Revisiting Uniform Computable Categoricity: For the Sixtieth Birthday of Prof. Rod Downey

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Abstract

Inspired by recent work of Csima and Harrison-Trainor and of Montalbán in relativizing the notion of degrees of categoricity, we return to uniform computable categoricity, as described in work of Downey, Hirschfeldt and Khoussainov. Our attempt to integrate these notions together leads to certain new questions about relativizing the concept of the jump of a structure, as well as to an idea of the structural information content of a countable structure, i.e., that information which can be recovered uniformly from copies of the structure.

1 Rod

For certain mathematicians, a sixtieth-birthday conference is mainly an opportunity to reflect on the body of their work and to start to view it as a whole. This is particularly true if one believes them to have mostly completed that work. Rod Downey, on the other hand, shows no signs whatsoever of slowing down, and one can hardly think of his oeuvre as completed when he keeps on churning out one paper after another. For Rod's sixtieth birthday, therefore, it seems more appropriate to try to create a present to give him. Once again, this is no easy task. However, the recent work of Csima and Harrison-Trainor on degrees of categoricity "on a cone" suggested connections to work by Rod, joint with Denis Hirschfeldt and Bakh Khoussainov

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in 2003, on uniform versions of computable categoricity. This paper is an attempt to integrate those two concepts together: the goal is not necessarily to produce a fully formed result, but rather to inspire questions which can serve as a birthday present, giving Rod and others something to play with. As with any birthday present, the author felt the need to play with it a bit himself first – just to test it out, of course – and so some theorems will be stated, along with examples, but even these serve mainly to illustrate the important points and to raise further questions, rather than to resolve them.

Happy birthday, Rod!

2 Introduction

The notion of *computable categoricity* has become absolutely standard in computable model theory. A computable structure $\mathcal{A}$ is computably categorical if every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. This does not mean that all isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ need be computable, of course; but it implies that, to determine whether or not $\mathcal{A}$ is isomorphic to an arbitrary computable structure $\mathcal{C}$, one need look no further than the computable functions to determine whether an isomorphism exists.

Computable model theorists have modified this definition in a number of ways. *Uniform computable categoricity* was promulgated, in two different versions, by Downey, Hirschfeldt, and Khoussainov in [5], and earlier by Kudinov [9, 10] and Ventsov [18]. For this property, we require not only that the computable isomorphism between the computable isomorphic structures $\mathcal{A}$ and $\mathcal{B}$ must exist, but that there must be an effective method of finding it. The main version demands a Turing functional Γ which, given the (computable) atomic diagrams of $\mathcal{A}$ and $\mathcal{B}$ as an oracle, always computes an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$; this version is equivalent (and very similar) to our Definition 3.1 below. A weaker version, in [5] and [9, 10], demands a computable function f which, given any i and j such that φ_i and φ_j compute the atomic diagrams of $\mathcal{A}$ and $\mathcal{B}$, outputs the index e of a computable isomorphism φ_e from $\mathcal{A}$ onto $\mathcal{B}$. One of the surprises of [5] was that these notions turned out to be distinct: the first one always implies the second, of course, but not vice versa.

Relative computable categoricity of $\mathcal{A}$ broadens the original definition in a different way, by extending it to all structures $\mathcal{B}$ on the domain ω, whether or not they are computable. Of course, requiring a computable isomorphism to map the computable structure $\mathcal{A}$ onto a noncomputable $\mathcal{B}$ would be untenable. Rather, we say that $\mathcal{A}$ is relatively computably categorical if, for every $\mathcal{B}$ with domain ω which is isomorphic to $\mathcal{A}$, there exists a $\mathcal{B}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. (It follows that, for every $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$, there is a $(\mathcal{B}\oplus\mathcal{C})$-computable isomorphism from $\mathcal{B}$ onto $\mathcal{C}$.) This version has been shown, in [1] and independently in [2], to be equivalent to a syntactic characterization using computable infinitary formulas, the *Scott family*, which we describe below. Moreover, relative computable categoricity of $\mathcal{A}$ is equivalent to the existence of a finite tuple of elements $\vec{a}$ from $\mathcal{A}$ such that $(\mathcal{A}, \vec{a})$ is uniformly computably categorical. However, it was soon shown, for instance in [9], that a computable structure can be computably categorical without being relatively computably categorical.

Finally, for computable structures which fail these criteria, we can ask how close they come to satisfying them. For example, for a computable ordinal α, a computable structure $\mathcal{A}$ is said to be *relatively* $\Delta_{1+\alpha}$*-categorical* if, for every $\mathcal{B}$ isomorphic to $\mathcal{A}$ with domain ω, there exists an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which is computable from the α-th jump of the degree of $\mathcal{B}$. (The irritating use of $(1+\alpha)$ is necessary to make this definition work for both finite and infinite ordinals α.) This too has a very pleasing syntactic characterization, by computable enumerability of a Scott family of $\Sigma^c_{\alpha+1}$-formulas. Plain $\Delta_{1+\alpha}$*-computable categoricity* is defined by analogy, restricting the relative definition to computable structures $\mathcal{B}$ only, and under this restriction to computable structures, a further generalization is explored in [7]: $\boldsymbol{d}$*-computable categoricity*, in which all computable copies $\mathcal{B}$ of $\mathcal{A}$ are required to have isomorphisms from $\mathcal{A}$ which are computable from the Turing degree $\boldsymbol{d}$. When $\boldsymbol{d} = \mathbf{0}^{(\alpha)}$, this is just $\Delta_{1+\alpha}$-computable categoricity, but the generalization to the relative version does not work smoothly when $\boldsymbol{d}$ is not of the form $\mathbf{0}^{(\alpha)}$.

The work [7] explored the possibility of a computable structure having a specific *degree of categoricity*, i.e., having a least degree $\boldsymbol{d}$ such that $\mathcal{A}$ is $\boldsymbol{d}$-computably categorical. Degrees of categoricity were shown there to include all c.e. and d.c.e. degrees, as well as degrees of the form $\mathbf{0}^{(\alpha)}$ with $\alpha \leq \omega$. The results there were nicely extended in [3], to all $\alpha < \omega_1^{CK}$, but the papers which largely inspired our approach here were [4] and [14]. In the first of these, Csima and Harrison-Trainor showed that every computable structure has a specific level of categoricity: relative to some fixed degree $\boldsymbol{d}$, its degree of categoricity is precisely some jump $\boldsymbol{d}^{(\alpha)}$ of $\boldsymbol{d}$. (In their language, a structure will have degree of categoricity $\mathbf{0}^{(\alpha)}$ *on the cone above* $\boldsymbol{d}$, i.e., with all definitions relativized to $\boldsymbol{d}$.) The results here through Section 3, and some of those beyond that section, are mostly implicit in their work and [14], if not explicitly stated there. Our goal, in addition to calling attention to their work, is to show how it can be integrated together with the notions of uniform computable categoricity.

With such a glut of definitions on hand, the newcomer to the subject may feel somewhat dazed. Nevertheless, each of these definitions arises out of reasonable questions. Here, to justify extending one of these definitions even further (below), we offer an example of a shortcoming in the foregoing catalogue, using two computable fields E and F.

