MATH 152, Calculus/Integration & Infinite Series, Spring 2024
Exam II by Scott Wilson

Name: ________________________________

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	10	
6	20	
Total	110	

Instructions: Read each problem carefully. Show all of your work, in order to receive full or partial credit. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work.
You are not permitted to use calculators that can perform symbolic differentiation or integration (e.g. TI-89 or TI-92). You may use TI-84.

(1) Do both parts:

(a) Write the partial fraction decomposition for the following rational function, in terms of unknown constants $A, B, C, \ldots$, etc.

$$\frac{x^2+x-1}{(x-1)^2(x^2+4)(x^2+9)^2}$$

You do not need to find the constants.

The above rational function can be written as

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{x^2+9}+\frac{Gx+H}{(x^2+9)^2}$$

(b) Compute

$$\int \frac{x^2+1}{x^3+81x}\,dx$$

First we write

$$\frac{x^2+1}{x^3+81x}=\frac{x^2+1}{x(x^2+81)}=\frac{A}{x}+\frac{Bx+C}{x^2+81}.$$

We solve for A, B, C from the equation

$$x^2+1=A(x^2+81)+(Bx+C)x=(A+B)x^2+Cx+81A,$$

and get $A=1/81$, $B=80/81$, and $C=0$. So,

$$\int \frac{x^2+1}{x^3+81x}\,dx=\frac{1}{81}\ln|x|+\frac{80}{81}\int \frac{x}{x^2+81}\,dx.$$

Letting $u=x^2+81$ so $du=2xdx$ we get a final answer of

$$\int \frac{x^2+1}{x^3+81x}\,dx=\frac{1}{81}\ln|x|+\frac{40}{81}\ln|x^2+81|+C.$$

(2) Answer both parts:

(a) Compute the improper integral:

$$\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$$

$$\int_1^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_1^t (3-x)^{-1/2}\,dx = \lim_{t\to 3^-} -2(3-x)^{1/2}\Big|_1^t$$

Evaluating and computing the limit, we get

$$\lim_{t\to 3^-} -2(3-t)^{1/2} + 2(3-1)^{1/2} = 0 + 2\sqrt{2} = 2\sqrt{2}$$

(b) Compute

$$\lim_{x\to 0}(1+\sin x)^{1/x}$$

Let $y = (1+\sin x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(1+\sin x)$ and

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = \lim_{x\to 0} \frac{\cos x/(1+\sin x)}{1} = 1.$$

In the second equality we used l'Hopital's Rule, and for the last, we can evaluate at $x=0$, since the function is continuous at zero. Finally, we have

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^1 = e.$$

where we used the fact that the exponential function is continuous.

(3) Do both parts:

(a) Find the area of the region between the following two curves:

$$y = 2x^2 \quad \text{and} \quad y = 4 + x^2$$

The curves intersect at $x = -2$ and $x = 2$, with the second curve lying above the first, so

$$A = \int_{-2}^{2} 4 + x^2 - 2x^2\, dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} = \frac{32}{3}.$$

(b) Find exact length of the curve:

$$y = \frac{x^3}{3} + \frac{1}{4x}, \qquad 1 \leq x \leq 2.$$

$$y' = x^2 - \frac{1}{4}x^{-2}$$

so that

$$1 + (y')^2 = 1 + x^4 - \frac{1}{2} + \frac{1}{16}x^{-4} = x^4 + \frac{1}{2} + \frac{1}{16}x^{-4}$$

and this factors as

$$1 + (y')^2 = \left(x^2 + \frac{1}{4}x^{-2}\right)^2$$

so that

$$\sqrt{1 + (y')^2} = x^2 + \frac{1}{4}x^{-2}.$$

The length is

$$L = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 x^2 + \frac{1}{4}x^{-2}\, dx = \frac{x^3}{3} - \frac{1}{4}x^{-1}\Big|_1^2 = \frac{59}{24}.$$

(4) Let R be the region enclosed by the curves $y = x$ and $y = x^3$, for $x \geq 0$. In the last two parts below, use the disk or washer method for finding volumes. Do not use the cylindrical shell method.

(a) Sketch the region enclosed by the two curves.

(b) Find the volume of the solid obtained by rotating the region R about the x-axis.

$$V = \int_0^1 \pi x^2 - \pi(x^3)^2 \, dx = \pi \int_0^1 x^2 - x^6 \, dx = \pi \left(\frac{1}{3} - \frac{1}{7} \right) = \frac{4\pi}{21}.$$

(c) Write down an integral that expresses the volume of the solid obtained by rotating R about the line $x = -1$. Do not compute the integral.

$$V = \int_0^1 \pi(1 + \sqrt[3]{y})^2 - \pi(1+y)^2 \, dy$$

(5) Use the method of cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region bound by the curves $y = \sqrt[3]{x}$, $x = 0$, $y = 2$. Do not use the disk or washer method. It may help to draw a picture.

$$V = \int_0^2 2\pi \cdot y \cdot y^3 \, dy = 2\pi \int_0^2 y^4 \, dy = 2\pi \frac{y^5}{5} \bigg|_0^2 = \frac{64\pi}{5}.$$

(6) Do both parts.

(a) Use Simpson's Rule with $n = 6$ to approximate the integral

$$\int_1^2 \frac{1}{x}\, dx$$

We use $\Delta x = (2-1)/6 = 1/6$, and $x_i = 1 + \frac{i}{6}$, for $i = 1, \ldots, 6$. Then

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right)$$

$$= \frac{1}{18}\left(\frac{1}{1} + \frac{4}{7/6} + \frac{2}{8/6} + \frac{4}{9/6} + \frac{2}{10/6} + \frac{4}{11/6} + \frac{1}{2}\right) \approx 0.69316979$$

(b) How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 \frac{1}{x}\, dx$ is accurate to within 0.0001?

(See page 355). If $f(x) = 1/x$ then $f^{(4)}(x) = 24/x^5$. Since $x \geq 1$, $1/x \leq 1$ so that $|f^{(4)}(x)| \leq 24$. Let's take $K = 24$ in the error bound for Simpson's Rule

$$|E_S| \leq \frac{24(2-1)^5}{180n^4|} = \frac{2}{15n^4}.$$

We want $|E_S| \leq 0.0001$ so it suffices to choose n large enough so that $\frac{2}{15n^4} \leq 0.0001$. Solving for n we get $n > \sqrt[4]{20000/15} \approx 6.0427$, so we better take $n = 8$, (since n must be even for Simpson's rule, and must be at bigger than 6.0427).