exists k \in \omega)(\exists z_0, \dots z_k \in \mathcal{A})[(y \subseteq \cup_i z_i \text{ or } x \subseteq y \cup z_0 \cup \dots \cup z_k) \ \& \\ (\forall i \leq k\ \forall t \subsetneq z_i[t \in \mathcal{A} \implies t = 0])].$$

Finally, Property (4) just says:

$$(\forall z \in \mathcal{A})[z \subseteq x \text{ or } z \cap x = 0 \text{ or } \exists t \in \mathcal{A}[0 \subsetneq t \subsetneq z]].$$

Each of these is in $\forall\exists\forall$ form over an oracle for $\mathcal{A}$, except Property (4), which merely $\forall\exists$ over $\mathcal{A}$. Moreover, for $x$ to be a $k$-fold $\mathcal{A}$-supremum is then a $\Sigma_4^{\mathcal{A}}$ property, uniformly in $k$, since it says that there exist $k$ pairwise-disjoint elements which satisfy these $\Pi_3^{\mathcal{A}}$ conditions and have union $x$.

We claim that every single $\mathcal{A}$-supremum $x$ satisfies all four properties. First, if $y \subseteq x$ is doubly dense, then no $\mathcal{A}$-atoms lie within $y$, and so $x - y = x$ by the minimality in Definition 2.5, forcing $y = 0$ and contradicting Definition 2.1. Thus Property (1) holds, and so $x \subseteq \cup_i [q_i, r_i)$. Indeed, since $x$ itself is a finite union of intervals and $\cup_i [q_i, r_i)$ is not, we get $x \subseteq \cup_{i \leq p} [q_i, r_i)$ for some finite $p$. Property (2) is immediate, and so Lemma 2.6 yields an interval $[m - \frac{1}{k}, m) \subseteq x$. Suppose $y \subseteq x$. Now $y$ is a finite union of intervals, so (increasing $k$ and/or replacing $y$ by $(x - y)$ if necessary) we may assume $[m - \frac{1}{k}, m) \subseteq y$. But we know $(y - x) \subseteq x \subseteq \cup_{i \leq p} [q_i, r_i)$. Moreover, since $y$ is a finite union of intervals, only finitely many of the $\mathcal{A}$-atoms in $\cup_{i \leq p} [q_i, r_i)$ can intersect both $y$ and $(x - y)$. Therefore, if $y$ intersected infinitely many $\mathcal{A}$-atoms, it would contain infinitely many $\mathcal{A}$-atoms, in which case it would also contain an interval $[m' - \frac{1}{k'}, m')$, by Lemma 2.6. But then, simply by disentangling $y$ and $(x - y)$ from the finitely many $\mathcal{A}$-atoms which they split, we would get two disjoint $\mathcal{A}$-suprema whose union equals $x$, and so $x$ would not be a single $\mathcal{A}$-supremum. (Let $a$ be the union of the finitely many $\mathcal{A}$-atoms which intersect both $y$ and $(x - y)$, and let $y' = y \cup a$, so $y'$ and $(x - y')$ would both be the least upper bounds of infinite sets of $\mathcal{A}$-atoms.) Therefore, Property (3) must also hold of $x$, and Property (4) is quickly seen: if $z$ were an $\mathcal{A}$-atom intersecting both $x$ and $(1 - x)$, then $z$ would not be among the $\mathcal{A}$-atoms whose supremum is $x$, and so $(x - z)$ would be a strictly smaller upper bound for those $\mathcal{A}$-atoms.