Our E is well-known: it is simply the algebraic closure of the purely transcendental extension $\mathbb{Q}(t_0, t_1, \ldots)$ of the rational numbers. Thus, E is the unique countable algebraically closed field of characteristic 0 with infinite transcendence degree over its prime field, and this field is well-known to be computably presentable. Ershov was the first to show that E is not computably categorical (see [6]). Indeed, there are computable presentations in which the algebraic dependence set

$$\{(x_0, \ldots, x_n) \in E^{<\omega} : (x_0, \ldots, x_n) \text{ is algebraically dependent over } \mathbb{Q}\}$$

can have arbitrary computably enumerable Turing degree, whereas a computable isomorphism between computable copies of E must preserve the Turing degree of this set. E is relatively Δ_2-categorical, however, since, for an arbitrary copy K of E, one can use a $(\deg(K))'$-oracle to pick out a transcendence basis in K and another in E (since $\mathbf{0}' \leq (\deg(K))'$), and every bijection between these bases extends effectively to an isomorphism from K onto E.

Our F requires a little more description, and uses the computably enumerable set $\emptyset'$, the Halting Problem. Let $p_0 < p_1 < \cdots$ enumerate the prime numbers $2 < 3 < \cdots$. F contains two square roots (arbitrarily named $\pm\sqrt{p_n}$) of each prime p_n. We now give a simplified version of the process for one number n. First, F also contains a square root of $+\sqrt{p_n}$. If $n \in \emptyset'$, then we adjoin a fourth root of $-\sqrt{p_n}$; in this case, both of $\pm\sqrt{p_n}$ have square roots, of course, but $+\sqrt{p_n}$ has no fourth root in F. If $n \notin \emptyset'$, then no such fourth root is ever adjoined, so $+\sqrt{p_n}$ has a square root of its own, whereas $-\sqrt{p_n}$ does not. So in both cases, the elements $\pm\sqrt{p_n}$ are in distinct orbits under automorphisms of F, but the reason for the distinction depends on whether $n \in \emptyset'$ or not.

Unfortunately, this exact procedure cannot be used for every n: once a square root of $-\sqrt{p_m}$ has been adjoined for one m, F will contain a square root of -1, and therefore any subsequent square root of any other $+\sqrt{p_n}$ would generate a square root of $-\sqrt{p_n}$ as well. However, one can follow the same plan used in [11], to give a process which accomplishes the same purpose for each single prime p_n without any interference between them. Start by adjoining $\pm\sqrt{p_n}$ to F for every n, and use the polynomials given in [11, Prop. 2.15] to "tag" them, as follows. First, picking one polynomial h_n (of a new prime degree) from that proposition, adjoin one root of $h_n(+\sqrt{p_n}, Y)$ to F. Then, if ever m enters $\emptyset'$, adjoin a root of $h_n(-\sqrt{p_n}, Y)$ to F; moreover, pick a new polynomial g_n (just like h_n, but of a new prime degree) from the proposition, and adjoin one root of $g_n(-\sqrt{p_n}, Y)$ to F. As long as all these g_n and h_n are chosen with distinct prime degrees, no extraneous roots of any of them will ever appear, as shown in Proposition 2.15 of [11], and so the procedure here will succeed. The root of $h_n(+\sqrt{p_n}, Y)$ is called the "initial tag" of $+\sqrt{p_n}$. If later n enters $\emptyset'$, the root of $h_n(-\sqrt{p_n}, Y)$ is the "balancing tag," and then the root of $g_n(-\sqrt{p_n}, Y)$ is the "secondary tag" of $-\sqrt{p_n}$.

Since $\emptyset'$ is computably enumerable, one can give a computable presentation of F in exactly this manner. However, there is another computable presentation $\widetilde{F} \cong F$ (in which we name the primes $\tilde{p}_n$, for clarity). Here again $+\sqrt{\tilde{p}_n}$ always has two square roots of its own, but if $n \in \emptyset'$, we adjoin both the initial tag and the secondary tag to $+\sqrt{\tilde{p}_n}$, with $-\sqrt{\tilde{p}_n}$ having only a balancing tag in $\widetilde{F}$. Therefore, the two fields are isomorphic, but each isomorphism f from F onto $\widetilde{F}$ must satisfy

$$f(+\sqrt{p_n}) = \begin{cases} +\sqrt{\tilde{p}_n}, & \text{if } n \notin \emptyset'; \\ -\sqrt{\tilde{p}_n}, & \text{if } n \in \emptyset'. \end{cases}$$

It follows that every such isomorphism f computes $\mathbf{0}'$.

On the other hand, this field F is relatively Δ_2-categorical. Given any field K isomorphic to F, we can use a $(\deg(K))'$-oracle to compute $\emptyset'$. Then, for each $n \in \emptyset'$, we wait until a secondary tag of one of $\pm\sqrt{p_n}$ appears in K. When we find it, we map it to the secondary tag of $-\sqrt{p_n}$ in F. For each $n \notin \emptyset'$, no secondary tags of $\pm\sqrt{p_n}$ will ever appear, and we simply find an initial tag of one of $\pm\sqrt{p_n}$ in K and map it to the initial tag of $+\sqrt{p_n}$ in F. Since these elements generate all of K, we can now extend our isomorphism effectively to all of K, proving relative Δ_2-categoricity.

None of the flavors of categoricity we have mentioned so far distinguishes E from F. Nevertheless, the proofs given here should feel different from each other: for E, the proof of relative Δ_2-categoricity made real use of the $(\deg(K))'$-oracle, whereas the proof for F only used this oracle to compute $\emptyset'$. To address this difference, in the next section, we will define yet another version of categoricity, which will distinguish these two situations. In essence it is the same definition used in [5], only allowing noncomputable structures as well as computable ones, as well as generalizing to consider $\Delta_{1+\alpha}$-categoricity for $\alpha > 0$. We believe it will strike the reader as a natural uniform version of the concept of effective categoricity.

3 Uniformly Computable Categoricity

The rationale behind the original definition of computable categoricity is standard in computable model theory, and has been used to define effective versions of many completely separate concepts as well. Roughly speaking the situation is this: we would like to investigate how difficult it is to compute isomorphisms among copies of the structure $\mathcal{A}$. Of course, the answer may be arbitrarily difficult, since (by a result of Knight in [8]) the copies of $\mathcal{A}$ themselves may be extremely difficult to compute, assuming that $\mathcal{A}$ satisfies a simple condition called automorphic non-triviality. In order to make the question about complexity of isomorphisms manageable, therefore, we restrict it: under the assumption that the copy $\mathcal{B}$ (and $\mathcal{A}$ itself) are computable structures, we ask how difficult it is to compute an isomorphism between them. This allows us to leave the structural complexity of $\mathcal{B}$ out of the question, and to focus on the difficulty of computing the isomorphisms themselves. (Requiring $\mathcal{B}$ to have domain ω is a similar restriction: it stops us from using the domain itself to encode complexity into $\mathcal{B}$. In this paper we will be able to continue to require all structures to have domain ω.)

A great deal of intriguing mathematics has arisen out of this original definition of computable categoricity, and it is certainly not our intention to disparage it. However, by reframing the question, we will be able to address the shortcoming exemplified by the example above with the fields E and F. In the definition below, we do not attempt to exclude any complexity from $\mathcal{B}$; instead, we assume that we have access to the entire atomic diagram of $\mathcal{B}$, no matter how complex it may be. The basic version of this definition was given in [5] and is shown there to be equivalent to their notion of uniform computable categoricity, and also (modulo use of parameters) to relative computable categoricity. Here we generalize first by adding an oracle X, and then (in Definition 3.3 below) by considering Δ_α-categoricity.

