A LOOP GROUP EXTENSION OF THE ODD CHERN CHARACTER

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ABSTRACT. We show that the universal odd Chern form, defined on the stable unitary group U, extends to the loop group LU as an equivariantly closed differential form. This provides an odd analogue to the Bismut-Chern form that appears in supersymmetric field theories. We also describe the associated transgression form, the so-called Bismut-Chern-Simons form, and explicate some properties it inherits as a differential form on the space of maps of a cylinder into the stable unitary group.

As one corollary, we show that in a precise sense the spectral flow of a loop of self adjoint Fredholm operators equals the lowest degree component of the Bismut-Chern-Simons form, and the latter, when restricted to cylinders which are tori, is an equivariantly closed extension of spectral flow. As another corollary, we construct the Chern character homomorphism from odd K-theory to the periodic cohomology of the free loop space, represented geometrically on the level of differential forms.

Contents

1. Introduction

The Chern character homomorphism plays a fundamental role in topology and geometry as it relates K-theory to ordinary cohomology. Several recent works have emphasized differential refinements of such cohomology theories [HS], [BS], where one works with geometric representatives, and the data of differential forms. In [TWZ3], the authors show that a careful study of the odd Chern Character, represented geometrically by an odd differential form on the stable unitary group U, leads to an explicit differential refinement of odd K-theory. A similar model was constructed in [HMSV] using loop group bundles, i.e. maps into BLU as opposed

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to U, and furthermore it was shown that this and the model in [TWZ3] do indeed define odd differential K-theory, which is unique up to unique isomorphism [BS].

In the setting of even K-theory, an analogous role is played by the even degree Chern form, which is associated to a connection ∇ on a bundle over a manifold M. The even Chern form on M has been shown by Bismut to extend to a differential form on the free loop space LM, which is closed with respect to an equivariant-type differential [B]. Bismut introduced this differential form as a contribution to the integrand of a path integral over the free loop space which calculates (non-rigorously) the index of a certain twisted Dirac operator. Subsequently, in [Ha] and [ST], the Bismut-Chern form has been given a rigorous field theoretic interpretation, appearing naturally in the dimensional reduction from a $1|1$ supersymmetric Euclidean field theory on M, to a $0|1$ field theory on LM.

This leads to the natural question of whether the odd Chern form on U extends to a differential form on the loop group LU in a way that is closed with respect to the equivariant-type differential. We answer this question affirmatively by constructing this *odd Bismut-Chern form* using new *equivariant* transgression techniques for free loop spaces that we establish below. Interestingly, this differential form on LU is given by an explicit iterated integral formula involving only the left invariant 1-form on U and its directional derivatives.

There are several reasons for studying this differential form on the loop group LU. One reasons is that it provides the universal form required to produced a differential refinement of odd K-theory by differential forms on the free loop space. This idea, or rather its even K-theoretic analogue, was initiated in [TWZ2] and more recently was completed in [BNV] using sheaves of spectra to obtain a full-fledged differential cohomology theory which is refined by differential forms on the free loop space of M, rather than M itself.

Another reason for interest in this differential form, and its transgression form, is for its connections with index theory and the spectral flow along families of Fredholm operators. This result is explained below. Finally, it is expected that this differential form on the loop groups LU will have a field theoretic interpretation as well.

The contents of this paper are as follows. In the next section we provide preliminary definitions and results concerning the complex of differential forms on free loop spaces that is used in the paper. These are somewhat scattered throughout the literature and it seems useful to have one comprehensive reference. In section 3 we provide background on the Bismut-Chern form, as well as a re-interpretation as a universal form on the free loop space LBU. The next section provides a general method for constructing transgression forms on free loop spaces, and defines the transgression form associated to the Bismut-Chern form. The existence of such a form, as the solution to a differential equation, was first shown in [B], with the first explicit formuls for the solution appearing [TWZ2]. We call this the Bismut-Chern-Simons form since it restricts along constant loops to the Chern-Simons form.

In section 5 we provide a brief review of the odd Chern form on U and in the subsequent sections define the odd Bismut-Chern form and prove several fundamental properties. In particular, we give an explicit formula for the transgression of this form, which is an even differential form on the space of maps of cylinder into U, whose value in degree zero is a lift of the "winding number" function. As we explain, it also agrees in lowest degree with the spectral flow of one parameter family of self adjoint Fredholm operators [P], and as a corollary we obtain an equivariantly-closed extension of the spectral flow to the torus mapping space of self adjoint Fredholm operators. We close with an application which produces a lift of the the odd Chern Character homomorphism $Ch : K^{-1} \to H^{odd}$ to a natural transformation with image in the the completed periodic cohomology of the free loop space.

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2. Preliminaries

For a smooth manifold M, let LM denote the space of smooth loops in M, considered as a diffeological (Chen) space [C1], [C2], or as a Fréchet space, [H]. In fact, these structures agree, [L], [Wa] Lemma A.1.7.

The free loop space LM has a natural vector field v, given by the circle action, whose induced contraction operator on differential forms is denoted by ι. We use the notation $\Omega^j_{d+\iota}$ to denote the vector space of invariant complex valued differential forms of degree j, i.e. those that are in the kernel of $L_v = d\iota + \iota d$.

Let

$$\Omega^{even}_{d+\iota}(LM) = \prod_{k\geq 0} \Omega^{2k}_{d+\iota}(LM), \quad \Omega^{odd}_{d+\iota}(LM) = \prod_{k\geq 0} \Omega^{2k+1}_{d+\iota}(LM).$$

The operator $d+\iota$ defines a $\mathbb{Z}_2$-graded complex and we denote the cohomology by

$$H^{\text{even}}_{d+\iota}(LM), \quad H^{\text{odd}}_{d+\iota}(LM).$$

The earliest reference for these groups are [W], [A], and [B], with more recent related work in [KM].