Finally, we show that every element of $\mathcal{B}$ satisfying these four properties is a single $\mathcal{A}$-supremum. Property (2) shows that the set $\{a \subseteq x : a \text{ is an } \mathcal{A}\text{-atom}\}$ is infinite. If $y \subseteq x$ is also an upper bound for that set, then $(x - y)$ contains no element within which $\mathcal{A}$ is doubly dense, by Property (1); moreover, any $\mathcal{A}$-atom $a$ intersecting $x$ must be bounded by $x$ (by Property (4)), hence is bounded by $y$ as well, so does not intersect $(x - y)$. But every nonzero interval in $\mathcal{B}$ either intersects a doubly dense interval, or else intersects an $\mathcal{A}$-atom. Therefore $x - y = 0$, so $x$ is the least upper bound for the set of $\mathcal{A}$-atoms within $x$. Finally, if $x_0$ and $x_1$ were disjoint $\mathcal{A}$-suprema with $x_0 \cup x_1 = x$, then Property (3) would fail, and so $x$ must be a single $\mathcal{A}$-supremum. ■

We now claim that there is no computable relation $R$ on $\mathcal{B}$ such that $(\mathcal{B}, R) \cong (\mathcal{B}, \mathcal{A})$. If there were, then the uniform $\Sigma_4^{\mathcal{A}}$ definition of $k$-fold $\mathcal{A}$-suprema from Lemma 2.7 would convert to a uniform $\Sigma_4^R$ definition on $(\mathcal{B}, R)$, which of course means just a $\Sigma_4^0$ definition, since $R$ is computable. But now we claim that for all $n$,

$$n \in C^{(4)} \iff \exists x \in \mathcal{A}\ [x \text{ is a } 2^n\text{-fold } \mathcal{A}\text{-supremum}].$$

The forward implication is clear from the use of the elements $n_0, n_1, \ldots$ of $C^{(4)}$ in the construction of $\mathcal{A}$: for each $i$, there is a $2^{n_i}$-fold $\mathcal{A}$-supremum in $\mathcal{A}$, namely the interval $[q_i, r_i)$. On the other hand, by Lemma 2.6, every single $\mathcal{A}$-supremum in $\mathcal{B}$ is the union of an interval of the form $[m - \frac{1}{k}, m)$ with finitely many more $\mathcal{A}$-atoms (and with $m \neq q_i$ for all $i$). So a $2^n$-fold $\mathcal{A}$-supremum must be the union of $2^n$ distinct such intervals with finitely many more $\mathcal{A}$-atoms, and such an element only entered $\mathcal{A}$ if $n = n_i$ for some $i$. (Here we come to understand the use of the powers $2^{n_i}$. For instance, if $n_0 = 0$ and $n_1 = 1$, then $\mathcal{A}$ contains a single $\mathcal{A}$-supremum $x_0$ and a separate 2-fold $\mathcal{A}$-supremum $x_1$. Being a Boolean subalgebra, $\mathcal{A}$ therefore contains $x_0 \cup x_1$, which is a 3-fold $\mathcal{A}$ supremum. However, every single $\mathcal{A}$-supremum in $\mathcal{A}$ differs from $x_0$ only by finitely many $\mathcal{A}$-atoms, and likewise every 2-fold $\mathcal{A}$-supremum differs from $x_1$ only by finitely many $\mathcal{A}$-atoms. Therefore, the only way $\mathcal{A}$ could contain a 4-fold $\mathcal{A}$-supremum is if $n_2 = 2$, i.e. if $2 \in C^{(4)}$. Since $\mathcal{A}$ contains essentially just a single $2^{n_i}$-fold $\mathcal{A}$-supremum for each $i$ (and since $n_i \neq n_j$ for $i \neq j$), there is no way that the smaller-fold $\mathcal{A}$-suprema can interfere with the use of $2^n$-fold $\mathcal{A}$-suprema to code whether $n \in C^{(4)}$.)

Of course, if $(\mathcal{B}, R) \cong (\mathcal{B}, \mathcal{A})$, then all of this analysis transfers over to $2^n$-fold $R$-suprema in $R$. But it is $\Sigma_4^0$ whether $R$ contains a $2^n$-fold $R$-supremum, and so this would imply $C^{(4)} \leq_T \emptyset^{(4)}$. Now the conditions on $C$ in Theorem 2.4 become clear: since $C$ is assumed not to be $\text{low}_4$, we know $C^{(4)} \not\leq_T \emptyset^{(4)}$, and so it is impossible for a computable unary relation $R$ on $\mathcal{B}$ to satisfy $(\mathcal{B}, R) \cong (\mathcal{B}, \mathcal{A})$.