Definition 3.1 In a computable language $\mathcal{L}$ with equality, a countable infinite $\mathcal{L}$-structure $\mathcal{A}$ is *uniformly computably categorical* if there exists a Turing functional Φ such that, for every pair of structures $\mathcal{B}$ and $\mathcal{C}$ both isomorphic to $\mathcal{A}$ (and with domains $\subseteq \omega$), the function

$$\Phi^{\mathcal{B}\oplus\mathcal{C}} : \omega \to \omega$$

defines an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$. More generally, for a subset $X \subseteq \omega$, $\mathcal{A}$ is *deg*(X)*-uniformly categorical* if there is some Φ such that, in the situation above,

$$\Phi^{X\oplus\mathcal{B}\oplus\mathcal{C}} : \omega \to \omega$$

always defines an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$. (Clearly this same property then holds of all sets $Y \geq_T X$.)

Finally, if there exists an $X \subseteq \omega$ for which the preceding holds, then we will call $\mathcal{A}$ *continuously categorical*, since the categoricity is witnessed by isomorphisms given continuously in the copies of $\mathcal{A}$.

Here the oracles $\mathcal{B}$ and $\mathcal{C}$ stand for the atomic diagrams of the structures, under some coding into ω of all atomic formulas in the language $\mathcal{L} \cup \{c_0, c_1, \ldots\}$ with a new constant c_n for each $n \in \omega$. We have momentarily allowed $\mathcal{B}$ and $\mathcal{C}$ to have domains $\subseteq \omega$, but this is immediately rectified: we have $n \in \mathrm{dom}(\mathcal{B})$ if and only if the formula $c_n = c_n$ lies in the atomic diagram of $\mathcal{B}$, and so we can decide the domain from the $\mathcal{B}$-oracle, and likewise for $\mathcal{C}$. With $\mathcal{A}$ being countably infinite, therefore, we will hereafter assume all structures to have domain ω.

Notice that this notion immediately distinguishes the fields E and F. F is $\emptyset'$-uniformly categorical, since the method given in the previous sections for computing an isomorphism onto F from an arbitrary copy $\mathcal{B}$ requires only $\emptyset'$ and $\mathcal{B}$ as oracles. On the other hand, E, the algebraically closed field of infinite transcendence degree over $\mathbb{Q}$, cannot be continuously categorical, no matter what oracle set X is used. It is not difficult to use Ershov's method, relativized to any X, to produce two X-computable copies $\mathcal{B}$ and $\mathcal{C}$ of E, one with an X-computable algebraic dependence set and the other without, and clearly no $\Phi^{X\oplus\mathcal{B}\oplus\mathcal{C}}$ could compute an isomorphism between them. Indeed, one can make the second copy have algebraic dependence set Turing equivalent to X', so X' is the degree of categoricity for X-computable copies.

One's intuition that categoricity of the field E requires precisely one jump – equivalently, one quantifier – over the atomic diagram is justified by its relative Δ_2-categoricity (along with the comments above). Indeed, relative Δ_2-categoricity without parameters will be exactly equivalent to the natural extension we now give of Definition 3.1. Recall first the definition of the *jump* of a structure $\mathcal{A}$, which was established by general agreement after initial work by Montalbán [12] and by Soskov and Soskova [17]. (From now on, in our notation, Σ^c_α denotes the set of computable infinitary formulas of complexity Σ_α.)

Definition 3.2 For a countable structure $\mathcal{A}$ in a language $\mathcal{L}$, the *jump of* $\mathcal{A}$ is another structure $\mathcal{A}'$ with the same domain, functions, relations, and constants as $\mathcal{A}$, but in an expanded language $\mathcal{L}'$. This $\mathcal{L}'$ contains an additional n-ary predicate R_φ for each infinitary Σ^c_1-formula φ in the free variables $v_1, \ldots, v_n$ (for all n), and

$$\models_{\mathcal{A}'} R_\varphi(a_1, \ldots, a_n) \iff \models_{\mathcal{A}} \varphi(a_1, \ldots, a_n).$$

This jump operation iterates through the computable ordinals. At a limit ordinal α, the result is a structure $\mathcal{A}^{(\alpha)}$ with reduct $\mathcal{A}$ in $\mathcal{L}$, but with predicates for all infinitary Σ^c_α $\mathcal{L}$-formulas (i.e., all infinitary Σ^c_β $\mathcal{L}$-formulas for all $\beta < \alpha$).

With this definition, it is now natural to extend continuous categoricity as follows. We use the ordinal $1 + \alpha$ here in order to accommodate the existing system of nomenclature: Δ_2-*categorical* means that the first jump $\mathcal{A}^{(1)}$ is computably categorical, whereas Δ_ω-*categorical* means that $\mathcal{A}^{(\omega)}$ is computably categorical.

Definition 3.3 A countable structure $\mathcal{A}$ is X-*uniformly* $\Delta_{1+\alpha}$-*categorical* if its α-th jump $\mathcal{A}^{(\alpha)}$ is X-uniformly categorical. If an $X \subseteq \omega$ exists for which this holds, then $\mathcal{A}$ is *continuously* $\Delta_{1+\alpha}$-*categorical.*

It is quickly seen that the field E is uniformly (i.e., $\emptyset$-uniformly) Δ_2-categorical, using the same argument as for relative Δ_2-categoricity. Of course, F is also uniformly Δ_2-categorical; the distinction between E and F occurs with the stronger notion of $\emptyset'$-uniform categoricity, as seen earlier.

One naturally asks, given a structure $\mathcal{A}$, for the smallest ordinal α such that $\mathcal{A}$ is continuously Δ_α-categorical. This question – and also the question of the existence of such an α – is readily addressed, using the existing notion of the Scott rank of a structure.

Definition 3.4 The *computable Scott rank* of a countable $\mathcal{L}$-structure $\mathcal{A}$ is the least ordinal $\alpha > 0$ such that, for every finite tuple $(a_1, \ldots, a_n)$ from $\mathcal{A}$, there exists a computable infinitary Σ^c_α $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$ for which, for all tuples $\vec{b} \in \mathcal{A}^n$,

$$\models_{\mathcal{A}} \varphi(\vec{b}) \iff (\exists f \in \mathrm{Aut}(\mathcal{A}))(\forall i \leq n) f(a_i) = b_i.$$

(That is, φ defines an orbit of n-tuples under the action of $\mathrm{Aut}(\mathcal{A})$.) A set $\mathfrak{F}$ of Σ^c_β-formulas all satisfying this condition, such that every tuple from $\mathcal{A}^{<\omega}$ realizes at least one formula in $\mathfrak{F}$, is called a *Scott family* (of *rank* β) for $\mathcal{A}$.

The *absolute Scott rank* of $\mathcal{A}$ is defined the same way, but with Σ^c_α replaced by Σ_α. That is, absolute Scott rank allows any $L_{\omega_1\omega}$ formula to be used, whether or not it is computable.

