MATH 320/620, Intro. to Topology, Spring 2024
Final Exam by Scott Wilson

Name: ______________________________ (★ Read the instructions ★)

Problem	Max points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	100 or 120	

Instructions: Read each problem carefully. If you need more space, you can use the back of the pages. In this case, make a clear reference to the continuation of your work. Give clear and thorough explanations for your solutions. You may use results from class or the textbook but make a clear reference to what you are using.

★ Students in 620 must do all problems.

★ Students in 320 may choose to omit exactly one problem, by X-ing it out, or may choose do to all problems. In either case, the test will be converted to a score out of 100. (If in doubt, leave one out!)

(1) Do all parts.

(a) Define what is a topology $\mathcal{T}$ on a set X.

A topology for X is a collection of subsets of X, including $\emptyset$ and X, that is closed under arbitrary unions and finite intersections.

(b) If X and Y are sets, and $f : X \to Y$ is a function, with $U \subset Y$, write down the defintion of $f^{-1}(U)$, i.e. the pre-image of U under f.

$$f^{-1}(U) = \{x \in X \,|\, f(x) \in U\}.$$

(c) Let $\mathcal{T}_X$ be a topology on a set X, and $\mathcal{T}_Y$ be a topology on a set Y. Define what is a continuous function from the space $(X, \mathcal{T}_X)$ to the space $(Y, \mathcal{T}_Y)$.

A function $f : X \to Y$ is continuous if for each open set U in Y, $f^{-1}(U)$ is open in X.

(2) Consider the real numbers $\mathbb{R}$ with the standard topology.

(a) Write down a basis for this topology, and explain *in terms of this basis* which subsets of $\mathbb{R}$ are open. (Your answer might start as "A subset of $\mathbb{R}$ is open if...")

A basis is given by all open intervals (a, b). A subset U of $\mathbb{R}$ is open if for each $x \in U$ there is an open interval (a, b) with $x \in (a, b) \subset U$. Alternatively, a set is open if and only if it is a union of open intervals.

(b) Is the set $(-\infty, 0)$ open in $\mathbb{R}$, closed in $\mathbb{R}$, both, or neither? Explain.

Open, not closed.

(c) Let $\mathbb{Z}^+$ denote the set positive integers. Is $\mathbb{Z}^+$ open in $\mathbb{R}$, closed in $\mathbb{R}$, both, or neither? Explain.

Closed, not open.

(d) Is the set $(-1, 4]$ open, closed, both, or neither? Explain.

Neither.

(3) For each part, give an example of a subset of $\mathbb{R}$ that has all of the properties listed, or explain why no such example exists.

(a) An infinite subset A of $\mathbb{R}$ whose closure is finite.

none exists, since $A \subset \bar{A}$.

(b) A subset A of $\mathbb{R}$ whose interior is empty, but closure is $\mathbb{R}$.

For example, $A = \mathbb{Q}$.

(c) A finite subset whose interior is non-empty.

None exists: any point has empty interior since it cannot contain a basis element (and so cannot contain an open set). Similarly, we can see that any finite set has empty interior.

(d) An open subset whose complement is connected.

For example $(-\infty, 0) \cup (1, \infty)$, whose complement is the interval $[0, 1]$, which is connected.

(e) A countably infinite subset whose closure is compact.

For example, $A = \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, whose closure is $A \cup \{0\}$, which is compact.

(4) Consider $\mathbb{R}$ with its standard topology. Do all parts and explain your answers.

(a) Let $\mathbb{Z}$ be the integers, so $\mathbb{Z} \subset \mathbb{R}$. Show that the subspace topology on $\mathbb{Z}$ is the same as the discrete topology on $\mathbb{Z}$.

It suffices to show every point in $\mathbb{Z}$ is open in the subspace topology, but this is true since $\{k\} = (k-1, k+1) \cap \mathbb{Z}$.

(b) What are the connected subsets of $\mathbb{Z}$?

Single point subsets (along with the empty set) are the only connected subsets. If a subset has more than one point has a separation (since every subset of $\mathbb{Z}$ is open).

(c) Show any continuous function $f : \mathbb{R} \to \mathbb{Z}$ must be constant.

$\mathbb{R}$ is connected, and the continuous image of a connected space is connected, so the image must be a single point.

(d) By the previous part, we can conclude the function $f : \mathbb{R} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is not continuous. Give an open subset of $\mathbb{Z}$ whose pre-image in $\mathbb{R}$ is not open.

For example $f^{-1}(\{1\}) = [0, \infty)$ which is not open in $\mathbb{R}$.

(5) Consider $\mathbb{R}^n$ with the metric topology determined by the standard Euclidean metric, denoted by d.

(a) Let

$$S^1 = \{x \in \mathbb{R}^2 | \, d(x, 0) = 1\}$$

be the unit circle. Is S^1 compact? Prove or disprove.

S^1 is compact since it a closed and bounded subset of $\mathbb{R}^2$.

(b) Is $\mathbb{R}^3$ Hausdorff? Explain.

Yes, every metric space is Hausdorff.

(c) Using the previous parts, where applicable, show if $f : S^1 \to \mathbb{R}^3$ is continuous, and C is closed in S^1, then $f(C)$ is closed in $\mathbb{R}^3$. [This shows, in particular, that the continuous image of circle in $\mathbb{R}^3$, a so-called "loop", is a closed subset of $\mathbb{R}^3$.]

The continuous image of a compact space is compact, so $f(S^1)$ is a compact subset of the Hausdorff space $\mathbb{R}^3$. A compact subset of a Hausdorff space is closed, so $Im(f) = f(S^1)$ is closed in $\mathbb{R}^3$.

(6) Do all parts, and explain your answers.

(a) Define what it means for a topological space X to be homeomorphic to another topological space Y.

We say X is homemorphic to Y is there is a homeomorphism from X to Y, i.e. a bijective function $f : X \to Y$ such that f and $f^{-1} : Y \to X$ are continuous.

(b) Are the spaces $\mathbb{R}$ and $(-3, 7)$ homeomorphic?

Yes, $\mathbb{R}$ is homeomorphic to any open interval (a, b), as shown in class. For example, $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a homeomorphism, and an appropriate linear function shows that any two open intervals are homeomorphic.

(c) Consider the unit circle S^1 as a subset of $\mathbb{R}^2$, where $\mathbb{R}^2$ has its standard topology, and S^1 is given the subspace topology. Draw a picture of S^1, and draw on it any non-empty subset of S^1 that is open in the subspace topology (other than S^1 itself).

For example, a little open arc on the circle.

(d) Consider the open unit ball at the origin of $\mathbb{R}^2$, defined by

$$B = \{(x, y) \in \mathbb{R}^2 |\, x^2 + y^2 < 1\},$$

and let $D = \overline{B}$ be the closure, i.e. the closed unit disk.
Exactly two of the spaces B, D and $\mathbb{R}^2$ (with the product topology) are homeomorphic. Explain which pair it is that are homeomorphic, and how you know that no others are homeomorphic.

D is compact (since it is closed and bounded), but B and $\mathbb{R}^2$ are not compact, so D is not homeomorphic to either of these.
B and $\mathbb{R}^2$ are homeomorphic, we described a homeomorphism in class. (If you like polar coordinates, take the function $f : \mathbb{R}^2 \to B$ given by $f(r, \theta) = (g(r), \theta)$ where $g : \mathbb{R} \to (0, \infty)$ is any homeomorphism.)