HERMITIAN IDENTITIES

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We list the identities that hold on a complex manifold with compatible metric. Identity (7) is proved in [Dem86] using a local argument, and it is stated there that the idea was implicit in [Gri66]. The remaining identities follow from algebraic manipulations, and all appear in [Dem86], except for identity (1) which is clear, and identities (4) and (8), which are used to prove a generalized hard Lefschetz duality below.

Let L be the Lefschetz operator, $\Lambda = L^*$, $\lambda = [\partial, L]$, $\bar\lambda = [\bar\partial, L]$, $\tau = [\Lambda, \lambda]$, $\bar\tau = [\Lambda, \bar\lambda]$, and $\Delta_\delta = [\delta, \delta^*]$ for any operator δ. Then

$$\begin{aligned} &\bar\partial^2 = 0 && \bar\lambda^2 = 0 && [\bar\lambda, \lambda] = 0 \\ &\partial^2 = 0 && [\bar\partial, \bar\lambda] = 0 && [\partial, \lambda] = 0 \\ &[\bar\partial, \partial] = 0 && [\bar\lambda, \partial] + [\bar\partial, \lambda] = 0 && \lambda^2 = 0, \end{aligned} \tag{1}$$

as well as the adjoint of these equations. Additional adjoint or conjugate equations may (or may not) be omitted.

$$\begin{aligned} &[\Lambda, \tau] = -2i\bar\tau^* && [L, \bar\tau] = 3\bar\lambda && [\Lambda, \lambda] = \tau && [L, \lambda] = 0 \\ &[\Lambda, \bar\tau] = 2i\tau^* && [L, \tau] = 3\lambda && [\Lambda, \bar\lambda] = \bar\tau && [L, \bar\lambda] = 0 \\ &[L, \tau^*] = -2i\bar\tau && [\Lambda, \bar\tau^*] = -3\bar\lambda^* && [L, \lambda^*] = -\tau^* && [\Lambda, \lambda^*] = 0 \\ &[L, \bar\tau^*] = 2i\tau && [\Lambda, \tau^*] = -3\lambda^* && [L, \bar\lambda^*] = -\bar\tau^* && [\Lambda, \bar\lambda^*] = 0. \end{aligned} \tag{2}$$

$$\begin{aligned} &[\bar\lambda, \tau] = -[\bar\tau, \lambda] && [\tau, \tau] = 2i[\lambda, \bar\tau^*] && 2[\bar\tau^*, \tau] = 3[\bar\lambda^*, \lambda] \\ & && [\bar\tau, \bar\tau] = -2i[\bar\lambda, \tau^*] && \\ &[\bar\lambda^*, \tau^*] = -[\bar\tau^*, \lambda^*] && [\tau^*, \tau^*] = -2i[\lambda^*, \bar\tau] && 2[\bar\tau, \tau^*] = 3[\bar\lambda, \lambda^*] \\ &[\lambda^*, \bar\tau^*] = -[\tau^*, \bar\lambda^*] && [\bar\tau^*, \bar\tau^*] = 2i[\bar\lambda^*, \tau] && \end{aligned} \tag{3}$$

$$\begin{aligned} &[L, \Delta_\tau + 3\Delta_\lambda] = -2i[\tau, \bar\tau] && [\Lambda, \Delta_\tau + 3\Delta_\lambda] = 2i[\tau^*, \bar\tau^*] \\ &[L, \Delta_{\bar\tau} + 3\Delta_{\bar\lambda}] = 2i[\tau, \bar\tau] && [\Lambda, \Delta_{\bar\tau} + 3\Delta_{\bar\lambda}] = -2i[\tau^*, \bar\tau^*] \\ &[L, \Delta_\tau + \Delta_{\bar\tau} + 3\Delta_{\bar\lambda} + 3\Delta_\lambda] = 0 && [\Lambda, \Delta_\tau + \Delta_{\bar\tau} + 3\Delta_{\bar\lambda} + 3\Delta_\lambda] = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} [\partial, \bar\tau^*] = -[\partial, \bar\partial^*] = [\tau, \bar\partial^*] \\ [\bar\partial, \tau^*] = -[\bar\partial, \partial^*] = [\bar\tau, \partial^*] \end{aligned} \tag{5}$$

