DENSITY OF NONCONTINUITY POINTS
WITH INFINITE ENTROPY

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Received June 16, 1984.

ABSTRACT

Let \( M \) be a compact Riemannian manifold with \( \dim M \geq 2 \). Denote by \( h \) the topological entropy function and let \( U_\alpha(M) = \{ f \in C^0(M, M) | h(f) = +\infty \text{ and } h \text{ is not continuous at } f \} \). In this paper, we prove that \( U_\alpha(M) \) is a dense subset of \( C^0(M, M) \).

I. INTRODUCTION

Let \( X \) be a compact metric space. Denote by \( C^0(X, X) \) the space of all continuous mappings of \( X \) into itself with the topology of uniform convergence. Suppose \( f \in C^0(X, X) \). Denote respectively by \( P(f) \), \( Q(f) \) and \( h(f) \) the set of all periodic points, the non-wandering set and the topological entropy of \( f \) (for definition of topological entropy, see [1]). \( h(f) \) is a nonnegative real number or \( +\infty \). Thus, we may consider the following function:

\[
h : C^0(X, X) \rightarrow \mathbb{R}^+ \cup \{ +\infty \},
\]

where \( \mathbb{R}^+ \) denotes the set of all nonnegative real numbers.

For the case of \( X = I = [0, 1] \), L. Block has proved the following

**Theorem A**. Let \( f \in C^0(I, I) \). If \( h(f) < +\infty \), then \( h \) is not continuous at \( f \).

In [3], Zhou Zou-Ling has generalized the above Theorem A to the case when \( X = |M| \) is a polyhedron. Recently in [4], Zhang Zhusheng generalized again the above Theorem A to a case when \( X \) is a compact topological manifold.

In this paper, let \( M \) be a compact Riemannian manifold, \( \langle \cdot, \cdot \rangle \) be a Riemannian structure on \( M \) and \( d \) be the induced topological metric of \( M \) by \( \langle \cdot, \cdot \rangle \).

References [2], [3], [4] have dealt with the noncontinuity problem of \( h \) at \( f \) when \( h(f) < +\infty \). In [5], Koichi Yano has studied the continuity of \( h \) at \( f \) with \( h(f) = +\infty \) and proved the following

**Theorem B**. Let \( C_\omega(M) = \{ f \in C^0(M, M) | h(f) = +\infty \text{ and } h \text{ is continuous at } f \} \). Then \( C_\omega(M) \) is a generic subset in \( C^0(M, M) \).

The aim of this paper is to study the noncontinuity problem of \( h \) at \( f \) with \( h(f) = +\infty \). In fact, we shall prove

**Theorem 1**. Let \( M \) be a compact Riemannian manifold with \( \dim M \geq 2 \) and let
II. LEMMAS

In case when $M$ is a connected finite polyhedron, Zhou Zuo-Ling has proved the following assertion in [3], i.e.

$$h: C^0(M, M) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$$

is surjective.

We prove here a similar result for the case when $M$ is a compact topological manifold as a lemma.

**Lemma 1.** Suppose $M$ is a compact topological manifold with $\dim M \geq 1$. Then the function

$$h: C^0(M, M) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$$

is surjective, i.e.

$$h(C^0(M, M)) = \mathbb{R}^+ \cup \{+\infty\}.$$

**Proof.** Take a local coordinate system $(U, \varphi)$ of $M$, where $\varphi$ maps $U$ homeomorphically onto $\mathbb{R}^n$, $n = \dim M$. Let $[a, b]$ be an interval in $\mathbb{R}$. For any $T \in \mathbb{R}^+ \cup \{+\infty\}$ we have a surjective continuous self-mapping $\tilde{f}_0$ of $[a, b]$ with $h(\tilde{f}_0) = T$ (a construction of such an $\tilde{f}_0$ was given in [3, proposition 1]).

Let $f_0 = \varphi^{-1} \circ \tilde{f}_0 \circ (\varphi^{-1} | [a, b])$. Then $f_0$ is a surjective continuous self-mapping of $\varphi^{-1} | [a, b]$ with $h(f_0) = h(\tilde{f}_0) = T$ (for $h$ is an invariant under topological conjugacy).

Since a compactly $T_3$ topological space is normal, by the Tietze extension theorem, there exists a continuous extension $f: M \rightarrow \varphi^{-1} | [a, b]$ of $f_0$ over $M$. Since $f(M) \subseteq M$, we find $f(f(M)) \subseteq f(M)$, i.e. $f(M)$ is an invariant set of $f$. In the same way as in the proof of propositions of [3], we see

$$h(f) = h(f|f(M)).$$

But

$$f(M) = \varphi^{-1} | [a, b].$$

We have

$$h(f|f(M)) = h(f_0).$$

Hence

$$h(f) = T.$$ Q.E.D.

