MATH 320/620 Point-Set Topology
by Scott Wilson

Here are solutions to Exercises $1, 2, 3, 4$ in Chapter 1.6 of Mendelson's book. The solutions assume that the reader is familiar with all relevant definitions.

(1) Let $f : A \to B$ be given.
  (a) If $x \in X$ then $f(x) \in f(X)$, so $x \in f^{-1}(f(X))$.
  (b) If $Y \subset B$, and $y \in f(f^{-1}(Y))$, then $y = f(x)$ for $x \in f^{-1}(Y)$, so $y = f(x) \in Y$.
  (c) If $f : A \to B$ is one-to-one, then for each $X \subset A$, we show $f^{-1}(f(X)) \subset X$. (The other containment always holds, by the first part.) Let $z \in f^{-1}(f(X))$, so that $f(z) \in f(X)$, i.e. $f(z) = f(x)$ for some $x \in X$. Since f is one-to-one, $z = x$, so $z \in X$.
  (d) If $f : A \to B$ is onto, then for each $Y \subset B$, we show $Y \subset f(f^{-1}(Y))$. (The other containment always holds, by the second part.) Let $y \in Y$. Since f is onto, $y = f(x)$ for some $x \in f^{-1}(Y) \subset A$. Then $y \in f(f^{-1}(Y))$.

(2) Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, each with two distinct elements. Let $f : A \to B$ be such that $f(x) = b_1$ for $x = a_1, a_2$.
  (a) $f^{-1}(f(\{a_1\})) = f^{-1}(\{b_1\}) = \{a_1, a_2\} \neq \{a_1\}$.
  (b) $f(f^{-1}(B)) = f(A) = \{b_1\} \neq B$.
  (c) $\emptyset = f(\emptyset) = f(\{a_1\} \cap \{a_2\}) \neq f(\{a_1\}) \cap f(\{a_2\}) = \{b_1\} \cap \{b_1\} = \{b_1\}$.

(3) Let $f : A \to B$ be given and let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of subsets of A.
  (a) We show $f\left(\bigcup_{\alpha\in I} X_\alpha\right) = \bigcup_{\alpha \in I} f(X_\alpha)$. We have: $y \in f\left(\bigcup_{\alpha\in I} X_\alpha\right)$ iff[1] $y = f(x)$ for some $x \in \bigcup_{\alpha\in I} X_\alpha$, iff $y = f(x)$ for some $x \in X_\alpha$ for some $\alpha \in I$, iff $y \in f(X_\alpha)$ for some $\alpha \in I$, iff $y \in \bigcup_{\alpha\in I} f(X_\alpha)$.
  (b) We show $f\left(\bigcap_{\alpha\in I} X_\alpha\right) \subset \bigcap_{\alpha\in I} f(X_\alpha)$. We have: $y \in f\left(\bigcap_{\alpha\in I} X_\alpha\right)$ implies $y = f(x)$ for some $x \in \bigcap_{\alpha\in I} X_\alpha$, which implies $y = f(x)$ for some x satisfying $x \in X_\alpha$ for all $\alpha \in I$. This implies $y \in f(X_\alpha)$ for all $\alpha \in I$, and so $y \in \bigcap_{\alpha\in I} f(X_\alpha)$.
  (c) Suppose $f : A \to B$ is one-to-one. We show $\bigcap_{\alpha\in I} f(X_\alpha) \subset f\left(\bigcap_{\alpha\in I} X_\alpha\right)$. If $y \in \bigcap_{\alpha\in I} f(X_\alpha)$ then for each $\alpha \in I$, $y = f(x_\alpha)$ for some $x_\alpha \in X_\alpha$. Since f is one-to-one, we must have $x_\alpha = x_\beta$ for any $\alpha, \beta \in I$, so there is a unique $x \in \bigcap_{\alpha\in I} X_\alpha$ such that $f(x) = y$. Then $y \in f\left(\bigcap_{\alpha\in I} X_\alpha\right)$.

(4) Let $f : A \to B$ be given and let $\{Y_\alpha\}_{\alpha\in I}$ be an indexed family of subsets of B.
  (a) We show $f^{-1}\left(\bigcup_{\alpha\in I} Y_\alpha\right) = \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$. We have: $x \in f^{-1}\left(\bigcup_{\alpha\in I} Y_\alpha\right)$ iff $f(x) \in \bigcup_{\alpha\in I} Y_\alpha$, iff $f(x) \in Y_\alpha$ for some $\alpha \in I$, iff $x \in f^{-1}(Y_\alpha)$ for some $\alpha \in I$, iff $x \in \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$.
  (b) We show $f^{-1}\left(\bigcap_{\alpha\in I} Y_\alpha\right) = \bigcap_{\alpha\in I} f^{-1}(Y_\alpha)$. We have: $x \in f^{-1}\left(\bigcap_{\alpha\in I} Y_\alpha\right)$ iff $f(x) \in \bigcap_{\alpha\in I} Y_\alpha$, iff $f(x) \in Y_\alpha$ for all $\alpha \in I$, iff $x \in f^{-1}(Y_\alpha)$ for all $\alpha \in I$, iff $x \in \bigcap_{\alpha\in I} f^{-1}(Y_\alpha)$.
  (c) The exercise[2] assumes $X \subset B$, but I will take $Y \subset B$, and show $f^{-1}(B - Y) = A - f^{-1}(Y)$. We have $a \in f^{-1}(B - Y) \subset A$ iff $f(a) \in B - Y$, iff $f(a) \in B$ and $f(a) \notin Y$, iff $a \in A$ and $a \notin f^{-1}(Y)$, or equivalently, $a \in A - f^{-1}(Y)$.
  (d) If $X \subset A$ and $Y \subset B$, we show $f(X \cap f^{-1}(Y)) = f(X) \cap Y$. By problem $3b$ and $1b$ we have $f(X \cap f^{-1}(Y)) \subset f(X) \cap f(f^{-1}(Y)) \subset f(X) \cap Y$. To prove the reverse containment, if $y \in f(X) \cap Y$ then $y = f(x)$ for some $x \in X \cap f^{-1}(Y)$, so that $y = f(x) \in f(X \cap f^{-1}(Y))$.

[1] "iff" means "if and only if"

[2] The author writes $C(Z)$ for the complement of Z in B. I prefer the notation $B - Z$.