MATH 152, Calculus/Integration & Infinite Series, Spring 2024
Exam I by Scott Wilson

Name: ________________________________

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	10	
5	10	
6	20	
Total	100	

Instructions: Read each problem carefully. Show all of your work, in order to receive full or partial credit. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work.
You are not permitted to use calculators that can perform symbolic differentiation or integration (e.g. TI-89 or TI-92). You may use TI-84.

(1) Consider the function

$$f(x) = 2x^3 + 5$$

(a) Show $f'(x) \geq 0$, and use this to explain why the function $f(x)$ is one-to-one.

$f'(x) = 6x^2 \geq 0$, and $f'(x) = 0$ only if $x = 0$, so $f(x)$ is a strictly increasing function and therefore $f(x)$ is one-to-one.

(b) Find a formula for the inverse of the function $f(x)$.

Let $y = 2x^3 + 5$. Solving for x we get $x = \sqrt[3]{\frac{y-5}{2}}$ so that

$$f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$$

(c) Recall that the derivative of an inverse function $f^{-1}(x)$ at $x = a$ is given by:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Use this formula and your answer to the first part to compute $(f^{-1})'(7)$.
[Hint: notice $f(1) = 7$.]

From the hint, $f^{-1}(7) = 1$, and $f'(1) = 6$ so

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(1)} = \frac{1}{6}.$$

(2) Do all parts:

(a) Solve for x.

$$\ln(3x - 10) = 2$$

$\ln(3x-10) = 2$ implies $e^2 = 3x - 10$, so that $3x = e^2 + 10$, and $x = (e^2 + 10)/3$.

(b) Differentiate

$$e^{\sin x}$$

$$\frac{d}{dx} e^{\sin x} = e^{\sin x} \cos(x)$$

(c) Differentiate, by first taking natural logarithm of both sides, then solving for y':

$$y = \frac{x^3 + 1}{(2x + 7)^5}$$

Taking natural logarithm of both sides and use properties of natural logarithms we get

$$\ln y = \ln(x^3 + 1) - 5 \ln(2x + 7)$$

Differentiating we get

$$\frac{1}{y} \frac{dy}{dx} = \frac{3x^2}{x^3 + 1} - 5 \frac{2}{2x + 7}$$

Solving for $\frac{dy}{dx}$ and substituting for y we get

$$\frac{dy}{dx} = \frac{x^3 + 1}{(2x + 7)^5} \left(\frac{3x^2}{x^3 + 1} - \frac{10}{2x + 7} \right)$$

(3) The half-life of the radioactive isotope "einsteinium-254" is about 276 days.

(a) Suppose a sample of einsteinium-254 has a mass of 100mg. Find a formula for the mass that remains after t days.

We know $m(t) = 100e^{kt}$ and can find k by solving the equation

$$1 = 2e^{276k},$$

and obtaining

$$k = \frac{\ln(\frac{1}{2})}{276} \approx -0.0025114.$$

So

$$m(t) = 100e^{\frac{\ln(1/2)}{276}t} \approx 100e^{-0.0025t}$$

(b) Find the mass after 500 days, correct to the nearest milligram.

$$m(500) = 100e^{\frac{\ln(\frac{1}{2})}{276}500} \approx 28.4876$$

To the nearest milligram, this is 28.

(c) After how many days will the mass be 30mg?

We solve the following equation for t

$$30 = 100e^{\frac{\ln(1/2)}{276}t}$$

and obtain

$$\frac{3}{10} = e^{\frac{\ln(1/2)}{276}t}$$

so that

$$t = \frac{\ln(\frac{3}{10}) * 276}{\ln(\frac{1}{2})} \approx 479.4$$

(4) Compute the following integral:

$$\int \sin^6 x \cos^3 x \, dx$$

We can rewrite the integral as

$$\int \sin^6 x \cos^2 x \cos x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx$$

Now let $u = \sin x$ so that $du = \cos x \, dx$ and we get

$$\int u^6 (1 - u^2) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C$$

Substituting back we get:

$$\int \sin^6 x \cos^3 x \, dx = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C.$$

(5) Compute the following integral:

$$\int t^2 e^t\, dt$$

We integrate by parts twice. First we let $u = t^2$ and $dv = e^t dt$, so that $du = 2tdt$ and $v = e^t$, and

$$\int t^2 e^t\, dt = t^2 e^t - 2\int te^t\, dt.$$

Then we let $u = t$ and $dv = e^t dt$, so that $du = dt$ and $v = e^t$, so that

$$\int te^t\, dt = te^t - \int e^t\, dt = te^t - e^t + C.$$

Putting this together we get

$$\int t^2 e^t\, dt = t^2 e^t - 2te^t + 2e^t + C.$$

You could also factor this and get, $e^t(t^2 - 2t + 2) + C$, but that's not neccesary.

(6) Use the appropriate trigonometric substitution to compute

$$\int \frac{1}{\sqrt{x^2-4}}\,dx$$

You may wish to use that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$. Write you final answer as a function of x.

We let $x = 2\sec\theta$ so that

$$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta.$$

Then $\sqrt{x^2-4} = 2\tan\theta$, and $dx = 2\sec\theta\tan\theta\,d\theta$, so

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{2\tan\theta} 2\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta.$$

We are given that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$, so need to substitute back to obtain a function of x. From $x = 2\sec\theta$ we have $\sec\theta = x/2$. Drawing the appropriate right triangle, with hypotenuse x, adjacent side 2 and side opposite angle θ with length $\sqrt{x^2-4}$, we have $\tan\theta = \sqrt{x^2-4}/2$, so that finally we obtain

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$