MATH 320, Intro. to Topology, Spring 2024
Exam II by Scott Wilson

Name: ____

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Instructions: Read each problem carefully. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Give clear and thorough explanations for your solutions. You may use results from class or the textbook but make a clear reference to what you are using.

(1) Let $X = \{a, b, c\}$ and consider the following three topologies on X,

$$\begin{aligned} \mathcal{C}_1 &= \{\emptyset, X, \{a\}, \{b, c\}\} \\ \mathcal{C}_2 &= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \\ \mathcal{C}_3 &= \{\emptyset, X, \{a, b\}, \{c\}\}. \end{aligned}$$

(a) Is $\mathcal{C}_1$ finer than $\mathcal{C}_2$?

No, for example $\{b\} \in \mathcal{C}_2$, but $\{b\} \notin \mathcal{C}_1$, so $\mathcal{C}_2 \nsubseteq \mathcal{C}_1$.

(b) What is $\mathcal{C}_2 \cap \mathcal{C}_3$? Is this a topology on X?

$$\mathcal{C}_2 \cap \mathcal{C}_3 = \{\emptyset, X, \{a, b\}\}$$

This is a topology on X since it contains $\emptyset$ and X, and is closed under arbitrary unions and finite intersections.

(c) What is $\mathcal{C}_1 \cup \mathcal{C}_3$? Is this a topology on X?

$$\mathcal{C}_1 \cup \mathcal{C}_3 = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b\}, \{c\}\}$$

This is not a topology on X since is not closed under finite intersections (e.g. $\{b, c\} \cap \{a, b\} = \{b\} \notin \mathcal{C}_1 \cup \mathcal{C}_3$), nor is it closed under unions (e.g. $\{a\} \cup \{c\} = \{a, c\} \notin \mathcal{C}_1 \cup \mathcal{C}_3$).

(2) Consider the following subsets of the topological space $\mathbb{R}$ (with its standard topology). For each of the following subsets, determine if it is open, closed, neither, or both, and explain your reasoning. Let $\mathbb{Z}$ denote the integers, and $\mathbb{Q}$ denote the rationals.

(a) $[1, 5)$

This set is not open or closed. It is not open since no interval containing 1 is contained in the set. Similarly, the complement is not open since no interval containing 5 is contained in the complement.

(b) $\mathbb{R} - \mathbb{Z}$

This set is open since it is the union of all open intervals of the form $(n, n+1)$ for $n \in \mathbb{Z}$. It is not closed since the complement $\mathbb{Z}$ is not open (by the same argument as in the previous part).

(c) $\mathbb{Q}$

This set is not open by the same argument as the first part. It is also not closed by the same argument applied to the complement $\mathbb{R} - \mathbb{Q}$. (Concisely, every open interval contains both rational and irrational numbers).

(3) Do each part. You must explain your answer for full credit.

(a) What is the closure of $\mathbb{Q}$ in $\mathbb{R}$?

The closure of $\mathbb{Q}$ is $\mathbb{R}$, since any open interval containing a given real number also contains a rational number.

(b) What is the interior of $[-1, 0) \cup (0, 3)$ in $\mathbb{R}$.

The interior is $(-1, 0) \cup (0, 3)$. This is an open set contained in the given set and no open set contained in the given set can contain -1.

(c) What is the closure of the set $\{x | \frac{1}{x} \in \mathbb{Z}\}$ in $\mathbb{R}$? What is the interior?

The closure of this set is $\{x | \frac{1}{x} \in \mathbb{Z}\} \cup \{0\}$, since 0 is the only limit point of $\{x | \frac{1}{x} \in \mathbb{Z}\}$ that is not contained in $\{x | \frac{1}{x} \in \mathbb{Z}\}$. The interior is empty.

(4) Let X be a set. Do both parts.

(a) Define what is a basis for a topology on X.

A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(i) For all $x \in X$, $x \in B$ for some $B \in \mathcal{B}$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

(b) Let $\mathcal{B}$ be the collection of all subsets of X which contain exactly one element:

$$\mathcal{B} = \{B \subseteq X \,|\, B \text{ has one element}\}.$$

Show that the topology generated by $\mathcal{B}$ is the discrete topology, i.e. every subset of X is open in this topology. (You do not have to prove $\mathcal{B}$ is a basis, but you can if you want).

Recall the topology $\mathcal{C}_{\mathcal{B}}$ generated by a basis $\mathcal{B}$ consists of all unions of elements of $\mathcal{B}$.

If U is any subset of X we can write U as the union of all elements $x \in U$,

$$U = \bigcup_{x \in U} \{x\}.$$

Since we are given that each $\{x\}$ is in $\mathcal{B}$ by assumption, $U \in \mathcal{C}_{\mathcal{B}}$, and therefore every subset of X is in $\mathcal{C}_{\mathcal{B}}$, so $\mathcal{C}_{\mathcal{B}}$ is the discrete topology.

(5) Let $(X, \mathcal{C}_X)$, $(Y, \mathcal{C}_Y)$ and $(Z, \mathcal{C}_Z)$ be topological spaces.

(a) Define what it means for a function $f : X \to Y$ to be continuous.

$f : X \to Y$ is continuous iff $f^{-1}(U) \in \mathcal{C}_X$ whenever $U \in \mathcal{C}_Y$. In words, the pre-image of any open set in Y is an open set in X.

(b) What is the product topology on $Y \times Z$?

The product topology is the one generated by the basis consisting of sets $U \times V$ where U is open in Y and V is open in Z.

(c) Suppose $f : X \to Y$ and $g : X \to Z$ are continuous functions. Show that the function $h : X \to Y \times Z$ defined by

$$h(x) = (f(x), g(x))$$

is continuous, if $Y \times Z$ has the product topology. [Hint: determine the set $h^{-1}(U \times V)$ from its definition, and express it in terms of $f^{-1}(U)$ and $f^{-1}(V)$.]

It suffices to show $h^{-1}(U \times V)$ is open in X whenever $U \times V$ is a basis element for the product topology on $Y \times Z$, i.e. U is open in Y and V is open in Z. We have

$$\begin{aligned} h^{-1}(U \times V) &= \{x \in X \,|\, h(x) \in U \times V\} \\ &= \{x \in X \,|\, (f(x), g(x)) \in U \times V\} \\ &= \{x \in X \,|\, f(x) \in U\} \cap \{x \in X \,|\, g(x) \in V\} \\ &= f^{-1}(U) \cap f^{-1}(V), \end{aligned}$$

which is open in X since $f^{-1}(U)$ and $g^{-1}(V)$ are open in X, since f and g are continuous.