HERMITIAN IDENTITIES

SCOTT O. WILSON

We list the identities that hold on a complex manifold with compatible metric. Identity (7) is proved in [Dem86] using a local argument, and it is stated there that the idea was implicit in [Gri66]. The remaining identities follow from algebraic manipulations, and all appear in [Dem86], except for identity (1) which is clear, and identities (4) and (8), which are used to prove a generalized hard Lefschetz duality below.

Let $L$ be the Lefschetz operator, $\Lambda = L^*$, $\lambda = [\partial, L]$, $\bar{\lambda} = [\bar{\partial}, L]$, $\tau = [\Lambda, \lambda]$, $\bar{\tau} = [\Lambda, \bar{\lambda}]$, and $\Delta_\delta = [\delta, \delta^*]$ for any operator $\delta$. Then

$$
\begin{align*}
\bar{\partial}^2 &= 0 & \bar{\lambda}^2 &= 0 & [\bar{\lambda}, \lambda] &= 0 \\
\partial^2 &= 0 & [\bar{\partial}, \lambda] &= 0 & [\partial, \lambda] &= 0 \\
[\bar{\partial}, \partial] &= 0 & [\bar{\lambda}, \partial] + [\bar{\lambda}, \partial] &= 0 & \lambda^2 &= 0,
\end{align*}
$$

(1)

as well as the adjoint of these equations. Additional adjoint or conjugate equations may (or may not) be omitted.

$$
\begin{align*}
[\Lambda, \tau] &= -2i\bar{\tau}^* & [L, \bar{\tau}] &= 3\bar{\lambda} & [\Lambda, \lambda] &= \tau & [L, \lambda] &= 0 \\
[\Lambda, \bar{\tau}] &= 2i\tau^* & [L, \tau] &= 3\lambda & [\Lambda, \bar{\lambda}] &= \bar{\tau} & [L, \bar{\lambda}] &= 0 \\
[L, \tau^*] &= -2i\bar{\tau} & [\Lambda, \bar{\tau}^*] &= -3\lambda^* & [L, \lambda^*] &= -\tau^* & [\Lambda, \lambda^*] &= 0 \\
[L, \bar{\tau}^*] &= 2i\tau & [\Lambda, \tau^*] &= -3\lambda^* & [L, \bar{\lambda}^*] &= -\bar{\tau}^* & [\Lambda, \bar{\lambda}^*] &= 0.
\end{align*}
$$

(2)

$$
\begin{align*}
[\lambda, \bar{\tau}] &= -[\bar{\tau}, \lambda] & [\tau, \bar{\tau}] &= 2i[\lambda, \bar{\tau}^*] & 2[\bar{\tau}^*, \tau] &= 3[\bar{\lambda}^*, \lambda] \\
[\bar{\lambda}, \tau] &= -[\bar{\lambda}, \bar{\tau}] & [\bar{\tau}, \tau] &= 2i[\bar{\lambda}, \tau^*] & 2[\tau^*, \bar{\tau}] &= 3[\lambda^*, \bar{\lambda}]
\end{align*}
$$

(3)

$$
\begin{align*}
[\lambda^*, \tau^*] &= -[\tau^*, \lambda^*] & [\tau^*, \tau^*] &= -2i[\lambda^*, \bar{\tau}] & 2[\bar{\tau}, \tau^*] &= 3[\bar{\lambda}, \lambda^*] \\
[\lambda^*, \bar{\tau}^*] &= -[\tau^*, \lambda^*] & [\bar{\tau}^*, \bar{\tau}^*] &= 2i[\lambda^*, \tau]
\end{align*}
$$

