MATH 152, Calculus/Integration & Infinite Series
Spring 2024
Exam III by Scott Wilson

Name: ______________________________

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	120	

Instructions: Read each problem carefully. Show all of your work, in order to receive full or partial credit. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work.
You are not permitted to use calculators that can perform symbolic differentiation or integration (e.g. TI-89 or TI-92). You may use TI-84.

(1) Do both parts.

(a) Solve the following differential equation

$$\frac{dy}{dx} = x^2 y \qquad y(0) = e^4$$

We can rewrite this as

$$\frac{dy}{y} = x^2 dx$$

and integrating we get

$$\ln|y| = \frac{x^3}{3} + C$$

so that

$$|y| = e^C e^{x^3/3},$$

Using $y(0) = e^4$, we have that $C = 4$ and $|y| = y > 0$, so that

$$y(x) = e^4 e^{x^3/3}.$$

(b) Use the Squeeze Lemma to determine the following limit:

$$\lim_{n\to\infty} \frac{\sin^2 n}{3^n}$$

Notice

$$0 \leq \frac{\sin^2 n}{3^n} \leq \frac{1}{3^n}$$

and $\lim_{n\to\infty} \frac{1}{3^n} = 0$, so by the Squeeze Lemma, $\lim_{n\to\infty} \frac{\sin^2 n}{3^n} = 0$.

(2) Do all parts.

(a) What is the exact value of

$$\sum_{n=0}^{\infty} 3 \cdot 5^{-n}$$

$$\frac{3}{1-(1/5)} = 15/4$$

(b) For which positive real numbers a does the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^a}$$

Explain in your own words which technique or ideas we used to conclude this. You don't have to give a complete argument or proof, but explain in some words how we came to this conclusion.

The series converges for $a > 1$ and diverges for $a \leq 1$. We used the integral test to establish this, by comparing the partial sums to the integral $\int_0^t \frac{1}{x^a}\,dx$ and computing the limit as $t \to \infty$.

(3) Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$$

(a) Explain, using the Alternating Series Test, why this series converges. Be sure to explain what are the conditions in the Alternating Series Test, and why they apply.

Let $b_n = \frac{1}{2n^2}$, then

$$b_{n+1} = \frac{1}{2(n+1)^2} \leq \frac{1}{2n^2} = b_n$$

for all $n \geq 1$, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n^2} = 0,$$

so the series converges.

(b) Use the Alternating Series Estimation Theorem estimate the value of the above series so that the error is less than 0.01.

Let s be the value of the series, and let s_n be the n^{th} partial sum. We can use Alternating Series Estimation Theorem

$$|s - s_n| \leq b_{n+1} = \frac{1}{2(n+1)^2} = \frac{1}{2n^2 + 4n + 2} \leq 0.01.$$

If we take $n = 7$ then this last inequality holds. So the 7^{th} partial sum s_7 has error less than 0.01 and this is equal to

$$\sum_{n=1}^{7} \frac{(-1)^n}{2n^2} = \frac{-1}{2} + \frac{1}{2 \cdot 2^2} - \cdots + \frac{-1}{2 \cdot 7^2} \approx -0.415621$$

(The actual value to 10 decimal places is -0.4112335167).

(4) Do both parts.

(a) Use the Comparison Test to determine if the following series converges or diverges. Explain your reasoning.

$$\sum_{n=1}^{\infty} \frac{3}{2n^2+n}$$

We have the inequality

$$0 \leq \frac{3}{2n^2+n} \leq \frac{3}{2n^2}$$

and $\sum_{n=1}^{\infty} \frac{3}{2n^2} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{3}{2n^2+n}$ converges.

(b) Use the Ratio Test or the Root Test to determine if the following series converges absolutely

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n\, n!}$$

Let $a_n = \frac{(-1)^n}{4^n\, n!}$. For the Ratio Test,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{4^n\, n!}{4^{n+1}\,(n+1)!} = \lim_{n\to\infty} \frac{1}{4\,(n+1)} = 0.$$

For the Root Test,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{4\sqrt[n]{n!}} = 0.$$

In either case, we have $L = 0 < 1$, so the series converges absolutely.

(5) For each part, find the radius and interval of convergence of the given power series.

(a)

$$\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

This is a geometric series, and converges for $|x/5| < 1$, so $|x| < 5$. So the radius of convergence is 5 and the interval of convergence is $(-5, 5)$.

(b)

$$\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$$

Let $a_n = \frac{x^n}{2n+1}$ and we can compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+1}{2(n+1)+1}|x| = |x|$$

To use the Ratio Test, we want this to be less than 1. So the radius of convergence is 1. The power series converges at $x = -1$, since

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

converges by the Alternating Series Test, and the power series diverges at $x = 1$ by comparison with the divergent series

$$\sum_{n=0}^{\infty} \frac{1}{2n+2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

So the interval of convergence is $[-1, 1)$.

(6) Do both parts.

(a) Write down the Maclaurin series for the function $f(x) = e^{-x^4}$. What is the interval of convergence?

We use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which converges for all real numbers, to obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!},$$

which also converges for all real numbers.

(b) Evaluate

$$\int \frac{1}{27 + x^3}\, dx.$$

using power series.

First,

$$\frac{1}{27 + x^3} = \frac{1}{27} \frac{1}{1 - (-x^3/27)} = \frac{1}{27} \sum_{n=0}^{\infty} \left(\frac{-x^3}{27} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{27^{n+1}} x^{3n}$$

so that

$$\int \frac{1}{27 + x^3}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{27^{n+1}} x^{3n}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)27^{n+1}} x^{3n+1}.$$