MATH 141, Calculus/Differentiation, Fall 2024
Exam II by Scott Wilson

Name: ____

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Instructions: **Read each problem carefully**. Show all of your work in order to receive full or partial credit. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Graphing calculators are permitted on this exam.
You may use any results that were proven in class or in the homework. In this case, clearly state what result you are using and how it applies.

(1) Find equation of the tangent line to the graph of

$$f(x) = x^3 - 2x^2 + 8$$

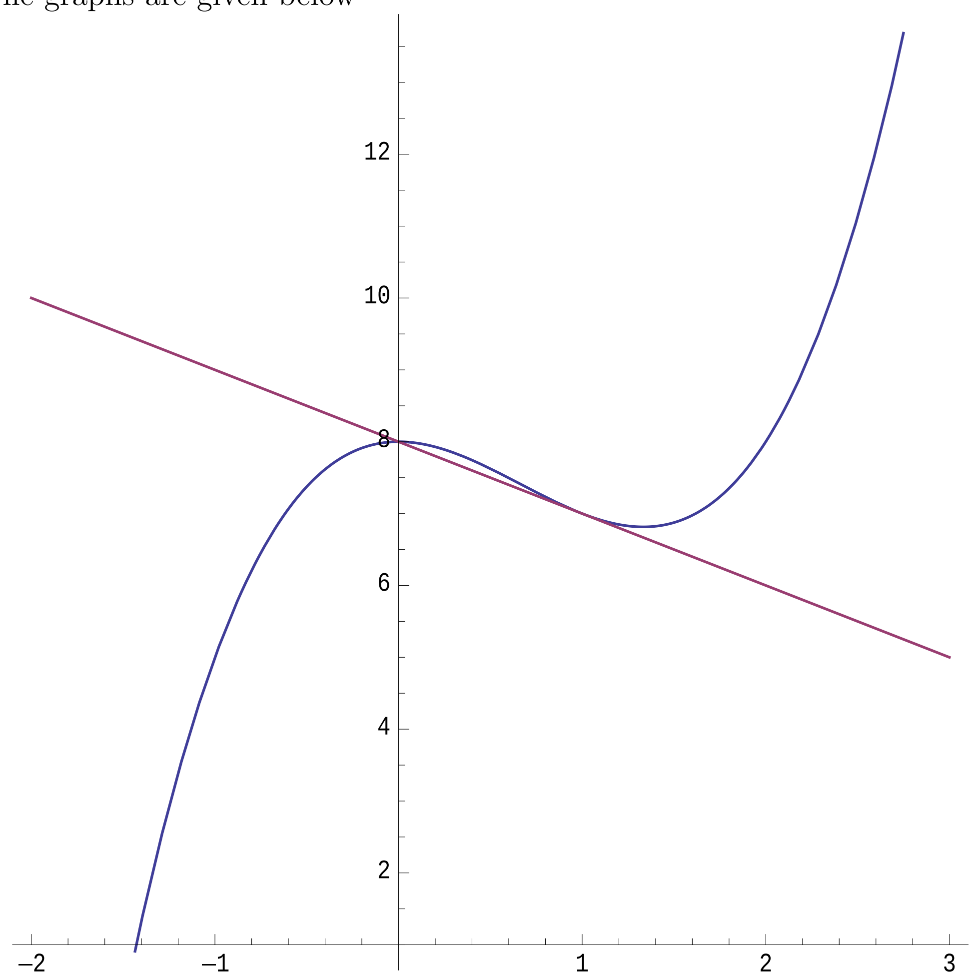
at $x = 1$.

Draw a picture of the graph of f and the tangent line at $x = 1$.

The derivative of f at x is $f'(x) = 3x^2 - 4x$, so the slope of the tangent line at $x = 1$ is $f'(1) = 3 - 4 = -1$. Since $f(1) = 1 - 2 + 8 = 7$, the tangent line at $x = 1$ is given by $y - 7 = -1(x - 1)$, or equivalently,

$$y = -x + 8$$

The graphs are given below



(2) Suppose the position of a particle at time t, where $0 \leq t \leq 4$, is given by

$$s(t) = -t^2 + 6t + 3$$

(a) What is the average velocity of the particle over the time interval $[0, 3]$?

$$\frac{s(3) - s(0)}{3 - 0} = \frac{12 - 3}{3 - 0} = 3$$

(b) For which values of t is the particle at rest?