We use the term *computable* Scott rank to emphasize that we only allow computable infinitary formulas. In Section 5, we will present some examples and questions regarding the use of arbitrary $L_{\omega_1\omega}$ formulas. It should be noted that several distinct definitions of Scott rank exist, and they do not all define the same ordinal for a single $\mathcal{A}$. Computable Scott rank is based on our needs here: it requires the individual formulas to be computable, but the family $\mathfrak{F}$ need not be given effectively. There is a connection: if $\mathcal{A}$ has an X-computably enumerable Scott family of rank α, and $X \leq_T \emptyset^{(\beta)}$, then $\mathcal{A}$ has a computably enumerable Scott family whose rank is $\max(\alpha, \beta + 1)$, built by folding the definition of X into the new Scott family.

Proposition 3.5 *Suppose that a computable structure $\mathcal{A}$ has computable Scott rank $\alpha + 1$, and that some Scott family of $\Sigma^c_{\alpha+1}$ formulas for $\mathcal{A}$ is X-computably enumerable. Then $\mathcal{A}$ is X-uniformly $\Delta_{1+\alpha}$-categorical.*

It is both important and difficult to get the indices correct here. First, for $\alpha = n \in \omega$, uniform Δ_n-categoricity corresponds to a Scott family of Σ^c_n formulas (since Δ_n means that we are given the $(n-1)$-st jump $\mathcal{A}^{(n-1)}$). However, in the case $\alpha = \omega$, uniform Δ_ω-categoricity means that we can compute an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, given the atomic diagrams of $\mathcal{A}^{(\omega)}$ and $\mathcal{B}^{(\omega)}$, i.e., given the Σ^c_n-diagrams of $\mathcal{A}$ and $\mathcal{B}$ uniformly for all n. Now a $\Sigma^c_{\omega+1}$-formula $\varphi(x)$ is an effective disjunction over k of formulas $\exists \vec{y}\, \psi_k(x, \vec{y})$, with each ψ_k in Π_ω. This means that, uniformly in k and $(a, \vec{a})$, we can decide whether $\models_{\mathcal{A}} \psi_k(a, \vec{a})$, and so, given $a \in \mathcal{A}$, we can find a formula φ with $\models_{\mathcal{A}} \varphi(a)$ in the X-c.e. Scott family of $\Sigma^c_{\omega+1}$ formulas. This sets up the usual argument for Scott families and categoricity, and so uniform Δ_ω-categoricity corresponds to a Scott family of $\Sigma^c_{\omega+1}$ formulas, just as stated in the Proposition with $\alpha = \omega$. This correspondence continues from $\alpha = \omega$ on up through the hyperarithmetical hierarchy: with the atomic diagram of $\mathcal{A}^{(\omega+1)}$, one can enumerate the $\Sigma^c_{\omega+2}$-statements true in $\mathcal{A}$, and thus use a Scott family of $\Sigma^c_{\omega+2}$-formulas to build an isomorphism, and so on. The Proposition states this for all α in one fell swoop, since $1 + \alpha = \alpha$ for $\alpha \geq \omega$ and $1 + \alpha = \alpha + 1$ for $\alpha < \omega$.

Proof. This is the standard use of Scott families to demonstrate categoricity. The Turing functional Φ, with oracle $X \oplus \mathcal{B}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}$, uses X to enumerate a Scott family for $\mathcal{A}$ until it finds a formula $\varphi(v_1)$ and atomic facts about $\mathcal{B}^{(\alpha)}$ (that is, $\Delta_{1+\alpha}$-facts about $\mathcal{B}$) showing that $\varphi(0)$ holds in $\mathcal{B}$. Then it searches in $\mathcal{C}^{(\alpha)}$ to find a y_0 and a tuple witnessing that $\varphi(y_0)$ holds in $\mathcal{C}$. With $\mathcal{A} \cong \mathcal{B} \cong \mathcal{C}$, the definition of Scott family shows that this search will eventually succeed, and when it does, Φ defines $\Phi^{X\oplus\mathcal{B}^{(\alpha)}\oplus\mathcal{C}^{(\alpha)}}(0) = y_0$. Next it goes backwards, finding a formula in the Scott family which holds in $\mathcal{C}$ of the tuple $(y_0, 0)$, and then finding an $x_0 \in \mathcal{B}$ such that the same formula holds of $(0, x_0)$. (If $y_0 = 0$, then $x_0 = 0$, of course.) Setting $\Phi^{X\oplus\mathcal{B}^{(\alpha)}\oplus\mathcal{C}^{(\alpha)}}(x_0) = 0$, it then proceeds to the tuple $(0, x_0, 1)$ from $\mathcal{B}$, and so on, by a back-and-forth procedure which ensures that $\Phi^{X\oplus\mathcal{B}^{(\alpha)}\oplus\mathcal{C}^{(\alpha)}}$ will be bijective and will be an isomorphism. ∎

For $\alpha = 0$, the converse also holds.

Proposition 3.6 *Suppose that a computable structure $\mathcal{A}$ is X-uniformly categorical. Then $\mathcal{A}$ has an X-computably enumerable Scott family of Σ^0_1 formulas.*

Proof. This follows from the methods used in [5], relativized to the degree of X. Notice that the proof there requires $\alpha = 0$: it does not consider Δ_2-categoricity or higher. Also, we can take the formulas in the Scott family to be finitary, so there is no need to worry that an individual formula might require an X-oracle to list out its disjuncts. ∎

One might expect this converse also to hold when $\alpha > 0$. It does, but we will first consider the example and the notions in Section 4. Before continuing there, we note that, in light of Propositions 3.5 and 3.6, it is more reasonable to define X-uniform categoricity for enumeration degrees, rather than for

Turing degrees. All we need is an enumeration of the Scott family, and so, if we can enumerate a Scott family from an enumeration of X, then we can do the same for every set in the enumeration degree of X. A natural question now arises.

Question 3.7 *Suppose a computable structure $\mathcal{A}$ has computable Scott rank α, as defined above: it has a Scott family of Σ^c_α-formulas, and α is least with this property. Is there a least enumeration degree $\boldsymbol{c}$ such that $\mathcal{A}$ is $\boldsymbol{c}$-continuously Δ_α-categorical? The e-degree of the Scott family itself is the obvious candidate for $\boldsymbol{c}$; the question really asks whether $\mathcal{A}$ could have another Scott family of Σ^c_α-formulas whose e-degree is incomparable (under $\leq_e$) with this $\boldsymbol{c}$.*

An analogous question could be asked in the next section for structures $\mathcal{A}$ which are countable but not computably presentable.

4 Continuity for Spectra of Structures

To see how the questions and definitions above lead into questions about spectra of structures, we now introduce another countable field L. Notice first that Definition 3.1 applies to all countable structures, not just computable ones, and indeed our L will have no computable presentation. It is simplest to view L as a sort of reverse of F, with the roles of $\emptyset'$ and its complement $\overline{\emptyset'}$ interchanged. Like F, L contains two square roots $\pm\sqrt{p_n}$ of each prime number p_n, and also contains an initial tag of $+\sqrt{p_n}$ for every n. For those $n \notin \emptyset'$, we adjoin to L a balancing tag of $-\sqrt{p_n}$, and also a secondary tag of $-\sqrt{p_n}$. It follows that, with an oracle for an arbitrary presentation of L, we could enumerate $\overline{\emptyset'}$, simply by enumerating those n for which either of $\pm\sqrt{p_n}$ has a secondary tag, and thus could compute $\emptyset'$. Indeed, the Turing degree spectrum of L is precisely the upper cone above (and including) $\mathbf{0}'$.

