Automorphism Spectra and Tree-Definability

Valentina Harizanov,

George Washington University

Russell Miller,

Queens College & Graduate Center of CUNY

Andrei Morozov,

Sobolev Institute of Mathematics

July 2008

The Automorphism Spectrum

Definition: For a computable structure $\mathcal{A}$, the *automorphism spectrum* of $\mathcal{A}$ is the set

$$\text{AutSpec}^*(\mathcal{A}) = \{\deg(f) : f \in \text{Aut}(\mathcal{A}) \ \& \ f \neq \text{id}\}.$$

So $\text{AutSpec}^*(\mathcal{A})$ measures the complexity of the nontrivial symmetries of $\mathcal{A}$.

Notice that the same definition makes sense for noncomputable structures $\mathcal{A}$ as well.

Singleton Automorphism Spectra

The following are known automorphism spectra:

- $\{\boldsymbol{d}\}$, for every c.e. degree $\boldsymbol{d}$, uniformly (HMM).
- $\{\boldsymbol{d}\}$, for every degree $\boldsymbol{d} \leq \mathbf{0}'$ (Hirschfeldt; Schmerl).
- $\{\boldsymbol{d}\}$, for every degree $\boldsymbol{d}$ with $\mathbf{0}^{(n)} \leq \boldsymbol{d} \leq \mathbf{0}^{(n+1)}$ (HMM, using a technique of Marker and others).

Theorem: If $\text{AutSpec}^*(\mathcal{A}) = \{\boldsymbol{d}\}$, then $\boldsymbol{d}$ is hyperarithmetical.
(This follows from the Perfect Set Theorem.)

Tree-Definability

Definition: A function $f : \omega \to \omega$ is *tree-definable* if there exists a computable subtree $T \subseteq \omega^{<\omega}$ such that f is the unique (infinite) path through T. We also say that $\deg(f)$ is *tree-definable*.

Theorem: The following are equivalent:

1. $\mathrm{AutSpec}^*(\mathcal{A})$ is at most countable.
2. $\mathrm{Aut}(\mathcal{A})$ is at most countable.
3. Every $\boldsymbol{d} \in \mathrm{AutSpec}^*(\mathcal{A})$ is tree-definable.

Clearly ($1 \iff 2$) and ($3 \implies 1$).
For ($2 \implies 3$), apply Kueker's Theorem to get a finite tuple $p_1, \ldots, p_n \in \mathcal{A}$ such that $(\mathcal{A}, \vec{p})$ is rigid. So for $f \in \mathrm{Aut}(\mathcal{A})$, build a computable tree of those partial automorphisms φ of $\mathcal{A}$ with $\varphi(p_i) = f(p_i)$ for all i.

Equivalence

Theorem: A degree $\boldsymbol{d}$ is tree-definable iff there is a computable $\mathcal{A}$ with $\mathrm{AutSpec}^*(\mathcal{A}) = \{\boldsymbol{d}\}$.

($\Longrightarrow$) follows from a construction by Morozov, 1993. (Cf. *Handbook of Recursive Mathematics*, vol. 1.)

Since all degrees $\mathbf{0}^{(\alpha)}$ with $\alpha < \omega_1^{CK}$ are tree-definable, this yields:
Corollary: For every $\alpha < \omega_1^{CK}$, there exists a computable $\mathcal{A}$ with $\mathrm{AutSpec}^*(\mathcal{A}) = \{\mathbf{0}^{(\alpha)}\}$.

Outside the Bubbles

Recall that each of the following degrees

forms a singleton automorphism spectrum.
What about arithmetic degrees outside this set?

New Automorphism Spectrum

Theorem: There exists a tree-definable Turing degree $\boldsymbol{d} \leq \mathbf{0}''$ which is incomparable with $\mathbf{0}'$.

Corollary: There exists a computable $\mathcal{A}$ whose sole nontrivial symmetry is arithmetical but incomparable with $\mathbf{0}'$.

Proof of theorem builds a computable $T \subseteq \omega^{<\omega}$ with unique path p satisfying:

$$\mathcal{R}_e : p \neq \Phi_e^{\emptyset'}$$

$$\mathcal{N}_e : \emptyset' \neq \Phi_e^p.$$

Satisfying $\mathcal{R}_e$

At some node ρ_e on p we make $p \neq \Phi_e^{\emptyset'}$. Let $n = |\rho_e|$ and start building T with $p(n) = d_e$. Each time $\Phi_{e,s}^{\emptyset'_s}(n)\downarrow$ at some s, pick a new large $y \neq \Phi_{e,s}^{\emptyset'_s}(n)$ and switch the construction so that $p(n) = y$. If later $\emptyset'$ changes on the use of this computation, switch back to $p(n) = d_e$.

So exactly one node above ρ_e is active at infinitely many stages.

Satisfying $\mathcal{N}_e$

To make $\emptyset' \neq \Phi_e^p$, use a Sacks preservation strategy at a node ν_e on p:

$$l(\nu_e, s) = \max\{x : (\forall y < x)\Phi^p_{e,s}(y)\downarrow = \emptyset'_s(y)\}$$
$$r(\nu_e, s) = \max\{\text{use}(\Phi^p_{e,s}(y)) : \ y \leq l(\nu_e, s)\}.$$

So if $\Phi_e^p = \emptyset'$, then $l(\nu_e, s) \to \infty$ and we would be able to compute $\emptyset'$.

For each $\nu_e \subset p$, T contains only one extension of ν_e up to the level $\lim l(\nu_e, s)$. (That extension will be ρ_{e+1}.) So a $\mathbf{0}''$-oracle can compute p, by finding the unique node at each level which is actively infinitely often.

Every time we add a new node σ to T_s, we choose $\sigma(|\sigma| - 1) > s$. So $T = \bigcup_s T_s$ is computable.

Outside Higher Bubbles

Corollary: For each $n \in \omega$, there exists a computable $\mathcal{A}$ with $\text{AutSpec}^*(\mathcal{A}) = \{\boldsymbol{d}\}$, where $\mathbf{0}^{(n)} \leq \boldsymbol{d} \leq \mathbf{0}^{(n+2)}$ but $\boldsymbol{d}$ is incomparable with $\mathbf{0}^{(n+1)}$.

Proof: Relativize the construction above to build $\mathcal{B} \leq \mathbf{0}^{(n)}$ with this automorphism spectrum; then apply Marker's technique to $\mathcal{B}$.

Known Automorphism Spectra

- Singletons $\{\boldsymbol{d}\}$: $\mathbf{0}^{(n)} \leq \boldsymbol{d} \leq \mathbf{0}^{(n+1)}$, $\boldsymbol{d} = \mathbf{0}^{(\alpha)}$, one $\boldsymbol{d}$ outside each bubble.
- All closures under join of finite unions of existing automorphism spectra.
- One set of three pairwise-incomparable degrees $\leq_T \mathbf{0}'$.
- All upward closures of existing automorphism spectra.
- The closure under join of $\{\deg(W_e) : W_e \in E\}$, for any uniformly c.e. family E of c.e. sets; likewise for properly Σ_{n+1} sets for other n.
- The set of all proper Σ_{n+1}-degrees; also the union of these for all n.