$$
\begin{aligned}
[\partial,\tau] &= -i[\bar\partial^* + \bar\tau^*,\lambda] \\
[\bar\partial,\bar\tau] &= i[\partial^* + \tau^*,\bar\lambda] \\
[\partial^*,\tau^*] &= i[\bar\partial + \bar\tau,\lambda^*] \\
[\bar\partial^*,\bar\tau^*] &= -i[\partial + \tau,\bar\lambda^*]
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
[\Lambda,\bar\partial] &= -i(\partial^* + \tau^*) \\
[\Lambda,\partial] &= i(\bar\partial^* + \bar\tau^*) \\
[L,\bar\partial^*] &= -i(\partial + \tau) \\
[L,\partial^*] &= i(\bar\partial + \bar\tau),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
[\Lambda,\Delta_\partial] &= [\partial,\lambda^*] + i[\partial^*,\bar\tau^*] \\
[\Lambda,\Delta_{\bar\partial}] &= [\bar\partial,\bar\lambda^*] - i[\bar\partial^*,\tau^*] \\
[L,\Delta_\partial] &= -[\partial^*,\lambda] + i[\partial,\bar\tau] \\
[L,\Delta_{\bar\partial}] &= -[\bar\partial^*,\bar\lambda] - i[\bar\partial,\tau].
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\Delta_\partial + [\partial,\tau^*] &= \Delta_{\bar\partial} + [\bar\partial,\bar\tau^*] \\
\Delta_\partial + [\partial^*,\tau] &= \Delta_{\bar\partial} + [\bar\partial^*,\bar\tau] \\
\Delta_d &= \Delta_\partial + \Delta_{\bar\partial} - [\partial,\bar\tau^*] - [\bar\partial,\tau^*] \\
\Delta_{\bar\partial} + \Delta_\lambda &= \Delta_{\partial+\tau} + [\Lambda,[\Lambda,\frac{i}{2}\partial\bar\partial\omega]]
\end{aligned} \tag{9}
$$

The following subspaces of d-harmonic forms,

$$\text{Ker }(\Delta_{\bar\partial} + \Delta_\partial + \Delta_{\bar\tau} + \Delta_\tau + \Delta_{\bar\lambda} + \Delta_\lambda) \cap \mathcal{A}^{p,q},$$

satisfy Hodge, Serre, and conjugation dualities, since the operator is real and self adjoint. By identities (4) and (8), it follows that there is an induced finite dimensional representation of $\mathfrak{sl}(2)$ on the direct sum of these (p,q)-spaces, and so hard Lefschetz duality holds. See [Wil19] for details and applications.

References

[Dem86] Jean-Pierre Demailly. Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne. In *Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984*, volume 1198 of *Lecture Notes in Math.*, pages 88–97. Springer, Berlin, 1986.

[Gri66] Phillip A. Griffiths. The extension problem in complex analysis. II. Embeddings with positive normal bundle. *Amer. J. Math.*, 88:366–446, 1966.

[Wil19] S.O. Wilson. Harmonic symmetries for hermitian manifolds. *Preprint arxiv:1906.02952*, 2019.