Next we note that since $\mathcal{B}$ is computably categorical, the foregoing suffices to show that the degree $\mathbf{0}$ does not lie in $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$. The details, which are not difficult, appear as Lemma 1.6 in [6], while the proof of computable categoricity for $\mathcal{B}$ is an easy back-and-forth argument, first established independently by Goncharov and Dzgoev in [4] and by Remmel in [10].

To prove the theorem, it remains to show that $\boldsymbol{c}$ itself, the degree of $C$, *does* lie in $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$. For this we use a $C$-oracle to compute a Boolean subalgebra $\mathcal{D}$ of $\mathcal{B}$, and then show that there is an automorphism of $\mathcal{B}$ mapping $\mathcal{D}$ onto $\mathcal{A}$.

First we make $\mathcal{D}$ doubly dense within the interval $(-\infty, 0)$ of $\mathcal{B}$. In doing so, we attend to one detail. For each $n$, using our $C$-oracle, we define the interval $[-2n-2, -2n-1)$ to lie in $\mathcal{D}$ iff $n \in C$. Clearly, it remains easy to make $\mathcal{D}$ doubly dense in $(-\infty, 0)$ after this is done, and by doing so, we ensure that $C \leq_T \mathcal{D}$. The rest of the construction can then be devoted to making $(\mathcal{B}, \mathcal{D}) \cong (\mathcal{B}, \mathcal{A})$ with $\mathcal{D} \leq_T C$.

Now $C^{(4)}$ can be expressed by a $\Sigma_4^C$ formula, say $\exists a \forall b \exists c \forall d R(n, a, b, c, d)$, with $R \leq_T C$. Notice that by [3, Corollary 5.14], we may assume that this $R$ has the property that for each $n$, there is at most one value of $a$ satisfying $\forall b \exists c \forall d R(n, a, b, c, d)$. We also have a reduction, uniform in $n$, $a$, and $b$:

$$[\exists c \forall d R(n, a, b, c, d)] \iff f(n, a, b) \in \mathrm{Fin}^C$$

using some total computable function $f$. Combining these gives the form we will use to approximate $C^{(4)}$ in our construction of $\mathcal{D}$:

$$\forall n \; [n \in C^{(4)} \iff \exists a \forall b f(n, a, b) \in \mathrm{Fin}^C]$$

along with the fact that for each $n \in C^{(4)}$, the corresponding $a$ is unique.

Having already built $\mathcal{D}$ within the interval $(-\infty, 0)$ of $\mathcal{B}$, we now extend it to the rest of $\mathcal{B}$. First, we divide the rest of $\mathcal{B}$ into intervals: each "even" interval $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ will be used to help code whether $n \in C^{(4)}$, while we make $\mathcal{D}$ doubly dense within each of the remaining "odd" intervals $[2i+1, 2i+2)$, every one of which is defined to lie in $\mathcal{D}$. Every even interval $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ is also defined to lie in $\mathcal{D}$, and is subdivided into $2^n$

subintervals. In each of these subintervals, we build a copy of the following structure $\mathcal{D}_{n,a}$, for the particular $n$ and $a$ corresponding to this even interval.