In [JP] the authors show these cohomology groups are isomorphic to the *completed periodic equivariant cohomology* $h^*_{S^1}(LM)$ in even and odd degrees, respectively. Recall this is the cohomology of the *completed periodic equivariant complex* which is defined to be the differential graded ring $(\Omega(LM)[[u,u^{-1}]], d+u\iota)$ consisting of formal power series in u, u^{-1} with coefficients in $\Omega_{d+\iota}(LM)$, where u is an indeterminant of degree 2. Note this allows for elements with arbitrary high powers of u^{-1}, but not u, since forms are concentrated in positive degrees. All of the results in this paper can be restated in terms in the completed periodic equivariant complex by introduction of this formal variables u^{-1}, and the complex with differential $d+\iota$ is recovered by setting $u=1$.

Finally, it follows from Theorem 2.1 of [JP] that the inclusion of constant loops $\rho : M \to LM$ induces an isomorphism

$$\rho^* : h^*_{S^1}(LM) \to u^{-1}H^*_{S^1}(M) = H^*(M) \otimes \mathbb{C}[u,u^{-1}]$$

where $u^{-1}H^*_{S^1}(M)$ is the localization with respect to u of the usual equivariant cohomology of M (with the trivial circle action). In summary, we have (for each

$n \geq 0$) that the restriction to constant loops induces an isomorphism

$$H^{even}_{d+\iota}(LM) \cong \left(H^*(M) \otimes \mathbb{C}[u,u^{-1}]\right)_{2n} \cong H^{even}(M)$$
$$H^{odd}_{d+\iota}(LM) \cong \left(H^*(M) \otimes \mathbb{C}[u,u^{-1}]\right)_{2n+1} \cong H^{odd}(M).$$

So, we obtain an expression of this equivariant-type cohomology of the free loop space in terms of the ordinary cohomology of M.

These results and their proofs have recently been generalized in [HM] for differential forms on LM which have a differential that is twisted by the data of an abelian gerbe with connection.

3. The even Bismut-Chern form

It was first shown in [B] that the even degree Chern form of a connection on a bundle extends to an $(d+\iota)$-closed even differential form on the free loop space of the base. This works was revisited in [GJP], in the context of cyclic homology. In [TWZ], and [TWZ2], the authors re-interpreted the now-called Bismut-Chern form as a $(d+\iota)$-closed extension of the trace of holonomy function on the loop space, and developed several properties with respect to sum and tensor products of connections.

We review this material in this section.

Proposition 3.1. *([B]) For any connection ∇ on a complex vector bundle $E \to M$, there is a differential form $BCh(\nabla) \in \Omega^{even}(LM)$ satisfying*

$$(d+\iota)BCh(\nabla) = 0,$$

which therefore determines a class $[BCh(\nabla)] \in \Omega^{even}_{S^1}(LM)$. Moreover, we have

$$\rho^* BCh(\nabla) = Ch(\nabla)$$

where $\rho^* : \Omega^{even}_{S^1}(LM) \to \Omega^{even}(M)$ is the restriction to constant loops and $Ch(\nabla) = Tr\left(e^R\right) \in \Omega^{even}(M)$ is the ordinary even Chern character.

Let us explain how the differential form $BCh(\nabla)$ is defined using local formulas. On any single local trivialization V of M, we can write the connection locally as a matrix A of 1-forms, with curvature R. There is a degree $2k$ form $Tr(hol^V_{2k})$ on LV given by

$$Tr(hol^V_{2k}) = Tr\left(\sum_{m\geq k}\ \sum_{1\leq j_1<\cdots<j_k\leq m} \int_{\Delta^m} X_1(t_1)\cdots X_m(t_m)dt_1\cdots dt_m\right), \tag{3.1}$$

where

$$X_j(t_j) = \begin{cases} R(t_j) & \text{if } j \in \{j_1,\ldots,j_k\} \\ \iota A(t_j) & \text{otherwise} \end{cases}$$

Here $R(t_j)$ is a 2-form taking in two vectors at $\gamma(t_j)$ on a loop $\gamma \in V$, and $\iota A(t_j) = A(\gamma'(t_j))$. Note that $Tr(hol_0)$ is simply the trace of the usual holonomy, and heustically $Tr(hol^V_{2k})$ is given by the same formula for the trace of holonomy, except with exactly k copies of the function ιA replaced by the 2-form R, summed over all possible places.

Summing over all degrees we obtain an expression that first appeared in [GJP], where it is shown in their Theorem 6.5 that this satisfies the differential equation which was originally formulated by Bismut [B], so that by the uniqueness theorem of ODE's the above local expression and the global formulation by Bismut agree.

More generally, a global form is defined on LM is defined using a open covering of LM induced by an open covering of M over which the bundle is locally trivialized. The formula above, essentially glued together by transition data, is shown explicitly to define an even differential form on LM, independent of choice of local data, which we denote by $Tr(hol_{2k})$, [TWZ]. Then, by definition,

$$BCh(\nabla) = \sum_{k\geq 0} Tr(hol_{2k}).$$

We have omitted the scalar factor $1/2\pi i$ that is sometimes used in the definition of the even Chern character for its relation to integral cohomology. By replacing R by $R/2\pi i$ in the definition of the Bismut-Chern form, we obtain a form that restricts to $Tr\left(e^{R/2\pi i}\right)$, and is closed with respect to $d + (2\pi i)\iota$. This operator produces a complex that is isomorphic to the one used here.

Remark 3.2. We note that a twisted version of this form has been introduced in [HM], also using local data. The twisting data comes from an abelian gerbe on M with local connection 2-form B, and in the case of $B = 0$, the trace of the degree $2k$ component of the twisted BCh form in ([HM], Equation (44)) agrees with Equation (3.1) above. (In [HM], curvature is denoted by F, wheresas here we denote it by R. We have chosen to collect here the summands by form degree, whereas in [HM] the connection terms A are gathered as parallel transport operators. Note that when $B = 0$, the 3-curvature H vanishes, and the twisted differential agrees with $d + \iota$ used here, so the underlying complexes are the same.)