The reason why L upsets our ideas about continuous categoricity is that, whereas F was only $\emptyset'$-uniformly categorical, L is uniformly computably categorical. To see this, suppose that $\widetilde{L}$ is an arbitrary copy of L with domain ω. For each n, we wait until either n enters $\emptyset'$ or a secondary tag of $\pm\sqrt{p_n}$ appears in $\widetilde{L}$. If $n \in \emptyset'$, we find an initial tag of one of $\pm\sqrt{\widetilde{p}_n}$ in $\widetilde{L}$ and map it to the corresponding initial tag in L (and map this $\pm\sqrt{\widetilde{p}_n}$ itself to $+\sqrt{p_n}$ in L). On the other hand, if we find a secondary tag $\tilde{x}$ of $\pm\sqrt{\widetilde{p}_n}$

in $\widetilde{L}$, then we wait for a secondary tag of $-\sqrt{p_n}$ to appear in L, and map $\tilde{x}$ to that secondary tag. In this case, the initial tag of $\pm\sqrt{\tilde{p}_n}$ in $\widetilde{L}$ will have be balanced by a tag of $\mp\sqrt{\tilde{p}_n}$, so we can find the balancing tag and map it to the balancing tag of $+\sqrt{p_n}$ in L. All of this can be determined from our oracles (the atomic diagrams of L and $\widetilde{L}$), and the map thus defined extends uniquely to an isomorphism from $\widetilde{L}$ onto L, since we have defined it on a generating set for $\widetilde{L}$. This proves the uniform computable categoricity of L.

This result does not contradict any previous statements, but it explains why the hypothesis of computable presentability of $\mathcal{A}$ was included in Propositions 3.5 and 3.6. Indeed, while L itself is not computably presentable, it does have a c.e. Scott family of Σ_1 formulas. For the elements $\pm\sqrt{p_n}$, this family includes all formulas saying that $\sqrt{p_n}$ has a secondary tag; it also includes, for those $n \in \emptyset'$, the formula saying that $\sqrt{p_n}$ has an initial tag. If $n \in \emptyset'$, then the formula saying that $\sqrt{p_n}$ has a secondary tag is never realized in L, and hence could have been eliminated from the Scott family, but in this case the family would no longer be c.e. On the other hand, even with these formulas included, the Scott family still allows computation of isomorphisms; unrealizable formulas clutter up the process but do not disrupt it.

We note, without going into details here, that the same process could be used with other sets in place of $\overline{\emptyset'}$. For instance, let A and B be Turing-incomparable c.e. sets. If a field J has initial tags of $+\sqrt{p_n}$ for every $n \in A \oplus \overline{B}$, and has balancing tags and secondary tags for every $n \in \overline{A} \oplus B$, then the degree spectrum of J is the upper cone above $\deg(A)$, and J has a B-c.e. Scott family of Σ_1^c formulas.

Finally, we combine two of our examples. Let $\mathcal{A}$ be the cardinal sum of the fields L (above) and F (from Section 2). That is, $\mathcal{A}$ is the disjoint union of these two structures, in the language of fields with one additional unary predicate R which holds of all elements of L but of no elements of F. So $\mathrm{Spec}(\mathcal{A}) = \mathrm{Spec}(F) \cap \mathrm{Spec}(L)$, which is the upper cone above $\mathbf{0}'$, and we get a Scott family $\mathfrak{F}$ of Σ_1^c formulas for $\mathcal{A}$ essentially just by taking the union of the Scott families for F and L, with obvious adjustments involving R. This $\mathfrak{F}$ is c.e. in $\emptyset'$ but not in any smaller or incomparable degree; indeed, $\mathfrak{F} \equiv_e \overline{\emptyset'}$. However, $\mathcal{A}$ is uniformly computably categorical! The process for computing isomorphisms between arbitrary copies of $\mathcal{A}$ is to use the L-side of one of the copies to enumerate $\overline{\emptyset'}$, and then to use that enumeration to enumerate $\mathfrak{F}$. Ultimately, therefore, the uniform computable categoricity of this $\mathcal{A}$ follows from the e-reduction $\mathfrak{F} \leq_e \mathrm{Th}_{\Sigma_1}(\mathcal{A})$.

Nevertheless, Proposition 4.1 and Theorem 4.3 below do *not* require an e-reduction such as $\mathfrak{F} \leq_e \mathrm{Th}_{\Sigma_1}(\mathcal{A})$; they actually show that our $\mathcal{A}$ has a c.e. Scott family of Σ_1-formulas. This family is not the $\mathfrak{F}$ described above; instead, it integrates the e-reduction $\mathfrak{F} \leq_e \mathrm{Th}_{\Sigma_1}(\mathcal{A})$ into its formulas. For each n, the new Scott family has one formula saying

if $n \in \emptyset'$, then use the appropriate $\exists$-formula on F,

and another one saying

if there exists a configuration in L showing that $n \notin \emptyset'$,
then use the other kind of $\exists$-formula on F.

So, somewhat surprisingly, we may extend Propositions 3.5 and 3.6 to all countable structures, and give the promised converse for the case $\alpha > 0$, without any use of $\mathrm{Th}_{\Sigma^c_\alpha}(\mathcal{A})$.

Proposition 4.1 *Fix any oracle set $X \subseteq \omega$ and any nonzero $\alpha < \omega_1^{CK}$. A countable structure $\mathcal{A}$ is X-uniformly $\Delta_{1+\alpha}$-categorical if and only if $\mathcal{A}$ has an X-c.e. Scott family of $\Sigma^c_{\alpha+1}$ formulas.*

A more precise statement is possible if we integrate e-reducibility into the notion of uniform Δ_α-categoricity. Proposition 4.1 follows from this version, which we now state as Theorem 4.3 and prove.

Definition 4.2 An *enumeration* of a set $S \subseteq \omega$ is a set $T \subseteq \omega$ such that S is the *projection of* T:

$$S = \mathrm{proj}(T) = \{m \in \omega : (\exists n)\langle m, n\rangle \in T\}.$$

So a set S is $\boldsymbol{d}$-c.e. if and only if it has a $\boldsymbol{d}$-computable enumeration.

Theorem 4.3 *Fix any oracle set $X \subseteq \omega$ and any nonzero $\alpha < \omega_1^{CK}$. A countable structure $\mathcal{A}$ has a Scott family of $\Sigma^c_{\alpha+1}$ formulas which is e-reducible to X if and only if $\mathcal{A}$ satisfies:*

> *There exists a Turing functional Φ such that, for all copies $\mathcal{B} \cong \mathcal{C} \cong \mathcal{A}$ with domain ω and all enumerations Y of X, the function $\Phi^{Y \oplus \mathcal{B}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}}$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$.*

Proof. Suppose first that $\mathcal{A}$ has such a Scott family $\mathfrak{F}$, and fix any Y as described and any $\mathcal{B} \cong \mathcal{A}$. (It is sufficient to show the second statement with $\mathcal{C} = \mathcal{A}$.) With its enumeration of X, Φ applies the given e-reduction to produce an enumeration of the Scott family $\mathfrak{F}$. From here it is standard to compute an isomorphism from $\mathcal{B}$ onto $\mathcal{A}$, going back and forth using the Scott family and the Σ^c_α-diagrams of the two structures.