To simply matters, we define $\mathcal{D}_{n,a}$ here on $2^n$-many copies of the interval $[0, +\infty)$, which is uniformly computably isomorphic to any left-closed, right-open interval of $\mathbb{Q}$. To begin with, every subinterval $U_i = [i, i+1)$ is placed into $\mathcal{D}_{n,a}$. These are the *original intervals*. (However, $[0, +\infty)$ itself does *not* belong to $\mathcal{D}_{n,a}$.) Then, using our $C$-oracle, we simultaneously enumerate all the $C$-enumerable sets $W^C_{f(n,a,b)}$, for this $n$ and $a$ and for every $b \in \omega$. We may assume that at most one such set receives an element at any single stage of our enumeration. At stage $s$, fix the unique $b$ (if any) for which $W_{f(n,a,b),s+1} \neq W_{f(n,a,b),s}$, and let $t_b$ be the union of the intervals $[b+1, +\infty)$ from each of the $2^n$-many copies of $[0, +\infty)$. We satisfy the $|W_{f(n,a,b),s}|$-th double-density requirement for $t_b$ (as defined in the proof of Lemma 2.3) by adding finitely many elements of $t_b$ to $\mathcal{D}_{n,a}$ and finitely many others to the complement $\overline{\mathcal{D}_{n,a}}$. We ensure that every element added to $\mathcal{D}_{n,a}$ only has end points whose denominators (in lowest terms) are $> s$. This completes the construction; the set $\mathcal{D}_{n,a}$ consists of all finite unions and complements of the intervals enumerated at all stages $s$ during this process.

Notice first that the Boolean subalgebra $\mathcal{D}_{n,a}$ of $\mathbf{Intalg}([0, +\infty))$ thus generated is $C$-computable. Every interval entering $\mathcal{D}_{n,a}$ at stage $s$ has end points with denominators $\geq s$, and so an arbitrary finite union of intervals lies in $\mathcal{D}_{n,a}$ iff it has been enumerated before the stage equal to the maximum of the denominators of its endpoints.

Next, suppose that $(\forall b) f(n, a, b) \in \text{Fin}^C$. Then every original interval $U_i$ had only finitely many of its subelements added to $\mathcal{D}_{n,a}$, since all the sets $W^C_{f(n,a,b)}$ with $b < i$ together had only finitely many elements enumerated into them. Therefore, in this case $\mathcal{D}_{n,a}$ is atomic and $[0, +\infty)$ is a single $\mathcal{D}_{n,a}$-supremum: it is the upper bound of all the atoms of $\mathcal{D}_{n,a}$, but if it is written as the union of two disjoint subelements, then only one of those elements can contain the right end of $[0, +\infty)$, and the other cannot contain infinitely many $\mathcal{D}_{n,a}$-atoms.

On the other hand, if there is a (least) $b$ for which $f(n, a, b) \notin \text{Fin}^C$, then the interval $t_b$ satisfies all of the double-density requirements, while the interval $[0, b+1)$ in each copy of $[0, +\infty)$ is the union of finitely many $\mathcal{D}_{n,a}$-atoms.

This completes our description of $\mathcal{D}_{n,a}$, for every $n$ and $a$. Recall now that the interval $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ of $\mathcal{B}$ consists of $2^n$ isomorphic copies of $\mathcal{D}_{n,a}$:

each lies in an interval $[2\langle n,a\rangle + \frac{i}{2^n}, 2\langle n,a\rangle + \frac{i+1}{2^n})$, with $i = 0, \ldots, 2^n - 1$. (We simply build an order isomorphism from $[0, +\infty)$ onto each $[2\langle n,a\rangle + \frac{i}{2^n}, 2\langle n,a\rangle + \frac{i+1}{2^n})$, and define $\mathcal{D}$ to contain everything in the image of $\mathcal{D}_{n,a}$ under this isomorphism.) We then close $\mathcal{D}$ under finite unions and complements; it remains $C$-computable, since each $\mathcal{D}_{n,a}$ was $C$-computable uniformly on each interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$. This parallels the same step in the construction of $\mathcal{A}$ above: certain elements from individual unit intervals are placed in $\mathcal{D}$, and then the entire set $\mathcal{D}$ is the Boolean subalgebra generated by those subelements of the unit intervals. It is important to note that this does *not* force the intervals $[2\langle n,a\rangle + \frac{i}{2^n}, 2\langle n,a\rangle + \frac{i+1}{2^n})$ to lie in $\mathcal{D}$, although the entire interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ does lie in $\mathcal{D}$.

