

**Survey of Degree Spectra of  
High<sub>n</sub> and Non-low<sub>n</sub> Degrees**

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## Spectrum of a Structure

**Defns:** For a countable structure  $\mathcal{S}$  with domain  $\omega$ , the *Turing degree of  $\mathcal{S}$*  is the Turing degree of the atomic diagram of  $\mathcal{S}$ . The *spectrum of  $\mathcal{S}$*  is

$$\text{Spec}(\mathcal{S}) = \{\text{deg}(\mathcal{A}) : \mathcal{A} \cong \mathcal{S}\}$$

of all Turing degrees of copies of  $\mathcal{S}$ .

For a relation  $R$  on a computable structure  $\mathcal{M}$ , the *spectrum of  $R$* ,  $\text{DgSp}_{\mathcal{M}}(R)$ , is

$$\{\text{deg}(f(R)) : f : \mathcal{M} \cong \mathcal{N} \text{ \& } \mathcal{N} \text{ is computable}\}.$$

## Non-low Degrees

**Theorem** (Slaman; Wehner; Hirschfeldt): There exists a structure whose spectrum contains every Turing degree  $> \mathbf{0}$ , but not the degree  $\mathbf{0}$ .

This also holds with an arbitrary  $\mathbf{d}$  in place of  $\mathbf{0}$ .

**Theorem** (GHKMMS): For each  $n \in \omega$ , there exists a structure whose spectrum contains precisely the non-low $_n$  degrees. Indeed, for each computable successor ordinal  $\alpha$ , there exists a structure with spectrum

$$\{\deg(X) : (\exists \mathbf{d} \notin \Delta_\alpha^0)[\mathbf{d} \text{ is } \Delta_\alpha^0 \text{ relative to } \mathbf{X}]\}.$$

## High Degrees

Given a structure  $\mathcal{A}$ , the technique of (GHKMMS) builds, for each successor ordinal  $\alpha$ , a structure  $\mathcal{B}$  such that

$$\mathbf{c} \in \text{Spec}(\mathcal{B}) \iff \mathbf{c}^{(\alpha)} \in \text{Spec}(\mathcal{A}).$$

**Fact:** For every  $\mathbf{d}$ , there is a structure  $\mathcal{A}_{\mathbf{d}}$  with spectrum  $\{\mathbf{c} : \mathbf{c} \geq_T \mathbf{d}\}$ .

So with  $\alpha = 1$  and  $\mathbf{d} = \mathbf{0}''$ , this gives a structure  $\mathcal{B}$  whose spectrum contains exactly the high-or-above degrees (those  $\mathbf{c}$  with  $\mathbf{c}' \geq_T \mathbf{0}''$ ). Likewise for  $\text{high}_n$ , and even  $\text{high}_\alpha$  (with  $\alpha \notin \mathbf{LOR}$ ). This extends a known result...

## Spectrum of high degrees

**Proposition** (Harizanov-Miller): There exists a relation  $R$  on a computable dense linear order  $\mathbb{Q}$  with

$$\text{DgSp}_{\mathbb{Q}}(R) = \{\mathbf{d} : \mathbf{d}' \geq_{\mathbf{T}} \mathbf{0}''\}.$$

**Corollary:** There exists a structure with this same spectrum.

**Corollary:** Not all spectra of unary relations on  $(\mathbb{Q}, <)$  can be realized as spectra of linear orders.

**Proof:** By a theorem of Knight (1986),  $\mathbf{0}'$  is the only possible jump degree of a linear order.

## How About Linear Orders?

For Boolean algebras, the spectrum  $\{d : d > \mathbf{0}\}$  is impossible.

- Jockusch-Soare: For every c.e.  $d > \mathbf{0}$ , there exists a linear order whose spectrum contains  $d$  but not  $\mathbf{0}$ .
- Downey/Seetapun: Extension to any  $d$  with  $\mathbf{0} < d \leq \mathbf{0}'$ .
- Knight: Extension to any  $d > \mathbf{0}$ .
- M.: There is a single linear order whose spectrum contains all  $d$  with  $\mathbf{0} < d \leq \mathbf{0}'$ , but not  $\mathbf{0}$ .
- Frolov: For each  $n \in \omega$ , there is a linear order whose spectrum contains all non- $\text{low}_n$  degrees  $\leq \mathbf{0}'$  but no  $\text{low}_n$  degrees.

**Question:** Can a linear order have a spectrum of precisely the non- $\text{low}_n$  degrees?

## Where Next?

Frolov's result builds an order  $\mathcal{L}$  by relativizing Miller's result to  $\mathbf{0}^{(n)}$ , so that  $\text{Spec}(\mathcal{L})$  contains all  $\mathbf{d}$  with  $\mathbf{0}^{(n)} < \mathbf{d} \leq \mathbf{0}^{(n+1)}$ , but not  $\mathbf{0}^{(n)}$ .

Then it applies the relativization of a theorem of Downey and Knight: A linear order  $\mathcal{L}$  is  $\Delta_2^0$  iff  $(\eta + 2 + \eta) \cdot \mathcal{L}$  is computable.

So the order  $\mathcal{L}_n = (\eta + 2 + \eta)^n \cdot \mathcal{L}$  has all non- $\text{low}_n$  degrees below  $\mathbf{0}'$  in its spectrum, but no  $\text{low}_n$  degree.