Now assume Φ is a functional satisfying the second statement. We run Φ simultaneously on input 0 with each binary string $(\sigma \oplus \beta \oplus \gamma) \in 2^{<\omega}$ as oracle; moreover, whenever $\Phi^{\sigma\oplus\beta\oplus\gamma}(0)$ converges, we then run $\Phi^{\sigma\oplus\beta\oplus\gamma}(1)$ until it converges (if ever), then $\Phi^{\sigma\oplus\beta\oplus\gamma}(2)$, and so on. Thus we produce an enumeration of those tuples $(\sigma \oplus \beta \oplus \gamma, \vec{y}) \in 2^{<\omega} \times \omega^{<\omega}$ such that, for all $i < |\vec{y}|$, $\Phi^{\sigma\oplus\beta\oplus\gamma}(i) = y_i$.

Each $(\sigma \oplus \beta \oplus \gamma, \vec{y})$ in this enumeration defines a set

$$X_0 = \{a : (\exists b)\ \sigma(\langle a, b\rangle) = 1\},$$

and yields a strong index for this finite set X_0. It also includes finite initial segments β and γ of the atomic diagrams of the structures $\mathcal{B}^{(\alpha)}$ and $\mathcal{C}^{(\alpha)}$. The convergence of Φ using this oracle means that, whenever $X_0 \subset \mathrm{proj}(Y)$ and the diagram of $\mathcal{B}$ realizes

$$\left(\bigwedge_{\beta(\varphi)=1} \varphi\right) \wedge \left(\bigwedge_{\beta(\varphi)=0} \neg\varphi\right),$$

and likewise for $\mathcal{C}$ and γ, then the map sending each $i < |\vec{y}|$ to y_i extends to an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$. If m is the largest domain element of $\mathcal{B}$ mentioned in the conjunction above, and n the largest domain element of $\mathcal{C}$, then we enumerate into our e-reduction an axiom saying that if $X_0 \subseteq X$, then the following formula (with $x_0, \ldots, x_{|\vec{y}|-1}$ free) is in the Scott family:

$$\begin{aligned}\exists x_{|\vec{y}|} \cdots \exists x_{m+n+1} &\left(\bigwedge_{\beta(\varphi)=1} \varphi(x_0, \ldots, x_m)\right) \wedge \left(\bigwedge_{\beta(\varphi)=0} \neg\varphi(x_0, \ldots, x_m)\right) \\ &\wedge \left(\bigwedge_{\gamma(\psi)=1} \psi^*\right) \wedge \left(\bigwedge_{\gamma(\psi)=0} \neg\psi^*\right) \\ &\wedge \left(\bigwedge_{|\vec{y}|\le i<j\le n} x_i \neq x_j\right) \wedge \left(\bigwedge_{m<i<j\le m+1+n} x_i \neq x_j\right)\end{aligned}$$

Here, if ψ is a sentence mentioning the domain elements $0, \ldots, n$ of $\mathcal{C}$, ψ^* is the same sentence, but with each domain element y_i replaced by the variable x_i, and with each domain element $j \notin \vec{y}$ replaced by the variable x_{m+1+j}. For φ, the replacements were simpler: each domain element i was replaced by the variable x_i. Thus, the formula defined here is $\Sigma_{\alpha+1}$ and says that $(x_0, \ldots, x_{|\vec{y}|-1})$ satisfies all the existential conditions given by β on $\mathcal{B}$ and all those given by γ on $\mathcal{C}$. (It is likely but not assumed that these conditions repeat each other; there is no danger in including conditions from β even if they are not in γ, or vice versa, but there would be a danger in excluding any of them, since these are the conditions which Φ actually checks before defining its isomorphism.) It is now clear from this definition and the conditions of the theorem that, given any enumeration of X, our e-reduction will enumerate a Scott family of $\Sigma^c_{\alpha+1}$-formulas for $\mathcal{A}$. ∎

In a certain sense, the reason why the field L can be uniformly computably categorical is that $\emptyset'$ is computable in every copy of L, and moreover, that this computation of $\emptyset'$ is uniform across all copies of L. This property wa studied in [8], and we give it a name here, which will only be used until we can demonstrate (in Proposition 4.5 below) its equivalence to a known condition. The reader may wish to try to identify this known condition right now, without skipping ahead to the proposition to peek.

Definition 4.4 A set $S \subseteq \omega$ is *uniformly intrinsically computable* from a countable infinite structure $\mathcal{A}$ if there exists a Turing functional Γ such that, for every $\mathcal{B} \cong \mathcal{A}$ with domain ω, $\Gamma^{\mathcal{B}}$ computes the characteristic function of S.

Likewise, S is *uniformly intrinsically computably enumerable* in $\mathcal{A}$ if there exists a Turing functional Θ such that, for every $\mathcal{B} \cong \mathcal{A}$ with domain ω, the function $\Theta^{\mathcal{B}}$ has domain S.

Clearly S is uniformly intrinsically computable from $\mathcal{A}$ if and only if both S and $\overline{S}$ are uniformly intrinsically c.e. in $\mathcal{A}$. The latter of these two properties will in fact be more natural and relevant; it is the property well-known to the Bulgarian school of computable model theory, where the collection of sets uniformly intrinsically c.e. in $\mathcal{A}$ is called the *co-spectrum of* $\mathcal{A}$. The following result was proven by Knight in [8].

Proposition 4.5 *A set $S \subseteq \omega$ is uniformly intrinsically c.e. in a countable structure $\mathcal{A}$ if and only if S is e-reducible to the existential theory $Th_{\Sigma_1}(\mathcal{A})$ of $\mathcal{A}$.*

Consequently, S is uniformly intrinsically computable in $\mathcal{A}$ if and only if both S and $\overline{S}$ are e-reducible to $Th_{\Sigma_1}(\mathcal{A})$.

Proof. The backwards direction is immediate. With an oracle for any copy of $\mathcal{A}$, we can (uniformly) enumerate $\mathrm{Th}_{\Sigma_1}(\mathcal{A})$, and therefore can enumerate S, uniformly, using its e-reduction to $\mathrm{Th}_{\Sigma_1}(\mathcal{A})$.

For the forwards direction, in order to show that $S \leq_e \mathrm{Th}_{\Sigma_1}(\mathcal{A})$, we enumerate a set Ψ of axioms $(n, \exists\vec{x}\beta(\vec{x}))$ for an e-reduction. Such an axiom represents the instruction "if $\models_{\mathcal{A}} \exists\vec{x}\beta(\vec{x})$, then enumerate n." The nature of the (finitary) Σ_1-theory is such that each axiom need contain only one formula, although e-reductions in general allow us to use a finite conjunction. (One can call our Ψ an *e-reduction of norm* 1.)

Recall the basics. We have a Gödel coding $\gamma \mapsto \ulcorner\gamma\urcorner$ of atomic sentences in the language $\mathcal{L}'$, which is the language of $\mathcal{L}$ extended by new constants $c_0, c_1, \ldots$ representing elements of the domain ω. A $\mathcal{B}$-oracle is simply the subset $\{\ulcorner\beta(c_{i_0}, \ldots, c_{i_n})\urcorner : \models_{\mathcal{B}} \beta(i_0, \ldots, i_n)\}$ of ω, and we know that whenever $\mathcal{B} \cong \mathcal{A}$, $\mathrm{dom}(\Phi^{\mathcal{B}}) = S$.

