MATH 620, Intro. to Topology, Spring 2024
Exam II by Scott Wilson

Name: ______________________________

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Instructions: Read each problem carefully. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Give clear and thorough explanations for your solutions. You may use results from class or the textbook but make a clear reference to what you are using.

(1) Consider the following subsets of the topological space $\mathbb{R}$ (with its standard topology). For each of the following subsets, determine if it is open, closed, neither, or both, and explain your reasoning. Let $\mathbb{Z}$ denote the integers, and $\mathbb{Q}$ denote the rationals.

(a) $[1, 5)$

This set is not open or closed. It is not open since no interval containing 1 is contained in the set. Similarly, the complement is not open since no interval containing 5 is contained in the complement.

(b) $\mathbb{R} - \mathbb{Z}$

This set is open since it is the union of all open intervals of the form $(n, n+1)$ for $n \in \mathbb{Z}$. It is not closed since the complement $\mathbb{Z}$ is not open (by the same argument as in the previous part).

(c) $\mathbb{Q}$

This set is not open by the same argument as the first part. It is also not closed by the same argument applied to the complement $\mathbb{R} - \mathbb{Q}$. (Concisely, every open interval contains both rational and irrational numbers).

(2) Do each part. You must explain your answer for full credit.

(a) What is the closure of $\mathbb{Q}$ in $\mathbb{R}$?

The closure of $\mathbb{Q}$ is $\mathbb{R}$. To see this, $if x \in \mathbb{R}$ then any open interval containing x contains a rational number, so x is in the the closure of $\mathbb{Q}$.

(b) What is the interior of $[-1,0) \cup (0,3)$ in $\mathbb{R}$.

The interior is $(-1,0) \cup (0,3)$. This is an open set contained in the given set, containing all points except -1, and no open set contained in the given set can contain -1.

(c) What is the closure of the set $\{x | \frac{1}{x} \in \mathbb{Z}\}$ in $\mathbb{R}$? What is the interior?

The closure of this set is $\{x | \frac{1}{x} \in \mathbb{Z}\} \cup \{0\}$, since 0 is the only limit point of $\{x | \frac{1}{x} \in \mathbb{Z}\}$ that is not contained in $\{x | \frac{1}{x} \in \mathbb{Z}\}$. The interior is empty.

(3) Let X be a set. Do both parts.

(a) Define what is a basis for a topology on X.

A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(i) For all $x \in X$, $x \in B$ for some $B \in \mathcal{B}$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

(b) Using an appropriate basis for each topology, show that the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the topology on $\mathbb{R} \times \mathbb{R}$ determined by the Euclidean metric. You can use drawings and words to explain your argument (rather than explicit formulas for variables, radii, lengths, widths, etc.)

We first show that any ϵ-ball B is open in the product topology. For any point x in an open ϵ-ball B, we can find an (open) rectangle $I \times J$ with $x \in I \times J \subset B$. Here I and J are open intervals in $\mathbb{R}$. So, an open ϵ-ball can be written as a union of rectangles, and so an ϵ-ball is open in the product topology. This shows the Euclidean topology is finer than the product topology.
Conversely, given a point x in an open rectangle $I \times J$, we can find an ϵ-ball B with $x \in B \subset I \times J$. So, an open rectangle can be written as a union of ϵ-balls, and thus is open in the Euclidean topology. This shows the product topology is finer than the Euclidean topology.
So, the two topologies are equal.

(4) Let $(X, \mathcal{C})$ be a topological space and let Y be a subset of X.

(a) What are the open sets in the subspace topology on Y? In other words, define the subspace topology. (You do not have to show that it is a topology).

A subset V of Y is open in Y if and only if $V = Y \cap U$ for some open subset U of X.

(b) Show that if X is Hausdorff then Y (with the subspace topology) is also Hausdorff.

Let $a, b \in Y \subset X$. Since X is Hausdorff, there are open sets U_a and U_b of X such that $a \in U_a$ and $b \in U_b$, and $U_a \cap U_b = \emptyset$. Then the sets $V_a = Y \cap U_a$ and $V_b = Y \cap U_b$ are open subsets of Y, with $a \in V_a$ and $b \in V_b$ and $V_a \cap V_b = \emptyset$. This shows Y is Hausdorff.

(5) Let $(X, \mathcal{C}_X)$, $(Y, \mathcal{C}_Y)$ and $(Z, \mathcal{C}_Z)$ be topological spaces.

(a) Define what it means for a function $f : X \to Y$ to be continuous.

$f : X \to Y$ is continuous iff $f^{-1}(U) \in \mathcal{C}_X$ whenever $U \in \mathcal{C}_Y$. In words, the pre-image of any open set in Y is an open set in X.

(b) Suppose $f : X \to Y$ and $g : X \to Z$ are continuous functions. Show that the function $h : X \to Y \times Z$ defined by

$$h(x) = (f(x), g(x))$$

is continuous, if $Y \times Z$ has the product topology.

It suffices to show $h^{-1}(U \times V)$ is open in X whenever $U \times V$ is a basis element for the product topology on $Y \times Z$, i.e. U is open in Y and V is open in Z. We have

$$\begin{aligned} h^{-1}(U \times V) &= \{x \in X \mid h(x) \in U \times V\} \\ &= \{x \in X \mid (f(x), g(x)) \in U \times V\} \\ &= \{x \in X \mid f(x) \in U\} \cap \{x \in X \mid g(x) \in V\} \\ &= f^{-1}(U) \cap f^{-1}(V), \end{aligned}$$

which is open in X since $f^{-1}(U)$ and $g^{-1}(V)$ are open in X, since f and g are continuous.