So, if $(\forall b) f(n,a,b) \in \mathrm{Fin}^C$, then $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ is the union of these $2^n$ intervals, each of which is a single $\mathcal{D}$-supremum, but no finite union of which (except $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ itself) lies in $\mathcal{D}$. In fact, in this case the interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ with the subalgebra $\mathcal{D}$ is order-isomorphic to the interval $[q_i, r_i)$ of $\mathcal{B}$ with the subalgebra $\mathcal{A}$ (where $n = n_i$). Thus $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ is a $2^n$-fold $\mathcal{D}$-supremum. Notice that since $(\forall b) f(n,a,b) \in \mathrm{Fin}^C$, this $a$ witnesses that $n \in C^{(4)}$. Moreover, this $a$ is the unique such witness; for every $a' \neq a$, there is some $b$ with $f(n,a',b) \notin \mathrm{Fin}^C$, the case we describe next.

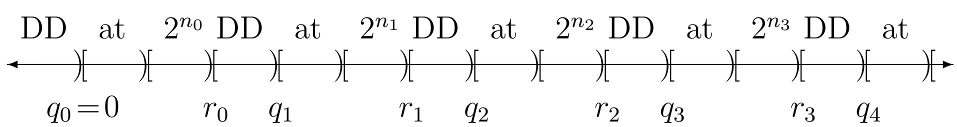
If it is not true that if $(\forall b) f(n,a,b) \in \mathrm{Fin}^C$, then we saw above that each copy of $\mathbf{Intalg}([0,+\infty))$ is the union of finitely many $\mathcal{D}_{n,a}$-atoms and one interval $[b+1, +\infty)$, and that $\mathcal{D}_{n,a}$ satisfied all the double-density requirements on the union $t_b$ of these $2^n$-many intervals $[b+1, +\infty)$. The structure of each copy of $\mathbf{Intalg}([0,+\infty))$ is transferred to each subinterval $[2\langle n,a\rangle + \frac{i}{2^n}, 2\langle n,a\rangle + \frac{i+1}{2^n})$. Thus, the entire interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ is the union of finitely many $\mathcal{D}$-atoms with the image of $t_b$. Since $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ lies in $\mathcal{D}$, we can split off the finitely many $\mathcal{D}$-atoms and see that $t_b$ is an element of $\mathcal{D}$, within which $\mathcal{D}$ is doubly dense, since this element satisfies all of the double-density requirements. Therefore, in this case no subinterval of $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ is even a single $\mathcal{D}$-supremum, let alone a severalfold one.

Now if $n \in C^{(4)}$, then there is a unique $a$ satisfying $(\forall b) f(n,a,b) \in \mathrm{Fin}^C$, and so there is a unique interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ which is a $2^n$-fold $\mathcal{D}$-supremum and lies in $\mathcal{D}$. Every other interval corresponding to this $n$ is the union of finitely many $\mathcal{D}$-atoms and one interval in which $\mathcal{D}$ is doubly dense. On the other hand, if $n \notin C^{(4)}$, then every interval $[2\langle n,a\rangle, 2\langle n,a\rangle + 1)$ (for every $a$) is the union of a $\mathcal{D}$-doubly-dense interval and finitely many $\mathcal{D}$-atoms. So this gives us our picture of the Boolean subalgebra $\mathcal{D}$ of $\mathcal{B}$, and explains how the coding of $C^{(4)}$ finally worked.

