MATH 310, Elem. Real Analysis, Fall 2024
Exam II by Scott Wilson

Name: ______________________________

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Instructions: Read each problem carefully. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Give clear and thorough explanations for your solutions. You may use results from class or the textbook but make a clear reference to what you are using.

(1) For each part, give an example of a sequence that has the stated property, or write "no example exists" and explain why no such example exists. Recall, the word "converges" means converges to a real number (not $+\infty$ or $-\infty$).

(a) A monotone sequence with no convergent subsequence.

For example, $a_n = n$.

(b) A bounded sequence with no convergent subsequence.

No such example exists. The "Bolzano-Weierstrass Theorem" that we proved states that every bounded sequence has a convergent subsequence.

(c) A bounded monotone sequence that does not converge.

No such example exists. A bounded monotone sequence converges to the supremum of all the values, as we showed.

(d) A convergent sequence with no monotone subsequence that converges.

No such example exists. We showed every sequence has a monotone subsequence, and that if a sequence converges then every subsequence converges to this same value.

(2) For each sequence below, determine the set of all subsequential limits. Then state the lim sup and lim inf of the sequence.

(a) $s_n = -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, \ldots$ (the pattern repeats indefinitely)

The set of all subsequential limits is $\{-1, 0, 1\}$, $\limsup s_n = 1$, and $\liminf s_n = -1$

(b) $s_n = \frac{(-1)^n}{2n+1}$.

The sequence converges to zero, so $\limsup s_n = \liminf s_n = 0$, and the set of subsequential limits is $\{0\}$.

(3) Do all parts. For the last two parts, explain your reasoning, using appropriate results from class or the textbook.

(a) Let a_k be a sequence. Define what it means for the series $\sum_{k=1}^{\infty} a_k$ to converge.

By definition, the series converges if the sequence $s_n = a_1 + \cdots + a_n$ of partial sums converges.

(b) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ converge absolutely?

Yes,

$$\left|\frac{(-1)^n}{n^2+n}\right| = \frac{1}{n^2+n} \leq \frac{1}{n^2}$$

so the series converges absolutely by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (which we showed to converge by the integral test).

(c) Does the series $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$ converge?

Yes, using the ratio test with $a_n = \frac{n^2}{(n+1)!}$ we have

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{(n+1)^2}{(n+2)!} \frac{(n+1)!}{n^2} = \limsup \frac{(n+1)^2}{n^2(n+2)} = 0 < 1$$

(4) Do all parts

(a) State the Intermediate Value Theorem.

If f is a continuous function on a closed interval $[a, b]$, and c is some value between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$.

(b) Let $f(x) = (x - 1)e^x$. Show there is a real number c so that $f(c) = 1$.
[You may assume that f is continuous and that $e > 2$.]

Since $f(0) = -1 < 1$ and $f(2) = e^2 > 1$, and f is continuous on $[0, 2]$, and $c = 1$ is between $f(0)$ and $f(2)$, we conclude by the Intermediate Value Theorem that there is some $x \in [0, 2]$ with $f(x) = 1$. (In fact, using Mathematica I find that $x \approx 1.2784645427610737\ldots$)

(5) Do both parts.

(a) Use the ϵ-δ definition of continuity to show that the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Let $\epsilon > 0$. We want to show that for some $\delta > 0$ we have that $|x - 0| < \delta$ implies $|x^2 \cos\left(\frac{1}{x}\right) - f(0)| < \epsilon$. We are given $f(0) = 0$.
Since we know $|\cos\left(\frac{1}{x}\right)| \leq 1$, it's enough to find a $\delta > 0$ so that $|x^2| < \epsilon$ whenever $|x| < \delta$. So we let $\delta = \sqrt{\epsilon}$. Then $|x| \leq \delta = \sqrt{\epsilon}$ implies $|x^2| < \epsilon$, and finally

$$\left| x^2 \cos\left(\frac{1}{x}\right) - f(0) \right| = \left| x^2 \cos\left(\frac{1}{x}\right) \right| = |x^2| \left| \cos\left(\frac{1}{x}\right) \right| \leq |x^2| < \epsilon.$$

(b) Use the limit definition of continuity to show that the following function is not continuous at $x = 0$

$$f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

You might want to draw a picture.

Let $x_n = \frac{1}{2\pi n}$. Then x_n converges to zero, but $f(x_n) = 1$ for all $n \in \mathbb{N}$, so $f(x_n)$ converges to 1, not to $f(0) = 0$. This shows f is not continuous at $x = 0$.