In the next section we will give a new global definition of BCh; it is not known to the author at the present how to give a global definition of the twisted BCh-form.

In [TWZ2] the following properties are proved.

Theorem 3.3. *([TWZ2], Theorem 3.3) Let $(E, \nabla) \to M$ and $(\bar{E}, \bar{\nabla}) \to M$ be complex vector bundles with connections. Let $\nabla \oplus \bar{\nabla}$ be the induced connection on $E \oplus \bar{E} \to M$, and $\nabla \otimes \bar{\nabla} := \nabla \otimes Id + Id \otimes \bar{\nabla}$ be the induced connection on $E \otimes \bar{E} \to M$. Then*

$$BCh(\nabla \oplus \bar{\nabla}) = BCh(\nabla) + BCh(\bar{\nabla})$$

and

$$BCh(\nabla \otimes \bar{\nabla}) = BCh(\nabla) \wedge BCh(\bar{\nabla}).$$

Remark 3.4. A complex vector bundle $E \to M$ of rank k with hermitian connection ∇ can always be given by the pullback along a map $M \to BU(k)$ of the universal bundle with universal connection over $BU(k)$, [NR], [NR2]. Here our model for $BU(k)$ is the limit of the Grassmann manifolds of k-planes in $\mathbb{C}^n$, with tautological bundles over them given by vectors in the k-planes. These bundles sit inside the trivial bundles with trivial connection given by exterior d, and the tautological sub-bundle has an induced connection given by the composition of projection P onto the k-plane composed with d, i.e. $\nabla = P \circ d$. The curvature of this connection is given by $R = P(dP)^2$. In this way we can describe the universal Bismut-Chern form on $LBU(k)$. An explicit formula is given by

$$Tr\left(\sum_{\substack{2k\geq 0\\ m\geq k}} \sum_{1\leq j_1<\cdots<j_{2k}\leq m} \int_{\Delta^m} X_1(t_1)\cdots X_m(t_m)dt_1\cdots dt_m\right), \tag{3.2}$$

where

$$X_j(t_j) = \begin{cases} P(dP)^2(t_j) & \text{if } j \in \{j_1, \dots, j_{2k}\} \\ \iota P(dP)(t_j) & \text{otherwise} \end{cases}$$

4. Transgression forms on free loop spaces

Assigning a differential form to some geometric data on a manifold often applies as well to a compact one-parameter family of the data. By integration along the family, we obtain a form of degree one lower whose exterior derivative is the difference between the differential forms assigned to the endpoint data. We explain now that this can also be accomplished for the free loop space LM and the operator $d + \iota$.

We will apply this to produce a *transgression form* for the BCh-form given above. In fact, such a differential form, assigned to a one parameter family of connections on a bundle, was shown by Bismut to exist in [B], as the solution to a certain differential equation. In fact, as we explain below, the transgression form we produce is the solution to this differential equation, see Remark 4.3.

Let $I = [0, 1]$. There is a map $j : LM \times I \to L(M \times I) = LM \times LI$ which is induced by the inclusion $I \to LI$ of constant loops. Note that $j^* : \Omega^*(L(M \times I)) \to \Omega^*(LM \times I)$ commutes with the contraction operators ι and exterior derivative d so there is a well defined map $j^* : \Omega^*_{d+\iota}(L(M \times I)) \to \Omega^*_{d+\iota}(LM)$.

Consider the composition

$$\begin{tikzcd} \Omega^*(LM \times I) \arrow[d, "\int_I"'] & \Omega^*(L(M \times I)) \arrow[l, "j^*"'] \\ \Omega^*(LM) & \end{tikzcd}$$

where $\int_I$ is integration along the fiber I. By Stokes' Theorem for integration along fiber, and the fact that for all forms ω we have

$$\iota \int_I j^*\omega = \int_I \iota j^*\omega = \int_I j^*\iota\omega,$$

it follows that we have the following *equivariant Stokes' formula* for integration along the fiber

$$(d + \iota) \int_I j^*\omega = \int_{\partial I} j^*\omega + \int_I j^*(d + \iota)\omega.$$

To our knowledge, this formula is new and of independent interest.

Definition 4.1. Let ∇_s be a path of connections on a bundle $E \to M$. Let $\tilde{E} \to M \times I$ be the pullback of E along the projection $M \times I \to M$, and let $\tilde{\nabla}$ be the connection on $\tilde{E}$ such that $\tilde{\nabla} = \partial_s + \nabla_s$ over the slice $M \times \{s\}$. We define the Bismut-Chern-Simons form $BCS(\nabla_s) \in \Omega^{odd}_{d+\iota}(LM)$ by

$$BCS(\nabla_s) = \int_I j^* BCh(\tilde{\nabla}).$$

From the discussion above, and the fact that BCh is $(d + \iota)$-closed, we have

$$(d + \iota)BCS(\nabla_s) = BCh(\nabla_1) - BCh(\nabla_0), \tag{4.1}$$

and this implies $(d\iota + \iota d)BCS(\nabla_s) = 0$.

An explicit formula for $BCS(\nabla_s)$ was previously given in [TWZ2], where Equation (4.1) was proved by hand ([TWZ2], Theorem 4.3). For completeness, we recall that formula here, and prove it equals the transgression form coming from Definition 4.1.