We still have to show that $(\mathcal{B}, \mathcal{D}) \cong (\mathcal{B}, \mathcal{A})$. Automorphisms of $\mathcal{B}$, even without any subalgebra involved, can be tricky. For instance, the bijection which interchanges each even interval $[2n, 2n+1)$ with its successor $[2n+1, 2n+2)$, for every $n \in \mathbb{Z}$, yields an automorphism of $\mathcal{B}$, whereas the bijection interchanging each odd interval $[2n+1, 2n+2)$ with $[-2n-1, -2n)$, for all $n \in \omega$, does not yield an automorphism. (In the latter, the image of $[0, +\infty)$ is not a finite union of intervals.) So we give here a careful description. The basic principle is that we will map the interval $(-\infty, 0)$ of $\mathcal{B}$ to itself, and that the complementary interval $[0, +\infty)$ is divided into subintervals, each of which has only finitely many others to its left. Each such subinterval in $(\mathcal{B}, \mathcal{D})$ is mapped to a similar subinterval in $(\mathcal{B}, \mathcal{A})$, which implies that our map sends each "tail" $[q, +\infty)$ in $(\mathcal{B}, \mathcal{D})$ to the union of a tail and finitely many other subintervals in $(\mathcal{B}, \mathcal{A})$. (Thus the key is that the subintervals of $[0, +\infty)$ appear in order type $\omega$ from left to right. If it were not so, more care would be required at the limit points in the sequence of subintervals.)

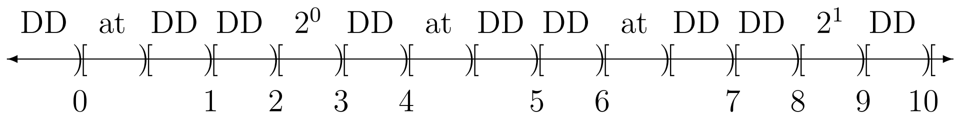
We offer the following helpful schematics of the two structures $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{B}, \mathcal{D})$. The first indicates, in $(\mathcal{B}, \mathcal{A})$, where to find the $\mathcal{A}$-doubly dense intervals (labeled by "DD"), the $2^{n_i}$-fold $\mathcal{A}$-suprema ("$2^{n_i}$"), and the first $\mathcal{A}$-atom ("at") of each $\mathcal{A}$-supremum. For reasons explained below, this first $\mathcal{A}$-atom is separated from the rest of the $2^{n_i}$-fold $\mathcal{A}$-supremum $[q_i + \frac{1}{2}, r_i)$.

$(\mathcal{B}, \mathcal{A})$:




The second schematic indicates, in $(\mathcal{B}, \mathcal{D})$, where the various $2^n$-fold $\mathcal{D}$-suprema in $\mathcal{D}$, the $\mathcal{D}$-doubly dense intervals, and the finite sequences of $\mathcal{D}$-atoms lie. Of course, with the construction of $\mathcal{D}$, we do not actually know which even intervals are $\mathcal{D}$-suprema, or how many atoms appear in those even intervals which are not $\mathcal{D}$-suprema, so the following represents a descriptive guess, in which $C^{(4)}$ contains both 0 and 1.

$(\mathcal{B}, \mathcal{D})$:




Now we describe the isomorphism from $(\mathcal{B}, \mathcal{D})$ onto $(\mathcal{B}, \mathcal{A})$. First, $\mathcal{A}$ is doubly dense in $(-\infty, 0)$, as is $\mathcal{D}$, so by Lemma 2.3, this interval can be mapped isomorphically onto itself. Next, recall that $C^{(4)} = \{n_0 < n_1 < \cdots\}$. For each $i \in \omega$, the unique interval $[2\langle n_i, a\rangle, 2\langle n_i, a\rangle + 1) \in \mathcal{D}$ which is a $2^{n_i}$-fold $\mathcal{D}$-supremum is mapped onto the interval $[q_i + \frac{1}{2}, r_i)$ by an order isomorphism. Notice that this leaves the $\mathcal{A}$-atom $[q_i, q_i + \frac{1}{2})$ out of the image, so that we have infinitely many $\mathcal{A}$-atoms still available, no two of which are adjacent to each other. (That is, no two of them share an end point.) These become the images of the $\mathcal{D}$-atoms from the intervals $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ for which some $b$ had $f(n, a, b) \notin \text{Fin}^C$; there were infinitely many such intervals, no two adjacent, and each one had finitely many $\mathcal{D}$-atoms in it. Finally, each such $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ was the union of those finitely many $\mathcal{D}$-atoms with one subinterval in which $\mathcal{D}$ was doubly dense, and our map pairs up those subintervals (along with the odd intervals $[2i+1, 2i+2)$) with the intervals $[r_i, q_{i+1})$ of $\mathcal{B}$ in which $\mathcal{A}$ was doubly dense. The $[r_i, q_{i+1})$ are pairwise nonadjacent, and each $\mathcal{D}$-doubly-dense interval in $\mathcal{B}$ was part of at most a finite chain of pairwise-adjacent such intervals, so we may safely map each of these $\mathcal{D}$-doubly-dense intervals to the $\mathcal{A}$-doubly-dense interval with which it was paired, using Lemma 2.3.

