Some algebra and applications related to mapping spaces

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(joint work with Thomas Tradler, Mahmoud Zeinalian)

In this talk we begin with a useful language for some elementary concepts in algebraic topology, and then show how these can be used to define generalizations of Hochschild homology. We also give some applications to invariants and constructions that appear in settings such as Chern characters, Witten deformations, and PDE's related to fluids.

Recall that a differential graded algebra is precisely a strict monoidal functor from the category of finite sets (denoted $\mathcal{F}$) to the category of chain complexes (denoted Ch). We can generalize this definition by asking for a weak functor, i.e. one that is monoidal only up to a coherent natural transformation. We'll refer to these as *partial algebras*.

The meaning of such functors is illuminated by a theorem proved by the author: such partial algebras can be functorially rectified to E_∞-algebras. With this in mind, though, one may prefer to deal with the apparently small package of a partial algebra.

There are versions of this functor approach in many other settings: modules over algebras, algebras over any operad, and their modules, co-versions of all of these, etc.

Examples of these structures are abundant. The author has proved the following conjecture of J. McClure: the chains of a PL space form a partial co-algebra, where the structure maps are generalized diagonal maps and the natural transformation is given by the cartesian product of chains. An appropriate dual of this gives a partial algebra on cochains. By a theorem of Mandell, it's reasonable that this tidy package determines the integral homotopy type of a nilpotent space.

Now, for any partial algebra $A : \mathcal{F} \to Ch$, and any finite simplicial set $Y : \Delta \to \mathcal{F}$, we obtain by composition a simplicial object of chain complexes, whose total complex[1] we denote by $CH^Y(A)$. In fact, this forms a partial algebra itself. And there are module versions, etc.

This construction generalizes the Hochschild complex of an algebra and the higher Hochschild complexes of Pirashvilli [4], [2]. More recently Ginot, Tradler and Zeinalian have shown that for the algebra A of differential forms on X there is, for any Y, an iterated integral map yielding a quasi-isomorphism from $CH^Y(A)$ to the forms on $Map(Y, X)$ (assuming certain connectivity hypothesese) [3]. The product in the domain is identified with a shuffle product and corresponds to the cup product on the mapping space.

For the case for $Y = S^1$, and (A, d) a strict dga, $CH^Y(A)$ is the usual Hochschild complex of A with differential D. The existence of the shuffle product, $*$, implies

[1]This construction, defined for any simplicial set Y and partial algebra A, should also be related to K. Walker's *Blob Homology*, which is defined for (at least) any manifold M and category C. See his abstract in this report.

the exponential map is defined, and we can compute

$$e^{1\otimes x} = 1 + 1\otimes x + 1\otimes x\otimes x + 1\otimes x\otimes x\otimes x + \cdots$$

Furthermore,

$$De^{1\otimes x} = (1\otimes(dx + x^2)) * e^{1\otimes x}$$

This implies that Maurer-Cartan elements of A give cycles on $CH^{S^1}(A)$ and, if we imagine A as matrices of forms on M, it reminds us of the formula for curvature of a connection.

This analogy has been taken further in Getzler, Jones, and Petrack [1] by constructing, from a bundle with connection, a closed equivariant form in cyclic chains agreeing with Bismut's analytic construction, which has the property that, upon restriction to the constant loops $M \subset LM$, it gives the classical Chern character. We are working now to similarly construct a cycle in $CH^{S^1\times S^1}$ which restricts appropriately to the class above and satisfies an equivariance condition[2].

Another interesting example is given by the path space, i.e. $Y = I$ is the interval. For $A = \Omega(M)$ there is a differential D on $CH^I(A)$ induced by the standard action of A on itself (on the left and right).

For M Riemmanian, the Hodge-star operator $\star$ induces a *dual module structure* given by $(x, y) \to \star^{-1}(x\wedge\star y)$, making (A, d^*) into a differential module over (A, d). Thus, on the same underlying vector space of $CH^I(A)$, we obtain a differential D^* corresponding to the usual right action and the dual left action. Clearly D^* is given by the transport of D by $id\otimes\cdots\otimes id\otimes\star$, so D^* is the formal adjoint of D. We call $\Delta = [D, D^*]$ the *Laplacian on the path space* and note that it has square root $D + D^*$.

For $x \in A$ of degree 1 and $s \in \mathbb{R}$ we compute

$$\Delta(e^{1\otimes s\cdot x\otimes 1}\cdot y) = e^{1\otimes s\cdot x\otimes 1}\cdot D^2_{x,s}(y) \tag{1}$$

where, letting L_x denote left multiplication by x, and $L^*_x = \star^{-1}L_x\star$ denote its adjoint, we have

$$D_{x,s} = d + d^* + sL_x + sL^*_x$$

This is the deformation of $d + d^*$ considered by Witten in [5], which can be used to prove the Poincaré-Hopf Index formula. It would be interesting to understand further properties of the operators $D + D^*$ and Δ on the path space, as well as their analogues defined on algebraic models of maps into a Riemannian manifold.

An interesting special case of (1) appears when we set $y = x$ and assume x is divergence free, $d^*x = 0$. Using $\star^{-1}(x\wedge\star x) = \|x\|^2$ we obtain

$$D^2_{x,s}(x) = d^*dx + s\left(\star^{-1}(x\wedge\star dx) + d\|x\|^2 + dx\wedge x\right) + s^2\|x\|^2 x$$

The *self-linking term* $dx \wedge x$ vanishes in dimension two and lower, though may be non-trivial in dimensions three and higher. The remaining terms are degree one,

[2] Some conference participants suggested that the data of gerbes with connections may be a more appropriate setting for this construction.

and modulo the s^2 term, can be seen in the Navier-Stokes equation for viscosity equal to one:

$$\dot{x} = \star^{-1}(x \wedge \star dx) + d\|x\|^2 + d^* dx + dp$$

Here x is now a time dependent 1-form (vector field) and the pressure p is determined uniquely (up to a constant) by the Hodge decomposition.

It may be fruitful to understand further connections between deformations of the Laplacian and non-linear PDE's such as this fluid equation.

References

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