The particle is at rest if and only if the instantaneous velocity is zero. The instantaneous velocity is given by

$$v(t) = s'(t) = -2t + 6$$

and this is zero if and only if $t = 3$. So the particle is at rest only when $t = 3$.

(c) What is the acceleration of the particle at time t? (Think about why your answer makes sense.)

The acceleration is given by

$$a(t) = v'(t) = -2.$$

This means the rate of change of velocity is always negative, i.e. the velocity is decreasing.

(d) What is the total distance traveled by the particle in the time interval $[0, 4]$?

The particle stops at $t = 3$ and changes direction (because the velocity is zero at $t = 3$ and the velocity is always decreasing). So the total distance traveled is

$$|s(4) - s(3)| + |s(3) - s(0)| = |11 - 12| + |12 - 3| = 1 + 9 = 10$$

(e) What is the displacement of the particle from $t = 0$ to $t = 4$?

The displacement is $s(4) - s(0) = -16 + 24 + 3 - 3 = 8$.

(3) Let f be a function.

(a) State the definition of the derivative of f at x. (Your answer should be expressed in terms of a limit).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use your answer from the first part to calculate

$$\lim_{h \to 0} \frac{\cos(\pi/2 + h) - \cos(\pi/2)}{h}$$

Explain your reasoning. (You may use the derivative of the cosine function.)

Using the first part: let $f(x) = \cos x$ and $x = \pi/2$, then this limit represents the derivative of $\cos x$ at $\pi/2$, which equals $-\sin(\pi/2) = -1$.

(c) Use your answer from the first part to calculate $f'(3)$ if $f(x) = \sqrt{x}$. (Do not use the power rule. Compute the limit directly.)

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}.$$

Here we multiplied the previous fraction by $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$. This equals

$$\lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}.$$

So, for $x = 3$ we have $f'(3) = \frac{1}{2\sqrt{3}}$.

(4) Calculate $\frac{dy}{dx}$ for the following functions. Do not simplify your answer.

(a) $y = \frac{1}{x^7} - x^4 \tan(x)$

Let's rewrite it as $y = x^{-7} - x^4 \tan(x)$, use the power rule on the first summand, and the product rule on the second summand:

$$\frac{dy}{dx} = -7x^{-8} - \left(4x^3 \tan x + x^4 \sec^2 x\right)$$

(b) $y = \frac{\cot(x)}{x^3 - 1}$

We use the quotient rule

$$\frac{dy}{dx} = \frac{(-\csc^2(x))(x^3 - 1) - (\cot(x))(3x^2)}{(x^3 - 1)^2}$$

Here we use that the derivative of $\cot(x)$ is equal to $-\csc^2(x)$.

(5) Do both parts. Do not simplify your answer.

(a) Calculate $\frac{dy}{dx}$ for the function: $y = 7\sin(\sqrt[3]{x})$

We can pull the constant 7 outside the derivative, and then use the chain rule where $f(x) = \sin x$ and $g(x) = \sqrt[3]{x}$. Then $f'(x) = \cos x$, $f'(g(x)) = \cos(\sqrt[3]{x})$ and $g'(x) = \frac{1}{3}x^{-2/3}$, so

$$\frac{dy}{dx} = 7f'(g(x))g'(x) = \frac{7}{3}\cos(\sqrt[3]{x})x^{-2/3}$$

(b) Use implicit differentiation to find the equation of the tangent line to the hyperbola

$$x^2 + 2xy - y^2 + x = 2$$

at the point $(1, 2)$.

Differentiating we get

$$2x + 2\left(y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} + 1 = 0$$

Simplifying we get

$$\frac{dy}{dx}(2x - 2y) = -1 - 2x - 2y$$

or

$$\frac{dy}{dx} = \frac{1 + 2x + 2y}{2(y - x)}.$$

At $(x, y) = (1, 2)$ we have $\frac{dy}{dx} = 7/2$, so the equation of the tangent line at the point $(1, 2)$ is $y - 2 = \frac{7}{2}(x - 1)$, or $y = \frac{7}{2}x - \frac{3}{2}$.