Lemma 4.2. *Let ∇_s be a path of connections on a bundle $E \to M$, and let $\tilde{\nabla} = \partial_s + \nabla_s$ be the induced connection on $\tilde{E} \to M \times I$.*

On any local trivialization $V \subset M$, the degree $2k+1$ component of $BCS(\nabla_s)$ on $LV \subset LM$ is given by

$$BCS^V_{2k+1}(A_s) = Tr\Bigg(\sum_{n \geq k+1} \sum_{\substack{r, j_1, \dots, j_k = 1 \\ \textit{pairwise distinct}}}^{n} \int_0^1 \int_{\Delta^n} \iota A_s(t_1) \dots R_s(t_{j_1}) \dots A'_s(t_r) \dots R_s(t_{j_k}) \dots \iota A_s(t_n) \quad ds\, dt_1 \dots dt_n \Bigg) \tag{4.2}$$

Here A_s is the local expression of the connection ∇_s, with curvature R_s and time derivative $A'_s = \frac{\partial A_s}{\partial s}$.

Proof. We calculate
$$\int_I j^* BCh^V_{2k+2}(\tilde{\nabla})$$
where $BCh^V_{2k+2}(\tilde{\nabla})$ denotes the degree $(2k+2)$-component of $BCh^V(\tilde{\nabla})$, given by the local expression in Equation (3.1). We have

$$BCh^V_{2k+2}(\tilde{\nabla}) = Tr\left(\sum_{m \geq k+1} \sum_{1 \leq j_1 < \dots < j_{k+1} \leq m} \int_{\Delta^m} X_1(t_1) \cdots X_m(t_m) dt_1 \cdots dt_m \right), \tag{4.3}$$

where
$$X_j(t_j) = \begin{cases} \tilde{R}_s(t_j) & \text{if } j \in \{j_1, \dots, j_{k+1}\} \\ \iota\tilde{A}_s(t_j) & \text{otherwise} \end{cases}$$

and $\tilde{A}_s$ and $\tilde{R}_s$ are the local expression for $\tilde{\nabla}$ and its curvature. The map j is an S^1-equivariant embedding with image in the loops which are constant in the I-factor, so that $j^*\iota\tilde{A}_s(t_j) = \iota A_s(t_j)$, where ι denote contraction by the vector field on the appropriate loop space. The curvature $\tilde{R}_s$ is locally given by
$$\tilde{R}_s = d_{M\times I}(A_s) + A_s \wedge A_s = A'_s ds + dA_s + A_s \wedge A_s = A'_s ds + R_s$$
Since the integration along the fiber is non-zero if and only if there is exactly one 'ds' term, we have that

$$\int_I j^* BCh^V_{2k+2}(\tilde{\nabla}) = Tr\Bigg(\sum_{m \geq k+1} \sum_{\substack{r, j_1, \dots, j_k = 1 \\ \text{pairwise distinct}}}^{m} \int_0^1 \int_{\Delta^m} \iota A_s(t_1) \dots R_s(t_{j_1}) \dots A'_s(t_r) \dots R_s(t_{j_k}) \dots \iota A_s(t_m) \quad ds\, dt_1 \dots dt_m \Bigg), \tag{4.4}$$

as claimed. $\square$

Remark 4.3. As noted in the beginning of this section, in Bismut's original proof that the $(d+\iota)$-cohomology class of $BCh(\nabla)$ does not depend on ∇, he introduces what might be called a "transgression form" as the (abstract) solution to a differential equation ([B], Equation 3.3). In ([TWZ2], Equation (4.4)) the author's show that the pre-integral local formula from Lemma 4.2 is the solution to this differential equation. Namely,

$$(d+\iota)\left(\sum_{k\geq 0} I_{2k+1}(\nabla_s)\right) = \frac{\partial}{\partial s} BCh(\nabla_s).$$

where

$$I_{2k+1} = Tr\Bigg(\sum_{m\geq k+1} \sum_{\substack{r,j_1,\dots,j_k=1 \\ \text{pairwise distinct}}}^{m} \int_{\Delta^m} \iota A_s(t_1)\dots R_s(t_{j_1})\dots A_s'(t_r)\dots R_s(t_{j_k})\dots \iota A_s(t_m) \quad dt_1\dots dt_m \Bigg) \tag{4.5}$$

Of course, this implies Equation (4.1) upon integrating over s.

Remark 4.4. In the proof of Theorem 2, Equation (52) of [HM], the authors introduce using local data a "twisted Bismut-Chern Simons transgression form". As mentioned in Remark 3.2, the twisting data comes from an abelian gerbe with connection. One would expect that if the connective data vanishes, i.e. $B=0$, then the trace of the formula in Equation (52) from [HM] would equal Equation (4.2), but it does not. The point is that in Equation (4.2) the terms A_s' appear in all possible positions, whereas in Equation (52) from [HM] they are all moved to the front (and denoted $\hat{A}$ for the straighline path of connections). The fact that trace is cyclic is not helpful; it is the time ordering of non-abelian matrices that makes them unequal.

We note that Equation (4.1) was proved in [TWZ2] using the local formula for BCS given here. In [HM] Theorem 2, the analogue of Equation (4.1) is correctly credited to Bismut [B] (see Equations 3.33 and 3.34 therein), though Bismut does not give explicitly the local formula that is proposed in [HM] Equation (52). Nevertheless, this does not affect any of the results in [HM], since the authors do not in fact use the formula they propose in (52), but rather requiring only the existence of some BCS-form satisfying the analogue of Equation (4.1), i.e. their equation (55).

A similar formula to Equation (4.2) can be obtained, using transition data, for expressing the entire odd form $BCS(\nabla_s)$ on LM, see [TWZ2], where the following is also proved.