We claim that the above description defines an isomorphism from $(\mathcal{B}, \mathcal{D})$ onto $(\mathcal{B}, \mathcal{A})$. Recall the structure of $(\mathcal{B}, \mathcal{D})$: $\mathcal{D}$ is doubly dense in $(-\infty, 0)$, in the odd intervals $[2i+1, 2i+2)$ (for $i \in \omega$), and in the finite union of the right ends of each of the $2^n$-many pieces of every even interval $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ for which some $b$ has $f(n, a, b) \notin \text{Fin}^C$. The remaining (left) portion of each piece of each such even interval consists of finitely many $\mathcal{D}$-atoms. Finally, each even interval $[2\langle n, a\rangle, 2\langle n, a\rangle + 1)$ for which all $f(n, a, b) \in \text{Fin}^C$ is a $2^n$-fold $\mathcal{D}$-supremum, containing $2^n$ single $\mathcal{D}$-suprema, in each of which the $\mathcal{D}$-atoms accumulate to the right. The preceding paragraph defined where all of these intervals should be mapped in $(\mathcal{B}, \mathcal{A})$, and also how all subintervals within them were to be mapped into $(\mathcal{B}, \mathcal{A})$. For every interval $[q, r)$ in $\mathcal{B}$ with $q \neq -\infty$ and $r \neq +\infty$, this defines the image of $[q, r)$ under this map: $[q, r)$ must be a finite union of elements whose images have been defined above, and the union of those images forms the image of $[q, r)$. (By the definitions of $\mathcal{A}$ and $\mathcal{D}$, $[q, r) \in \mathcal{D}$ iff all its intersections with individual unit intervals lie in $\mathcal{D}$, in which case its image is the union of elements of $\mathcal{A}$, hence lies in $\mathcal{A}$, and conversely.) The same holds if $q = -\infty$, since $(-\infty, 0)$ mapped isomorphically onto $(-\infty, 0)$. If $r = +\infty$ and $q \geq 0$, then the interval $[q, r)$ contains cofinitely many of the $\mathcal{D}$-supremum intervals, cofinitely many of the $\mathcal{D}$-atoms from even intervals, and cofinitely many of the subintervals in which $\mathcal{D}$ is doubly dense (both odd intervals, and the right ends of even intervals which are not $\mathcal{D}$-suprema). The finitely many subintervals from $[0, q)$ have all been mapped to subintervals within $[0, +\infty)$, none of which extends to $+\infty$, and we define $[q, r)$ to map to the complement of the image of $[0, q)$ within $[0, +\infty)$. To complete the description, we map any interval $[q, +\infty)$ in $\mathcal{B}$ with $q < 0$ to the union of the images of $[q, 0)$ and $[0, +\infty)$, and we map each finite union of intervals in $\mathcal{B}$ to the union of the images of those intervals.

