MATH 320/620, Point Set Topology, Spring 2024
Exam I by Scott Wilson

Name: ____

Problem	**Max points**	**Grade**
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Instructions: Read each problem carefully. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Give clear and thorough explanations for your solutions. You may use results from class or the textbook but make a clear reference to what you are using.

(1) Consider the following sets: $A = \{a, b\}$, $B = \{b, x, y\}$, and $S = \{a, b, c, x, y, z\}$. Answer the following, using proper set-theoretic notation.

(a) What is $A \cup B$?

$$A \cup B = \{a, b, x, y\}$$

(b) What is $A \cap B$?

$$A \cap B = \{b\}$$

(c) What is the complement of B in S?

$$\{a, c, z\}$$

(d) How many elements are in the power set of S?

$$2^6 = 64$$

(2) Let I be a set, and let A_α be a set for each $\alpha \in I$.

(a) Prove: for each $\beta \in I$, $A_\beta \subset \bigcup_{\alpha \in I} A_\alpha$.

If $x \in A_\beta$, then $x \in A_\alpha$ for some $\alpha \in I$, so $x \in \bigcup_{\alpha \in I} A_\alpha$.

(b) Is the following true or false, in general: for each $\beta \in I$, $A_\beta \subset \bigcap_{\alpha \in I} A_\alpha$.

False. Even for two sets, i.e. $I = \{1, 2\}$, we do not have $A_1 \subset A_1 \cap A_2$ in general.

(c) Is the following true or false, in general:
for each $\beta \in I$, $A_\beta \cap \left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} (A_\beta \cap A_\alpha)$.

True, proof: $x \in A_\beta \cap \left(\bigcup_{\alpha \in I} A_\alpha\right)$, if and only if $x \in A_\beta$ and $x \in A_\alpha$ for all $\alpha \in I$, if and only if $x \in A_\beta \cap A_\alpha$ for all $\alpha \in I$.

(d) Consider an example, with $I = (0, \infty)$, and let $X_\alpha = (-\alpha, \alpha) \subset \mathbb{R}$, for $\alpha \in I$. Determine the sets

$$\bigcap_{\alpha \in I} X_\alpha \quad \text{and} \quad \bigcup_{\alpha \in I} X_\alpha$$

$$\bigcap_{\alpha \in \mathbb{R}} X_\alpha = \{0\} \quad \text{and} \quad \bigcup_{\alpha \in \mathbb{R}} X_\alpha = \mathbb{R}$$

(3) Let X and Y be sets, and let $f : X \to Y$ be a function.

(a) Let $A \subset X$. Define the image of A, which we denote by $f(A)$.

$$f(A) = \{y \in Y \,|\, y = f(a) \text{ for some } a \in A\}.$$

(b) Let $B \subset Y$. Define the pre-image of B, which we denote by $f^{-1}(B)$.

$$f^{-1}(B) = \{x \in X \,|\, f(x) \in B\}.$$

(c) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$. What is $g^{-1}(g([2,3]))$? You might want to graph the function.

$$g^{-1}(g([2,3])) = g^{-1}([4,9]) = [-3,-2] \cup [2,3]$$

(d) Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2 - 1$. What is $h(h^{-1}([-2,0]))$? You might want to graph the function.

First graph the function. Notice that $x^2 - 1 \geq -1$ and the image of h is $[-1, \infty)$. Also $h(1) = h(-1) = 0$ and $h(0) = -1$. First, $h^{-1}([-2,0]) = h^{-1}([-1,0]) = [-1,1]$, so

$$h(h^{-1}([-2,0])) = h([-1,1]) = [-1,0].$$

(4) Do all parts.

(a) Define what is a metric space.

A metric space is a set X with a function $d : X \times X \to \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(b) Consider the set $\mathbb{R}^2$ with the metric

$$d(a, b) = \max\{|a_1 - b_1|, |a_2 - b_2|\},$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

(i) Compute $d(a, b)$ where $a = (2, 4)$ and $b = (-1, 2)$.

$$d(a, b) = \max\{|2 + 1|, |4 - 2|\} = 3.$$

(ii) Draw a picture of $B_d(c, 1)$, the ball of radius 1 around the point $c = (2, 2)$.

This is the region inside the square with vertices at $(1, 1)$, $(3, 1)$ $(3, 3)$, and $(1, 3)$.

(5) Do all parts.

(a) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a function $f : (X, d_X) \to (Y, d_Y)$ to be continuous at $a \in X$. (Your answer should involve δ and ϵ.)

A function $f : (X, d_X) \to (Y, d_Y)$ is continuous if, for any $\epsilon > 0$ there is a number $\delta > 0$ so that if $d_X(x, a) < \delta$ then $d_Y(f(x), f(a)) < \epsilon$.
Equivalently, for any $\epsilon > 0$ there is a number $\delta > 0$ so that if $x \in B_{d_X}(a, \delta)$ then $f(x) \in B_{d_Y}(f(a), \epsilon)$.
Equivalently, for any $\epsilon > 0$ there is a number $\delta > 0$ so that $B_{d_X}(a, \delta) \subset f^{-1}(B_{d_Y}(f(a), \epsilon))$.

(b) Consider $\mathbb{R}^n$ with the two metrics

$$d_E(a, b) = \left((a_1 - b_1)^2 + \cdots + (a_n - b_n)^2\right)^{1/2}$$
$$d(a, b) = \max_{i=1,\ldots,n} |a_i - b_i|$$

for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$.

Show that the identity function $i : (\mathbb{R}^n, d_E) \to (\mathbb{R}^n, d)$ is continuous at every point $a \in \mathbb{R}^n$.

Given $\epsilon > 0$ let $\delta = \epsilon$. If $d(a, x) < \delta = \epsilon$, then

$$(a_1 - x_1)^2 + \cdots + (a_n - x_n)^2 < \epsilon^2,$$

so that $(a_i - x_i)^2 < \epsilon^2$ for each $i = 1, \ldots, n$, and therefore

$$d(i(a), i(x)) = d(a, x) = \max_{i=1,\ldots,n} |a_i - x_i| < \epsilon.$$