Proposition 4.5. *For any path of connections ∇_s on $E\to M$, the differential form $BCS(\nabla_s)$ is an element of $\Omega^{odd}_{d+\iota}(LM)$ and satisfies*

$$\rho^* BCS(\nabla_s) = CS(\nabla_s),$$

where $\rho^*:\Omega^*_{d+\iota}(LM)\to\Omega^*(M)$ is the restriction to constant loops and

$CS(\nabla_s)$

$$= \mathrm{Tr}\left(\int_0^1 \sum_{n\geq 0} \frac{1}{(n+1)!} \sum_{i=1}^{n+1} \overbrace{R_s \wedge \cdots \wedge R_s \wedge \underbrace{(\nabla_s)'}_{i^{th}} \wedge R_s \wedge \cdots \wedge R_s}^{n+1\ factors}\right) ds$$

is the Chern-Simons form associated to ∇_s.

Moreover the following multiplicative properties hold, see [TWZ3].

Theorem 4.6. *Let* $E \to M$ *and* $\bar{E} \to M$ *be two complex vector bundles, each with a path of connections* (E, ∇_s) *and* $(\bar{E}, \bar{\nabla}_s)$ *for* $s \in [0,1]$*, respectively. Let* $\nabla_s \oplus \bar{\nabla}_s$ *be the induced path of connections on* $E \oplus \bar{E}$*, and let* $\nabla_r \otimes \bar{\nabla}_s := \nabla_r \otimes Id + Id \otimes \bar{\nabla}_s$ *be the induced connections on* $E \otimes \bar{E}$ *for any* $r, s \in [0,1]$*. Then*

$$BCS(\nabla_s \oplus \bar{\nabla}_s) = BCS(\nabla_s) + BCS(\bar{\nabla}_s)$$

and

$$BCS(\nabla_0 \otimes \bar{\nabla}_s) = BCh(\nabla_0) \wedge BCS(\bar{\nabla}_s) \qquad BCS(\nabla_s \otimes \bar{\nabla}_1) = BCS(\nabla_s) \wedge BCh(\bar{\nabla}_1).$$

Finally, for two paths of connections ∇_s *and* $\bar{\nabla}_s$ *such that* $\nabla_1 = \bar{\nabla}_0$*, if* $\nabla_s * \bar{\nabla}_s$ *denotes the path of connections obtained by path-composition of* ∇_s *and* $\bar{\nabla}_s$*, then we have*

$$BCS(\nabla_s * \bar{\nabla}_s) = BCS(\nabla_s) + BCS(\bar{\nabla}_s).$$

Remark 4.7. As in Remark 3.4, we may regard this Bismut-Chern-Simons form $BCS(\nabla_s)$ as induced by a map $\phi : M \times I \to BU(k)$, and by the adjoint property there is a universal form on the space of cylinders in $BU(k)$. Namely, letting P denote the smooth path-space functor, $BCS \in \Omega^{\mathrm{odd}}_{d+\iota}(PLBU(k))$ is defined by

$$BCS = \int_I ev_t^* BCh$$

where $ev_t : PLBU \times I \to LBU$ is evaluation at time $t \in I$. To see that this agrees with Definition 4.1, let $\tilde{\phi} : M \to PBU$ denote the adjoint mapping and note that the following diagram commutes

$$\begin{tikzcd}
\Omega^*(LBU) \arrow[r, "ev_t^*"] \arrow[d, "(L(\phi))^*"] & \Omega^*(LPBU \times I) \arrow[r, "\int_I"] \arrow[d, "(L(\tilde{\phi}) \times id)^*"] & \Omega^*(LPBU) \arrow[d, "(L(\tilde{\phi}))^*"] \\
\Omega^*(L(M \times I)) \arrow[r, "j^*"] & \Omega^*(LM \times I) \arrow[r, "\int_I"] & \Omega^*(LM)
\end{tikzcd}$$

Remark 4.8. In [TWZ3] the authors show the Bismut-Chern-Simons form may be used to define a refinement of differential K-theory. We note that the construction applies to the category of S^1-spaces mapping into $L(BU \times \mathbb{Z})$ with universal Bismut-Chern form BCh, c.f. Remark 3.4. Then the loop differential K-theory of [TWZ3] is the special case of taking the free loop of a bundle with connection $M \to BU \times \mathbb{Z}$. A sheafified version of this construction was also given in [BNV], producing a differential cohomology theory which refines K-theory of M by differential forms on the free loop space of M.

5. The odd Chern form

Let U denote the stable unitary group defined as the limit of the finite unitary groups $U(n)$. This is filtered by finite dimensional manifolds and so defines a diffeological space where a finite dimensional plot is a smooth map with image in some $U(n)$. The group U has a left invariant 1-form with values in the Lie algebra $\mathfrak{u}$, which we denote by ω. This generates the canonical bi-invariant closed odd differential form on U given by

$$Ch^{odd} = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} \omega^{2n+1} \in \Omega^{odd}_{cl}(U; \mathbb{C}).$$

For a smooth map $g : M \to U$, the odd Chern form of g is defined by pullback to be

$$Ch^{odd}(g) = g^*(Ch^{odd}) = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} (g^{-1}dg)^{2n+1}.$$

We remark that the scalar factor of $1/(2\pi i)^{n+1}$ is omitted to be consistent with our convention for the even Chern form. These factors may be introduced as mentioned above, where for LM below we modifying the complex of forms to have operator $d + (2\pi i)\iota$. The following lemma explains how the odd Chern form is related to the Chern-Simons form of Proposition 4.5.

Lemma 5.1. *([G], Prop. 1.2) For any $g \in Map(M, U)$ and any n such that $g(M) \subset U(n)$ we have*

$$Ch^{odd}(g) = CS(d + sg^{-1}dg),$$

where $d + sg^{-1}dg$ is the straight path of connections on the trivial $\mathbb{C}^n$-bundle over M, from the trivial connection d to the gauge equivalent flat connection $g^*(d) = d + g^{-1}dg$.

Remark 5.2. The universal version of the statement is as follows: the trivial $\mathbb{C}^\infty$-bundle over the stable unitary group comes with a gauge transformation given by left action by g on the fiber over g. The universal Chern form Ch^{odd} on U equals the Chern Simons form of the straightline path of connections from the trivial connection d to the transform of d by the gauge transformation.