To build Ψ, we simply run $\Phi_s^{\sigma}(n)$ for all $n, s \in \omega$ and all $\sigma \in 2^{<\omega}$. If this computation halts within the allotted s steps, we enumerate into Ψ the axiom

$$(n, \exists\vec{x}\beta) = \left(n, \exists\vec{x}\left(\left(\bigwedge_{\sigma(\ulcorner\gamma\urcorner)=1} \gamma_{\vec{x}}^{\vec{c}}\right) \wedge \left(\bigwedge_{\sigma(\ulcorner\gamma\urcorner)=0} \neg\gamma_{\vec{x}}^{\vec{c}}\right)\right)\right),$$

where $\gamma_{\vec{x}}^{\vec{c}}$ has x_i substituted for each c_i in γ. This is less complicated than it appears: β is simply the configuration described by σ, where σ is seen as a (partial) characteristic function deeming certain atomic facts to be true and certain others to be false.

Now if $(n, \exists\vec{x}\beta) \in \Psi$, say with $\vec{x} = (x_0, \ldots, x_m)$, and if $\models_{\mathcal{A}} \exists\vec{x}\beta$, then we can easily build a structure $\mathcal{B} \cong \mathcal{A}$ whose elements $0, \ldots, m$ realize β: let $\mathcal{B}$ be the image of $\mathcal{A}$ under an isomorphism which permutes a finite subset of ω to make this happen. It follows that $n \in S$, since now $n \in \mathrm{dom}(\Phi^{\mathcal{B}})$ for this $\mathcal{B}$, and so our e-reduction Ψ only ever enumerates elements of S when we run it using an arbitrary enumeration of $\mathrm{Th}_{\Sigma_1}(\mathcal{A})$. Of course, it may happen that $(n, \exists\vec{x}\beta) \in \Psi$ yet $\not\models_{\mathcal{A}} \exists\vec{x}\beta$; but in this case the instruction $(n, \exists\vec{x}\beta) \in \Psi$ will have no effect when we use $\Psi^{\mathrm{Th}_{\Sigma_1}(\mathcal{A})}$ to enumerate S.

On the other hand, if $n \in S$, then $\Phi^{\mathcal{A}}(n)$ itself halts, after examining only a finite initial segment σ of its oracle $\mathcal{A}$, i.e., of the atomic diagram of the structure $\mathcal{A}$. Our construction of Ψ will have found this σ and will have enumerated a corresponding axiom $(n, \exists\vec{x}\beta)$ into Ψ. Since $\models_{\mathcal{A}} \exists\vec{x}\beta$, we certainly have $(\exists\vec{x}\beta) \in \mathrm{Th}_{\Sigma_1}(\mathcal{A})$, and so, when Ψ runs using any enumeration of $\mathrm{Th}_{\Sigma_1}(\mathcal{A})$, it will enumerate n. This completes the proof. ∎

As with categoricity, this proposition reflects various concepts and facts already known about countable structures, such as relative intrinsic computable enumerability (see e.g. [13]). The obvious distinction is that here we consider information content (that is, arbitrary subsets of ω given uniformly in copies of $\mathcal{A}$) rather than definable subsets of the structure $\mathcal{A}$ itself. One naturally asks whether this distinction is significant, but we leave that question for future study.

By way of piquing interest in uniform intrinsic computability, we recall a theorem of Richter from [15]. This theorem is usually quoted as saying that for countable infinite linear orders and for countable infinite trees (as partial orders) $\mathcal{A}$, the only possible least degree in the Turing degree spectrum of $\mathcal{A}$ is the degree $\mathbf{0}$. (Richter did not mention Boolean algebras, but her proof is quickly seen to apply to them as well.) In fact, Richter proved slightly more: that every such structure $\mathcal{A}$ has spectrum containing a minimal pair of Turing degrees, and thus the spectrum cannot be contained within any nontrivial upper cone. One might say that the structure $\mathcal{A}$ cannot *intrinsically compute* any noncomputable set.

Since uniform intrinsic computation is a form of intrinsic computation, Richter's result immediately implies the following special case as a corollary. However, our notions yield a far more direct proof.

Corollary 4.6 *For any countable infinite linear order, tree (viewed as a partial order), or Boolean algebra $\mathcal{A}$, only the computably enumerable sets are uniformly intrinsically computably enumerable in $\mathcal{A}$.*

Proof. Apply Proposition 4.5, since the existential theory of any such $\mathcal{A}$ is decidable. (For the trees, this decidability requires an application of Kruskal's Theorem – as did Richter's original result.) ∎

Uniform categoricity is in much the same spirit as past investigations into intrinsic computability, as in [1] and [13], for example. If the images $f(R)$ of a subset R of the domain of a countable structure $\mathcal{A}$ are computably enumerable relative to $\mathcal{B}$ under every isomorphism f from $\mathcal{A}$ onto any copy $\mathcal{B}$ with domain ω, then R must be defined in $\mathcal{A}$ by a Σ_1^c formula, possibly using finitely many parameters $\vec{a}$ from $\mathcal{A}$. The parameters create a nonuniformity, but in the structure $(\mathcal{A}, \vec{a})$, this definition yields a Turing functional Γ such that $f(R) = \text{dom}(\Gamma^{(\mathcal{B}, f(\vec{a}))})$ under every isomorphism f from $\mathcal{A}$ onto any $\mathcal{B}$. That is, relative intrinsic computable enumerability is equivalent (up to those parameters) to the uniform version, i.e., to the existence of such a Γ, since the latter clearly implies the former. Proposition 4.5 is a natural extension of these results.

5 Noncomputable Infinitary Formulas

So far, the only infinitary formulas we have used have been computable ones, in the classes Σ_α^c (for various $\alpha < \omega_1^{CK}$). The main point of this section is to suggest that these formulas are not sufficient: we give examples of structures which would have lower levels of categoricity if one allowed certain noncomputable infinitary formulas. In the general setting of [4], working on a cone, one simply chooses the base degree of the cone to include sufficient information to be able to compute the necessary formulas. This is improved a bit further in [14]. However, it appears that our Definition 3.3 could be improved by adding a parameter Y, representing a Turing degree, and allowing Y-jumps (that is, jumps of structures defined by adding predicates for all Y-computable infinitary Σ_1-formulas). This section is mostly conjectural; we would welcome proofs of precise results about the examples described here.

Fix an arbitrary set $Y \subseteq \omega$, and define the following (symmetric, irreflexive) graph $\mathcal{A}_Y$. We start with a single node u, with countably many nodes z_{n0} (for all $n \in \omega$) adjacent to u. Each z_{n0} is then adjacent to z_{n1}, which is adjacent to z_{n2} and so on, so that countably many "ω-chains" are attached to u. For identification purposes, we also attach to u a single loop of length 3 (that is, we make u adjacent to one of the three nodes in this loop), and attach a loop of length $2i + 5$ to each node z_{ni}, (that is, one unique loop for each pair $\langle n, i \rangle$).

We now use loops of even length to add the desired complexity to $\mathcal{A}_Y$. Write $Y^{[n]} = \{j : \langle n, j \rangle \in Y\}$ for the n-th column of Y. To each node

z_{ni}, we attach one loop of length $2j+4$ for every $j \notin Y^{[n]}$. Finally, writing $Y^{[n]} = \{k_{n0} < k_{n1} < \cdots\}$, we attach to z_{ni} a single loop of length $2k_{ni}+4$. Hereafter this one will be known as the special loop for z_{ni}.