[,]	L	Λ	H	∂	$\bar\partial$	∂^*	$\bar\partial^*$	λ	$\bar\lambda$	λ^*	$\bar\lambda^*$	τ	$\bar\tau$	τ^*	$\bar\tau^*$
deg	$(1,1)$	$(-1,-1)$	$(0,0)$	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$	$(2,1)$	$(1,2)$	$(-2,-1)$	$(-1,-2)$	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$
ord	0	2	1	1	1	2	2	0	0	3	3	1	1	2	2
L	0	H	$-2L$	$-\lambda$	$-\bar\lambda$	$i(\bar\partial+\bar\tau)$	$-i(\partial+\tau)$	0	0	$-\tau^*$	$-\bar\tau^*$	3λ	$3\bar\lambda$	$-2i\bar\tau$	$2i\tau$
Λ	-H	0	2Λ	$i(\bar\partial^*+\bar\tau^*)$	$-i(\partial^*+\tau^*)$	λ^*	$\bar\lambda^*$	τ	$\bar\tau$	0	0	$-2i\bar\tau^*$	$2i\tau^*$	$-3\lambda^*$	$-3\bar\lambda^*$
H	$2L$	-2Λ	0	∂	$\bar\partial$	$-\partial^*$	$-\bar\partial^*$	3λ	$3\bar\lambda$	$-3\lambda^*$	$-3\bar\lambda^*$	τ	$\bar\tau$	$-\tau^*$	$-\bar\tau^*$
∂	λ	$-i(\bar\partial^*+\bar\tau^*)$	$-\partial$	0	0	Δ_∂	(5)	0	$\partial\bar\partial\omega$	(8)	(6)	(6)	(8)	(9)	(5)
$\bar\partial$	$\bar\lambda$	$i(\partial^*+\tau^*)$	$-\bar\partial$	0	0	(5)	$\Delta_{\bar\partial}$	$\bar\partial\partial\omega$	0	(6)	(8)	(8)	(6)	(5)	(9)
∂^*	$-i(\bar\partial+\bar\tau)$	$-\lambda^*$	∂^*	Δ_∂	(5)	0	0	(8)	(6)	0	$(\partial\bar\partial\omega)^*$	(9)	(5)	(6)	(8)
$\bar\partial^*$	$i(\partial+\tau)$	$-\bar\lambda^*$	$\bar\partial^*$	(5)	$\Delta_{\bar\partial}$	0	0	(6)	(8)	$(\bar\partial\partial\omega)^*$	0	(5)	(9)	(8)	(6)
λ	0	$-\tau$	-3λ	0	$\partial\bar\partial\omega$	(8)	(6)	0	0	Δ_λ	$\frac{2}{3}[\bar\tau^*,\tau]$	0	$-[\bar\lambda,\tau]$	$-[L,\Delta_\lambda]$	$[\tau,\tau]/2i$
$\bar\lambda$	0	$-\bar\tau$	$-3\bar\lambda$	$\bar\partial\partial\omega$	0	(6)	(8)	0	0	$\frac{2}{3}[\tau^*,\bar\tau]$	$\Delta_{\bar\lambda}$	$-[\bar\tau,\lambda]$	0	$i[\bar\tau,\bar\tau]/2$	$-[L,\Delta_{\bar\lambda}]$
λ^*	τ^*	0	$3\lambda^*$	(8)	(6)	0	$(\bar\partial\partial\omega)^*$	Δ_λ	$\frac{2}{3}[\tau^*,\bar\tau]$	0	0	$[\Lambda,\Delta_\lambda]$	$i[\tau^*,\tau^*]/2$	0	$-[\bar\lambda^*,\tau*]$
$\bar\lambda^*$	$\bar\tau^*$	0	$3\bar\lambda^*$	(6)	(8)	$(\partial\bar\partial\omega)^*$	0	$\frac{2}{3}[\bar\tau^*,\tau]$	$\Delta_{\bar\lambda}$	0	0	$[\bar\tau^*,\bar\tau^*]/2i$	$[\Lambda,\Delta_{\bar\lambda}]$	$-[\bar\tau^*,\lambda^*]$	0
τ	-3λ	$2i\bar\tau^*$	$-\tau$	(6)	(8)	(9)	(5)	0	$-[\bar\tau,\lambda]$	$[\Lambda,\Delta_\lambda]$	$[\bar\tau^*,\bar\tau^*]/2i$	$2i[\lambda,\bar\tau^*]$	$\frac{i}{2}[L,\Delta_\tau+3\Delta_\lambda]$	Δ_τ	$3[\bar\lambda^*,\lambda]/2$
$\bar\tau$	$-3\bar\lambda$	$-2i\tau^*$	$-\bar\tau$	(8)	(6)	(5)	(9)	$-[\bar\lambda,\tau]$	0	$i[\tau^*,\tau^*]/2$	$[\Lambda,\Delta_{\bar\lambda}]$	$\frac{i}{2}[L,\Delta_\tau+3\Delta_\lambda]$	$-2i[\bar\lambda,\tau^*]$	$3[\lambda^*,\bar\lambda]/2$	$\Delta_{\bar\tau}$
τ^*	$2i\bar\tau$	$3\lambda^*$	τ^*	(9)	(5)	(6)	(8)	$-[L,\Delta_\lambda]$	$i[\bar\tau,\bar\tau]/2$	0	$-[\bar\tau^*,\lambda^*]$	Δ_τ	$3[\lambda^*,\bar\lambda]/2$	$-2i[\lambda^*,\bar\tau]$	$\frac{i}{2}[\Lambda,\Delta_\tau+3\Delta_\lambda]$
$\bar\tau^*$	$-2i\tau$	$3\bar\lambda^*$	$\bar\tau^*$	(5)	(9)	(8)	(6)	$[\tau,\tau]/2i$	$-[L,\Delta_{\bar\lambda}]$	$-[\bar\lambda^*,\tau*]$	0	$3[\bar\lambda^*,\lambda]/2$	$\Delta_{\bar\tau}$	$\frac{i}{2}[\Lambda,\Delta_\tau+3\Delta_\lambda]$	$2i[\bar\lambda^*,\tau]$

The graded Lie algebra of differential operators on a Hermitian manifold

(compiled by Scott O. Wilson)