## Spectral Universality

An embedding  $f : \mathcal{A} \hookrightarrow \mathcal{B}$  *preserves the spectrum* if  $\text{Spec}(\mathcal{A}) = \text{DgSp}_{\mathcal{B}}(f(\mathcal{A}))$ .

A computable model  $\mathcal{B}$  of a theory  $T$  is *spectrally universal* for  $T$  if every countable model  $\mathcal{A}$  of  $T$  embeds into  $\mathcal{B}$  via some  $f$  preserving the spectrum.

Many (but not all!) computable Fraïssé limits of theories turn out to be spectrally universal for those theories. Examples:

- Countable dense linear order.
- Random graph.
- Countable atomless Boolean algebra.

Counterexample:

- Algebraic closure of the field  $\mathbb{Z}/(p)$ .

## Structures vs. Relations

**Corollary** (of the spectral universality of the random graph): The spectra of countable graphs are precisely the spectra of unary relations on the random graph.

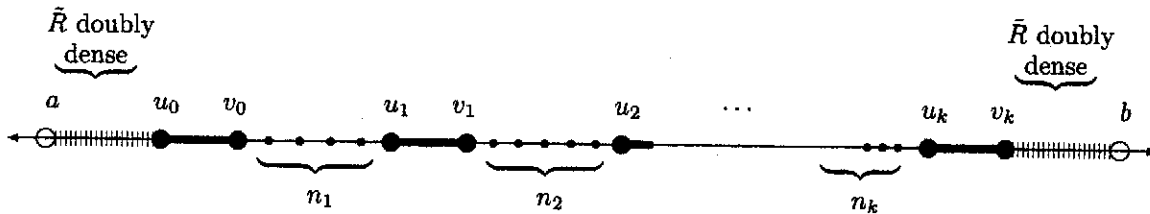
We saw above that this does *not* hold of linear orders. Spectral universality of the countable DLO shows that every spectrum of a LO is the spectrum of a unary relation on the DLO, but the converse is false.

## Candidate for a Counterexample

**Theorem (M.):** There exists a unary relation  $\tilde{R}$  on the countable DLO  $(\mathbb{Q}, <)$  whose degree spectrum contains the non-low degrees:

$$\text{DgSp}_{\mathbb{Q}}(\tilde{R}) = \{\mathbf{d} : \mathbf{d}' >_{\mathbf{T}} \mathbf{0}'\}.$$

For a given finite set  $F = \{n_1, n_2, \dots, n_k\} \subset \omega$  and  $a < b$  in  $\mathbb{Q}$ , define  $\tilde{R}$  on  $(a, b)$  for  $F$  as follows:



Wehner's construction gives a family  $\mathcal{F}$  of finite sets  $F$ , uniformly enumerable by any degree  $> \mathbf{0}'$ , but not by  $\mathbf{0}'$ .

For each  $F \in \mathcal{F}$ , in each order, define  $\tilde{R}$  as above for this  $F$  on densely many intervals  $(a, b)$  in  $\mathbb{Q}$ .

## Does this work?

This  $\tilde{R}$  on  $(\mathbb{Q}, \prec)$  gives a potential second counterexample to the converse of spectral universality for linear orders.

**Question:** Does there exist a linear order with spectrum  $\{\mathbf{c} : \mathbf{c}' >_T \mathbf{0}'\}$ ?

Notice that the restriction of  $\prec$  to  $\tilde{R}$  does *not* yield such an order.

## Fields

**Example:** Fix  $r_0 = e$  and  $r_{i+1} = e^{r_i}$ . Given  $S \subseteq \omega$ , let  $F_S$  be the closure of  $\mathbb{Q}(r_t \mid t \in S)$  under square roots of positive elements. We claim that

$$\text{Spec}(F_S) = \{\mathbf{d} : S \text{ is } \Sigma_2^0 \text{ in } \mathbf{d}\}.$$

**Cor.:** For any  $A \subseteq \omega$ , there is a field whose spectrum contains precisely those  $\mathbf{d}$  with  $A \leq \mathbf{d}'$ .

$$\begin{aligned} \text{Spec}(F_{A'}) &= \{\mathbf{d} : (\exists D \in \mathbf{d}) A' \leq_1 D''\} \\ &= \{\mathbf{d} : (\exists D \in \mathbf{d}) A \leq_T D'\} \end{aligned}$$

As a relation on its algebraic closure,  $F_S$  has the same spectrum  $\{\mathbf{d} : S \text{ is } \Sigma_2^0 \text{ in } \mathbf{d}\}$ .

So the high degrees form the spectrum of a field, and also the spectrum of a subfield of the algebraic closure.

$$\text{Spec}(F_S) = \{d : S \text{ is } \Sigma_2^0 \text{ in } d\}.$$

$\subseteq$ : If  $E \cong F_S$ , then  $S$  is the set

$$\{t \in \omega : (\exists x \in E)(\forall q \in \mathbb{Q})[q < r_t \leftrightarrow q \prec x \text{ in } E]\}.$$

The order  $\prec$  on  $E$  is  $E$ -computable, by the closure of  $E$  under square roots of positive elements.

$\supseteq$ : If  $S \leq_1 \text{Fin}^D$ , let  $t \in S$  iff  $W_{h(t)}^D$  is finite. Start building  $F_\omega$  (the field containing all  $r_t$ ). Each time  $W_{h(t)}^D$  receives an element, make the old  $r_t$  become rational and add a new  $r_t$  to replace it.