Proof. Let $A = g^{-1}dg$ and $A^s = sg^{-1}dg$ so that $(A^s)' = A$ and $R^s = -s(1-s)A \wedge A$. Then

$$\begin{aligned}
CS(d + sg^{-1}dg) &= \mathrm{Tr} \sum_{n \geq 0} \frac{1}{(n+1)!} \sum_{i=1}^{n+1} \int_0^1 R^s \dots \underbrace{(A^s)'}_{i\text{-th}} \dots R^s ds \\
&= \sum_{n \geq 0} \frac{1}{n!} \mathrm{Tr} \int_0^1 (-s(1-s))^n \underbrace{A \dots A}_{2n+1 \text{ factors}} \; ds \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{n! n!}{(2n+1)!} \mathrm{Tr}(A^{2n+1}) \\
&= \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} \mathrm{Tr}(A^{2n+1}) \\
&= Ch^{odd}(g)
\end{aligned}$$

where the identity $\int_0^1 s^k(1-s)^\ell ds = \frac{k!\ell!}{(k+\ell+1)!}$ was used. $\square$

It is straightforward the check that the odd Chern map

$$Ch^{odd} : Map(M,\ U) \to \Omega^{odd}_{cl}(M)$$

is a monoid homomorphism, i.e.

$$Ch^{odd}(g \oplus h) = Ch^{odd}(g) + Ch^{odd}(h),$$

and furthermore satisfies $Ch^{odd}(g^{-1}) = -Ch^{odd}(g)$.

The odd Chern form $Ch^{odd}(g)$ depends on g, but for a smooth family $g_t : M \times I \to U(n) \subset U$, there is a smooth even differential form that interpolates between $Ch^{odd}(g_1)$ and $Ch^{odd}(g_0)$, in the following way. We define $CS^{even}(g_t) \in \Omega^{even}(M)$ by

$$CS^{even}(g_t) = \int_I Ch^{odd}(g_t), \tag{5.1}$$

where $Ch^{odd}(g_t) \in \Omega^{odd}_{cl}(M \times I)$ and $\int_I$ is the integration along the fiber

$$\begin{tikzcd} M \times I \arrow[r, "g_t"] \arrow[d, "\int_I"'] & U \\ M & \end{tikzcd}$$

By Stokes' theorem we have

$$dCS^{even}(g_t) = d\int_I Ch^{odd}(g_t) = \int_{\partial I} Ch^{odd}(g_t) - \int_I dCh^{odd}(g_t) = Ch^{odd}(g_1) - Ch^{odd}(g_0)$$

since $dCh^{odd}(g_t) = 0$. By the usual adjunction, our definitions above give a universal form $CS^{even} \in \Omega^{even}(PU)$ where PU is the smooth pathspace of U. It follows that the restriction of this form along the inclusion $LU \to PU$ of the free loop space gives a closed even degree form $CS^{even} \in \Omega^{even}(LU)$.

The following explicit formula will be useful. For a proof, see for example [TWZ3].

Lemma 5.3. *For any $g_t \in Map(M \times I, U)$ the* Chern-Simons form *$CS^{even}(g_t) \in \Omega^{even}(M)$ associated to g_t is*

$$CS^{even}(g_t) = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n)!} \int_0^1 (g_t^{-1} g_t') \cdot (g_t^{-1} dg_t)^{2n} dt.$$

We restate the fundamental property for $CS(g_t)$ here, along with several others, whose proofs are immediate from the definitions and Lemma 5.3.

Proposition 5.4. *For any paths $g_t, h_t \in Map(M \times I, U)$ we have*

$$\begin{aligned} dCS^{even}(g_t) &= Ch^{odd}(g_1) - Ch^{odd}(g_0) \\ CS^{even}(g_t \oplus h_t) &= CS^{even}(g_t) + CS^{even}(h_t) \\ CS^{even}(g_t^{-1}) &= -CS^{even}(g_t). \end{aligned}$$

*If g_t and h_t can be composed (*i.e. *if $g_1 = h_0$), then the composition $g_t * h_t$ satisfies*

$$CS^{even}(g_t * h_t) = CS^{even}(g_t) + CS^{even}(h_t).$$

Finally, if $g_t : M \to \Omega_* U$, then the degree zero component of $CS^{even}(g_t)$ is the integer equal to the winding number.

Defining odd K-theory by $K^{-1}(M) = [M, U]$, it follows that there is an induced homomorphism $Ch : K^{-1}(M) \to H^{\text{odd}}(M)$, which is a geometric representation of the odd Chern Character, which is an isomorphism after tensoring with $\mathbb{C}$, see e.g. [G]. In particular, this implies that the components of Ch^{odd} generate the cohomology of U, and the transgression CS-forms generate the even cohomology of ΩU, [Bo].

6. The odd Bismut-Chern form

Let LU be the free loop space of U. Motivated by Lemma 5.1, we make the following definition.

Definition 6.1. Let $BCh^{odd} \in \Omega^{odd}(LU)$ be defined by

$$BCh^{odd} = BCS(d + s\omega),$$

where $d + s\omega$ is the path of connections on the trivial $\mathbb{C}^n$-bundle over $U(n)$. We refer to this as the universal odd Bismut-Chern form. In particular, for any map $g : M \to U$, there is an induced odd Bismut-Chern form given by $BCh^{odd}(g) \equiv (Lg)^* BCh^{odd} \in \Omega^{odd}(LM)$.

We note that this is well defined since it passes to the limit with respect to n. We also note that by Remark 4.3, this form can equivalently be described as the solution to a certain differential equation.