Now $\mathcal{A}_Y$ has a Scott family of infinitary Y-computable Σ_2 formulas (in fact, Π_1 formulas). The principal difficulty is to distinguish the nodes z_{n0} for different n; everything else is well labeled by loops. (To compute an isomorphism between copies, clearly it would suffice to map the z_{n0}'s to correct images.) A Y-oracle allows us to specify exactly what loops should be attached to each z_{ni} in the n-th ω-chain. Specifically, each $z = z_{n0}$ satisfies a formula saying,

$$(\forall i)(\forall \text{ loops } L \text{ attached to } z_i \text{ in } z\text{'s chain})\ [2 \cdot |L| + 4 = k_{ni} \text{ or } \notin Y^{[n]}],$$

along with the statements specifying that each of these z_i is connected to z by a chain of length i and is attached to a loop of length $2i+5$, and that z is adjacent to some u adjacent to a loop of length 3.

One might therefore expect $\mathcal{A}_Y$ to be Y-uniformly Δ_3-categorical. In fact, though, this can fail for certain Y. (Thanks are due to an anonymous referee for the following proof of this fact!) Let $\varphi_0, \varphi_1, \ldots$ be a list of all computable infinitary formulas. Set $\mathcal{Y}_0 = 2^\omega$. For each n, if only countably many $Z \in \mathcal{Y}_n$ have the property that $\models_{\mathcal{B}_Z} \varphi_n$, then let $\mathcal{Y}_{n+1}$ be $\mathcal{Y}_n$ with these countably many Z deleted. Otherwise, let $\mathcal{Y}_{n+1} = \mathcal{Y}_n$. By induction, every $\mathcal{Y}_n$ is co-countable, so there exists some $Z \in \cap_n \mathcal{Y}_n$, and this Z has the property that, for every n, either $\not\models_{\mathcal{B}_Z} \varphi_n$ or else uncountably many U have $\models_{\mathcal{B}_U} \varphi_n$.

Now, for each n, choose some $Z_n \neq Z$ for which

$$\models_{\mathcal{B}_{Z_n}} \varphi_n \iff \models_{\mathcal{B}_Z} \varphi_n.$$

Now set $Y = Z \oplus (\oplus_{(j,k)\in\omega^2} Z_j)$ to be the set with Z and infinitely many copies of each Z_n as its columns. We claim that $\varphi_n(x)$ cannot identify the node z_{00} at the top of the chain for Z. Indeed, if $\varphi_n(z_{00}$ holds in $\mathcal{A}_Y$, then $\varphi_n(z_{\langle n,0\rangle+1,0})$ (the node at the top of one of the Z_n-chains) must also hold there, since $\models_{\mathcal{B}_{Z_n}} \varphi_n$ if and only if $\models_{\mathcal{B}_Z} \varphi_n$. Since $Z \neq Z_n$, this means that φ_n cannot be part of a Scott family for $\mathcal{A}_Y$: it holds of two nodes not in the same orbit under automorphisms. So in fact this $\mathcal{A}_Y$ has no Scott family of computable formulas at all, and thus cannot be continuously Δ_α-categorical for any α.

The problem with the Scott family of infinitary Y-computable Σ_2 formulas is that those formulas are not computable infinitary; they are only Y-computable infinitary. Consequently, the second jump $(\mathcal{A}_Y)''$ generally does not give information about whether a specific node satisfies such a formula or not: the predicates in the second jump of a structure only describe satisfaction of computable infinitary Σ_2-formulas.

For certain structures, one can convert a Scott family of Y-computable infinitary formulas into an Y-computably enumerable family of computable infinitary formulas, possibly of higher rank. However, the $\mathcal{A}_Y$ here in general is sufficiently complex to preclude this possibility, with the use of the Y-oracle hidden within the Π_1 part of the Σ_2 formulas. So we have here structures $\mathcal{A}_Y$ which do have absolute Scott rank 2, yet do not appear to satisfy any of our versions of continuous Δ_3-categoricity.

On the bright side, there does exist a single Turing functional Γ such that, for every $Y \subseteq \omega$ and every $\mathcal{B} \cong \mathcal{A}_Y$, the function

$$\Gamma^{(\mathcal{A}_Y \oplus \mathcal{B})''}$$

is an isomorphism from $\mathcal{A}_Y$ onto $\mathcal{B}$. With this oracle, Γ searches for some $z \in \mathcal{B}$ adjacent to the u in $\mathcal{B}$ such that, for every loop attached to every z_i below z, there is a loop of the same length attached to the same z_{ni} below z_{n0} (in $\mathcal{A}_Y$), and likewise with the roles of $\mathcal{A}_Y$ and $\mathcal{B}$ reversed. So we conjecture that using the jump(s) of the join of (the atomic diagrams of) $\mathcal{A}_Y$ and $\mathcal{B}$, rather than the join of their jump(s), may allow us to extend unform notions of categoricity to other computably non-presentable structures.

Finally, we note that Y is not in general e-reducible to the existential theory of the structure $\mathcal{A}_Y$. Indeed, given any Y, every set U with the same columns as Y (and having each column occur with the same multiplicity) will yield an $\mathcal{A}_U$ with the same existential theory, indeed with $\mathcal{A}_U \cong \mathcal{A}_Y$. However, unless cofinitely many columns of U are all equal to each other, there will be uncountably many distinct sets U with the same columns as Y, and all but countably many of those U will have no e-reduction to the existential theory of their structures $\mathcal{A}_U$.

It is the case that each individual column $Y^{[n]}$ is decidable from the jump $(\mathcal{A}_Y)'$: the i-th element $k_{ni} \in Y^{[n]}$ has the property that there is a loop of length $2k_{ni}+4$ attached to z_{ni} but no such loop attached to $z_{n(i+1)}$. However, this procedure is not uniform: starting this process in the jump of a copy $\mathcal{B} \cong \mathcal{A}_Y$ requires knowing the image of z_{n0} in $\mathcal{B}$, i.e., knowing which ω-chain in $\mathcal{B}$ to use.

To sum up all the loose ends in this section is a challenge, but in general they suggest that it would be useful to define a relative notion of the jump of a structure, using Y-computable infinitary Σ_1-formulas in place of computable ones. Definition 3.3 could likely be sharpened by using such jumps, and/or by allowing the Turing functional to use a jump of the join $(X \oplus \mathcal{A})^{(\alpha)}$ in place of the join $(X' \oplus \mathcal{A}^{(\alpha+1)})$ of their jumps. The conjectures and examples in this section make it appear that under Definition 3.3, there exist countable structures $\mathcal{B}$ which cannot be continuously Δ_α-categorical, no matter what α one chooses: if $\mathcal{B}$ has noncomputable Scott rank β, then the jump $\mathcal{B}^{(\beta)}$ is not even defined; and even for structures such as many of the $\mathcal{A}_Y$ constructed here, it seems likely that no jump $(\mathcal{A}_Y)^{(\alpha)}$ with $\alpha < \omega_1^{CK}$ is continuously categorical. This contradicts one's intuition, based on the original results of Scott, that categoricity should be continuous for every countable structure, although at arbitrarily high countable levels β. So it would be natural to develop a relativized notion of the jump of a structure – presumably using X-computable infinitary Σ_1 formulas to relativize to X – and to extend our notion of uniform Δ_α-categoricity to a uniform Δ_α^X-categoricity which includes this relativization.

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