To see that this map is onto, one simply follows the description from the point of view of $(\mathcal{B}, \mathcal{A})$, which was the range of the isomorphism described above. Doing so makes it clear that every $[q_i, r_i)$ is the isomorphic image of the unique $2^{n_i}$-fold $\mathcal{D}$-supremum in $\mathcal{D}$ (along with a single $\mathcal{D}$-atom from elsewhere in $(\mathcal{B}, \mathcal{D})$, which maps to $[q_i, q_i + \frac{1}{2})$); that every $[r_i, q_{i+1})$ is the isomorphic image of some interval in which $\mathcal{D}$ is doubly dense, and that $(-\infty, 0)$ is the isomorphic image of $(-\infty, 0)$. The argument then extends to all subintervals of $(\mathcal{B}, \mathcal{A})$, and all finite unions thereof, just as in the preceding paragraph. Once again, the key point is that each of $(\mathcal{B}, \mathcal{D})$ and $(\mathcal{B}, \mathcal{A})$ is divided into countably many subintervals, appearing in order type $\omega$.

This isomorphism shows that $\boldsymbol{c}$, the degree of the set $C$, does lie in $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$, even though that spectrum has already been shown not to contain the degree $\mathbf{0}$. So we have proven Theorem 2.4. ■

# 3 Further Questions

The immediate question to follow Theorem 2.4 is whether one can adapt the construction to produce a Boolean algebra whose spectrum (as a structure) likewise contains a $\text{low}_5$ degree without containing $\mathbf{0}$. A fairly quick examination of the Boolean subalgebra $\mathcal{A}$ constructed in the theorem shows that $\mathcal{A}$ itself does not have such a spectrum. In particular, the set of $n$ such that $\mathcal{A}$ contains an $n$-fold $\mathcal{A}$-supremum must now be an initial subset of $\omega$: in order to be a 3-fold $\mathcal{A}$-supremum, for example, an $x \in \mathcal{A}$ must be the union of three single $\mathcal{A}$-suprema, meaning that $\mathcal{A}$ contains both a single and a 2-fold $\mathcal{A}$-supremum as well. Since $\mathcal{A}$ contained $k$-fold $\mathcal{A}$-suprema for arbitrarily large $k$, it is actually just the union of countably many single $\mathcal{A}$-suprema with countably many dense intervals interspersed between them, hence has a computable copy. As in the example from [3] on linear orders, discussed in Section 1, the ambient structure $\mathcal{B}$ allowed our construction to code information into $\mathcal{A}$ which could not have been coded in the same way without the larger framework of $\mathcal{B}$. Possibly one could produce a related Boolean algebra $\mathcal{A}^*$ which codes the structure $(\mathcal{B}, \mathcal{A})$, in which case the coding of $\mathcal{C}^{(4)}$ could be deciphered from an arbitrary copy of $\mathcal{A}^*$ in five or more jumps: one or more to decode $(\mathcal{B}, \mathcal{A})$, and then four more to decode $C^{(4)}$ itself. In this case, a low$_n$ set $C$ for appropriate $n$ might have its degree in Spec$(\mathcal{A}^*)$, yet preclude any computable copy of $\mathcal{A}^*$ from existing.

Turning to spectra of Boolean subalgebras, it is natural to ask whether the construction in Theorem 2.4 for non-low$_4$ degrees can be modified to work for a degree which is not low$_3$, or not low$_2$, or not low. From the opposite side, it would be natural to attempt first to reproduce the Downey-Jockusch argument on low Boolean subalgebras of $\mathcal{B}$. Both of these are the subjects of current investigations. A positive answer to the question for non-low$_3$ degrees would entail a Boolean subalgebra $\mathcal{C}$ of $\mathcal{B}$ whose spectrum (as a relation on $\mathcal{B}$) is not the spectrum of any Boolean algebra (as a structure), thus resolving the question asked by Montalbán and quoted at the end of [6]. Furthermore, a positive answer might involve a construction of $\mathcal{C}$ simpler than that of $\mathcal{D}$ in Theorem 2.4, since fewer jumps would be involved (although the coding might be more complicated), and therefore might be more readily adapted to design a Boolean algebra $\mathcal{C}^*$ to code $(\mathcal{B}, \mathcal{C})$, as described above.

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