Theorem 6.2. *The universal odd Bismut-Chern form is closed, i.e.*

$$(d + \iota)BCh^{odd} = 0,$$

so that $BCh^{odd} \in \Omega^{odd}_{d+\iota}(LU)$.

Also, BCh^{odd} *is an extension to* LU *of the universal odd Chern form* $Ch^{odd} \in \Omega^{odd}_{cl}(U)$*, in the sense that*

$$\rho^*(BCh^{odd}) = Ch^{odd},$$

where $\rho : U \to LU$ *is the inclusion of constant loops.*

Proof. From Equation (4.1) we have

$$(d + \iota)BCh^{odd} = (d + \iota)\left(BCS(d + s\omega)\right) = BCh(d + \omega) - BCh(d) = 0$$

where the last equality holds since $d = d + 0$ and $d + \omega$ are gauge equivalent via the left action, see Remark 5.2. (This statement becomes more familiar in the notation $\omega = g^{-1}dg$, where $g^{-1}dg = g^{-1}0g + g^{-1}dg$.) For the second statement we have

$$\rho^* BCS(d + s\omega) = CS(d + s\omega) = Ch^{odd}$$

where the first equality is Proposition 4.5 and the second equality is Lemma 5.1. □

Letting BCh^{odd}_γ denote the value of BCh^{odd} at $\gamma \in LU$, we see from Equation (4.2) where $A_s = s\omega$, $R_s = -s(1-s)\omega \wedge \omega$ and $A'_s = \omega$, that an explicit formula for the degree $2k+1$ part of BCh^{odd} is given by

$$Tr\left(\sum_{n \geq k+1} \frac{(-1)^k (n-1)! k!}{(n+k)!} \sum_{\substack{r, j_1, \dots, j_k = 1 \\ \text{pairwise distinct}}}^{n} \int_{\Delta^n} \iota\omega(t_1) \dots \omega^2(t_{j_1}) \dots \omega(t_r) \dots \omega^2(t_{j_k}) \dots \iota\omega(t_n) \quad dt_1 \dots dt_n \right). \tag{6.1}$$

In this equation, and in Equation 7.1 below, $\omega(t_i) := \gamma(t_i)^*\omega$ is the 1-form with values in $\mathfrak{u}$ given by pullback of the left invariant 1-form ω on U along the loop γ at time t_i, so that $\iota\omega(t_i) := \omega(\gamma'(t_i)) \in \mathfrak{u}$ is given by contraction of ω by the tangent vector $\gamma'(t_i)$ to the loop $\gamma \in U$ at time t_i. Similarly, $\omega^2(t_{j_i})$ is the 1-form with values in $\mathfrak{u}$ given by contracting $\omega^2 = \omega \wedge \omega$ by the vector $\gamma'(t_{j_i})$.

Lemma 6.3. *Let BCh^{odd}_γ denote the value of BCh^{odd} at $\gamma \in LU$. Then the following properties hold*

$$BCh^{odd}_{\gamma_1 \oplus \gamma_2} = BCh^{odd}_{\gamma_1} + BCh^{odd}_{\gamma_2}$$

and

$$BCh^{odd}_{\gamma_1 * \gamma_2} = BCh^{odd}_{\gamma_1} + BCh^{odd}_{\gamma_2}.$$

Proof. These are immediate from Definition 6.1 and Theorem 4.6. □

7. Transgression, spectral flow, and odd K-theory

The form $BCh^{odd} \in \Omega^{odd}_{d+\iota}(LU)$ has a transgression form $BCS^{even} \in \Omega^{even}_{d+\iota}(PLU)$ defined by

$$BCS^{even} = \int_I ev_s^* BCh^{odd},$$

where $ev_s : PLU \times I \to LU$ is evaluation in the path-space variable $s \in I$.

By a commutatitive diagram argument similar to Remark 4.7 we see that, for a map $g_s : M \times I \to U$, with adjoint $\tilde{g}_s : M \to PU$, and induced map $L(\tilde{g}_s) : LM \to LPU = PLU$, we have $BCS^{even}(g_s) \equiv (L(\tilde{g}_s))^* BCS^{even} \in \Omega^{even}_{d+\iota}(LM)$ is given by

$$\int_I j^*(L(g_s))^* BCh^{odd},$$

where $j : LM \times I \to L(M \times I)$ is inclusion of constant loops on I.

It follows from the equivariant Stokes' formula of section 3 that

$$(d+\iota)BCS^{even} = ev_1^* BCh^{odd} - ev_0^* BCh^{odd}.$$

Notice that this equation implies that $BCS^{even} \in \Omega^{even}_{d+\iota}(PLU)$. It follows that the restriction to $L^2U = LLU \subset PLU$ produces a $(d+\iota)$-closed form of even degree, which we also denote by $BCS^{even} \in \Omega^{even}_{d+\iota}(L^2U)$.

An explicit formula for the degree $2k$ component of BCS^{even}_γ can be given using the same notation as in Equation 6.1, with an additional sum over possible contractions $\iota_{\partial/\partial s}$ of any $\omega^2(t_i, s)$ or $\omega(t, s)$:

$$\begin{aligned}(7.1)\quad Tr\Bigg(&\sum_{n \geq k+1} \frac{(-1)^k (n-1)! k!}{(n+k)!} \sum_{\substack{r, j_1, \dots, j_k = 1 \\ \text{pairwise distinct}}}^{n} \\ &\int_I \int_{\Delta^n} \iota\omega(t_1) \dots \omega^2(t_{j_1}) \dots \iota_{\partial/\partial s}\omega(t_r) \dots \omega^2(t_{j_k}) \dots \iota\omega(t_n) \\ + \sum_{1 \leq i \leq k} &\int_I \int_{\Delta^n} \pm \iota\omega(t_1) \dots \omega^2(t_{j_1}) \dots \omega(t_r) \dots \iota_{\partial/\partial s}\omega^2(t_{j_i}) \dots \omega^2(t_{j_k}) \dots \iota\omega(t_n) \quad ds dt_1 \dots dt_n \Bigg)\end{aligned}$$

where the sign $\pm$ is positive if $r < j_i$ and negative if $r > j_i$. For ease of reading, we have dropped the dependence of $\omega^2(t_i, s)$ or $\omega(t, s)$ on the variable s. To be clear, for $2k$-many tangent vectors at a point in PLU, i.e. $2k$-many vector fields along a cylinder in U, this formula produces a number.