(4)

$$
\begin{align*}
[L, \Delta_\tau + 3\Delta_\lambda] &= -2i[\tau, \bar{\tau}] & [\Lambda, \Delta_\tau + 3\Delta_\lambda] &= 2i[\tau^*, \bar{\tau}^*] \\
[L, \Delta_\bar{\tau} + 3\Delta_{\bar{\lambda}}] &= 2i[\tau, \bar{\tau}] & [\Lambda, \Delta_{\bar{\tau}} + 3\Delta_{\bar{\lambda}}] &= -2i[\tau^*, \bar{\tau}^*] \\
[L, \Delta_\tau + \Delta_{\bar{\tau}} + 3\Delta_{\lambda} + 3\Delta_{\bar{\lambda}}] &= 0 & [\Lambda, \Delta_\tau + \Delta_{\bar{\tau}} + 3\Delta_{\lambda} + 3\Delta_{\bar{\lambda}}] &= 0
\end{align*}
$$

(5)

$$
\begin{align*}
\partial^* &= [\partial, \partial^*] = [\partial, \partial^*] \\
\bar{\partial}^* &= [\bar{\partial}, \bar{\partial}^*] = [\bar{\partial}, \bar{\partial}^*]
\end{align*}
$$
\[ [\partial, \tau] = -i[\bar{\partial}^* + \tau^*, \lambda] \]
\[ [\bar{\partial}, \bar{\tau}] = i[\partial^* + \tau^*, \bar{\lambda}] \]
\[ [\partial^*, \tau^*] = i[\bar{\partial} + \bar{\tau}, \lambda^*] \]
\[ [\bar{\partial}^*, \bar{\tau}^*] = -i[\partial + \tau, \lambda^*] \]
\[ [\Lambda, \bar{\partial}] = -i(\partial^* + \tau^*) \]
\[ [\Lambda, \partial] = i(\bar{\partial}^* + \bar{\tau}^*) \]
\[ [L, \bar{\partial}^*] = -i(\partial + \bar{\tau}) \]
\[ [L, \partial^*] = i(\bar{\partial} + \bar{\tau}) \]
\[ [\Lambda, \Delta_{\partial}] = [\partial, \lambda^*] + i[\partial^*, \bar{\tau}^*] \]
\[ [\Lambda, \Delta_{\bar{\partial}}] = [\bar{\partial}, \bar{\lambda}^*] - i[\bar{\partial}^*, \tau^*] \]
\[ [L, \Delta_{\partial}] = -[\partial^*, \lambda] + i[\partial, \bar{\tau}] \]
\[ [L, \Delta_{\bar{\partial}}] = -[\bar{\partial}^*, \bar{\lambda}] - i[\bar{\partial}, \tau] \]
\[ \Delta_{\partial} + [\partial, \tau^*] = \Delta_{\bar{\partial}} + [\bar{\partial}, \bar{\tau}^*] \]
\[ \Delta_{\bar{\partial}} + [\partial^*, \tau] = \Delta_{\bar{\partial}} + [\bar{\partial}^*, \bar{\tau}] \]
\[ \Delta_d = \Delta_{\partial} + \Delta_{\bar{\partial}} - [\partial, \tau^*] - [\bar{\partial}, \tau^*] \]
\[ \Delta_{\partial} + \Delta_{\bar{\partial}} = \Delta_{\partial + \tau} + [\Lambda, \frac{i}{2} 2\bar{\partial} \omega] \]

The following subspaces of $d$-harmonic forms,
\[ \text{Ker} (\Delta_{\partial} + \Delta_{\bar{\partial}} + \Delta_{\tau} + \Delta_{\lambda} + \Delta_{\bar{\lambda}}) \cap A^{p,q}, \]
satisfy Hodge, Serre, and conjugation dualities, since the operator is real and self adjoint. By identities (4) and (8), it follows that there is an induced finite dimensional representation of $\mathfrak{sl}(2)$ on the direct sum of these $(p, q)$-spaces, and so hard Lefschetz duality holds. See [Wil19] for details and applications.

REFERENCES
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The graded Lie algebra of differential operators on a Hermitian manifold (compiled by Scott O. Wilson)