Proposition 7.1. *Let $CS^{even} \in \Omega^{even}(PU)$ denote the universal (even) Chern-Simons form and let $\rho : Map(I,U) \to Map(S^1 \times I, U)$ be the inclusion of paths in U which are constant in the S^1-direction. Then*

$$\rho^*(BCS^{even}) = CS^{even}.$$

Consequently, if $\gamma : S^1 \times S^1 \to U$ is independent of the first S^1-factor, then the degree 0 component BCS_0 of $BCS^{even}(\gamma)$ equals the winding number of the induced map $S^1 \to U$ obtained by collapsing the first factor of the torus, i.e. the following diagram commutes

$$\begin{tikzcd} L^2U \arrow[r, "BCS_0"] & \mathbb{R} \\ LU \arrow[u, hook] \arrow[r, "W"] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

where W is the winding number of the determinant.

Proof. The first statement follows from the fact that restriction is compatible with integration along the fiber, as well as Theorem 6.2 and Equation (5.1). The second statement follows from the first, in light of the last result in Proposition 5.4. □

Remark 7.2. At the classical level of Chern and Chern-Simons forms on a manifold M, it is known that these differential forms (and not just the cohomology classes) are 2-periodic modulo exact forms ([TWZ4], Theorem 3.17). It is an interesting question as to whether a similar result holds for the Bismut-Chern and Bismut-Chern-Simons forms. We note that the spectrum $\{LU, BLU\}$ is 2-periodic since $\Omega LU = L\Omega U \cong LBU \cong BLU$.

We now relate the even degree Bismut-Chern-Simons forms to spectral flow. One knows from [AS] and [P] that there is a well defined "spectral-flow" function

$$\mathrm{sf} : L(\mathcal{F}^*_{sa}) \to \mathbb{Z}$$

where $\mathcal{F}^*_{sa}$ is the space of self-adjoint Fredholm operators on a Hilbert space whose essential spectrum contain positive and negative parts, and L is the free loop space (functor). Moreover, there is a homotopy equivalence $\phi : \mathcal{F}^*_{sa} \to U$ which makes the diagram

$$\begin{tikzcd} L(\mathcal{F}^*_{sa}) \arrow[dd, "L(\phi)"'] \arrow[dr, "\mathrm{sf}"] & \\ & \mathbb{Z} \\ LU \arrow[ur, "W"'] & \end{tikzcd}$$

commute, where W is the winding number of the determinant. The map ϕ, whose details we needn't reproduce here, is essentially the exponential map composed with a nice weak deformation retraction.

Corollary 7.3. *The spectral flow $sf : L(\mathcal{F}^*_{sa}) \to \mathbb{Z}$, regarded as a function*

$$sf \in \Omega^0(L(\mathcal{F}^*_{sa}))$$

extends to a $(d + \iota)$-closed differential form on $LL(\mathcal{F}^*_{sa}) \supset L(\mathcal{F}^*_{sa})$.

Proof. Consider the diagram

$$\begin{tikzcd} LL\left(\mathcal{F}^*_{sa}\right) \arrow[d, "LL(\phi)"'] & L\left(\mathcal{F}^*_{sa}\right) \arrow[l, hook'] \arrow[d, "L(\phi)"] \\ LLU & LU \arrow[l, hook'] \end{tikzcd}$$

where the horizontal maps are the inclusions of maps which are constant on the first S^1-factor. The differential form $BCS^{even} \in \Omega^{even}_{d+\iota}(LLU)$ is closed and its restriction to LU equals the winding number in degree zero, which pulls back to the spectral flow on $L\left(\mathcal{F}^*_{sa}\right)$. Since the diagram commutes, the pullback of $BCS^{even} \in \Omega^{even}_{d+\iota}(LLU)$ to $LL\left(\mathcal{F}^*_{sa}\right)$ is a differential form which satisfies the claim. □

It would be interesting to have explicit calculations of these higher degree forms for such families of self adjoint Fredholm operators.

As an additional corollary to the construction of the even degree Bismut-Chern-Simons form, we have the following

Corollary 7.4. *There is a well defined group homomorphism*

$$[BCh^{odd}] : K^{-1}(M) \to H^{odd}_{d+\iota}(LM)$$

defined by $[g] \mapsto BCh^{odd}(g)$, *for* $g : M \to U$, *making the following diagram commute*

$$\begin{tikzcd} & H^{odd}_{d+\iota}(LM) \arrow[d, "\rho^*"] \\ K^{-1}(M) \arrow[ur, "{[BCh^{odd}]}"] \arrow[r, "{[Ch^{odd}]}"] & H^{odd}(M) \end{tikzcd}$$

where $[Ch^{odd}] : K^{-1}(M) \to H^{odd}(M)$ *is the odd Chern character, and* ρ *is the restriction to constant loops.*

There is an even K-theoretic analogue of this statement, first proved in [Z], by completely different methods.

Proof. If g_t is a homtopy from g_0 to g_1 then

$$(d+\iota)BCS^{even}(g_t) = BCh^{odd}(g_1) - BCh^{odd}(g_0),$$

so $[BCh^{odd}([g])] \in \Omega^{odd}_{d+\iota}(LM)$ is well defined. The map is a group homomorphism by Lemma 6.3 and the diagram commutes by Theorem 6.2. □

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