In the following, we denote by $C^1(M, M)$ the space of all $C^1$ self-mapping of $M$.

**Lemma 2.** Let $n = \dim M$. If $f \in C^1(M, M)$, then

$$h(f) \leq \max\{0, n \log a\},$$
where \( a = \sup_{x \in \mathcal{M}} \|Df(x)\| \).

For a proof, see [1, p. 181, Theorem 7.15].

**Lemma 3.** \( C^r(\mathcal{M}, \mathcal{M}) \) is a dense subset of \( C^r(\mathcal{M}, \mathcal{M}) \).

For a proof, see [6, p. 49, Theorem 2.6].

**Lemma 4.** Let \( n = \dim \mathcal{M} \geq 1 \) and \( f \in C^r(\mathcal{M}, \mathcal{M}) \). Again let \( \varepsilon > 0 \). There exists \( p \in \mathcal{M} \) and a positive integer \( m \geq 1 \) such that if \( p_i = f^i(p) \), \( i = 0, 1, \ldots, m \) then \( p_i \neq p_j \), \( i, j = 0, 1, \ldots, m-1 \) and there exists \( g \in C^r(\mathcal{M}, \mathcal{M}) \cap B_\varepsilon(f) \) where \( B_\varepsilon(f) = \{ x \in C^r(\mathcal{M}, \mathcal{M}) \mid \rho(f, x) < \varepsilon \} \) and there exist local coordinate systems \( U_i, u_i \) near \( p_i \), \( i = 0, 1, \ldots, m-1 \), satisfying

(i) \( U_i \cap U_j = \emptyset \), \( i \neq j \), \( i, j = 0, 1, \ldots, m-1 \).

(ii) \( u_i(U_i) = E^*_\mathcal{M} \), where \( E^*_\mathcal{M} = \{ x \neq (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \| x \| < \frac{1}{4} \} \), \( i = 0, 1, \ldots, m-1 \).

(iii) \( g_i(U_i) = U_{i+1} \mod m \).

(iv) The local representation \( g_i = u_{i+1} \circ g \circ u_i \) of \( g \) is the identity self-mapping of \( E^*_\mathcal{M} \), \( i = 0, 1, \ldots, m-1 \).

**Proof.** Since \( \mathcal{M} \) is a compact Riemannian manifold, \( \mathcal{O}(f) \neq \emptyset \). Fix \( g \in \mathcal{O}(f) \).

For \( \varepsilon > 0 \), take \( 0 < \varepsilon < \varepsilon/2 \) such that \( D_\varepsilon(g) = \{ p \in \mathcal{M} \mid d(p, g) < \varepsilon \} \) is contained in a local coordinate neighborhood around \( g \). Since \( g \in \mathcal{O}(f) \), there is a \( P \in D_\varepsilon(g) \) such that \( A = \{ k \in N \mid f^k(p) \in D_\varepsilon(g) \} \neq \emptyset \), where \( N \) denotes the set of all natural numbers. Denote \( m = \min(A) \). Then \( m \geq 1 \).

For \( i = 0, 1, \ldots, m-1 \), denote \( p_i = f^i(p) \). It is easy to verify \( p_i \neq p_j \) when \( i \neq j \), \( i, j = 0, 1, \ldots, m-1 \), for otherwise there would exist \( m-1 \geq i > j \geq 0 \) such that \( f^i(p) = f^j(p) \). Thus we have \( f^{-i+j}(p) = f^m(p_i) \in D_\varepsilon(g) \). This contradicts \( m = \min(A) \).

For \( i = 0, 1, \ldots, m-1 \), take \( D_{\varepsilon_i}(p_i) = \{ p \in \mathcal{M} \mid d(p, p_i) < \varepsilon_i \} \) to be such that \( 0 < \varepsilon_i < \varepsilon \) and

(i) \( D_{\varepsilon_i}(p_i) \) is contained in a local coordinate neighborhood around \( p_i \), \( i = 0, 1, \ldots, m-1 \), and \( D_{\varepsilon_i}(p_i) \subset D_\varepsilon(g) \);

(ii) \( D_{\varepsilon_i}(p_i) \cap D_{\varepsilon_j}(p_j) = \emptyset \) for \( i \neq j \), \( i, j = 0, 1, \ldots, m-1 \), and \( D_{\varepsilon_i}(p_i) \cap D_{\varepsilon_j}(p_j) = \emptyset \);

(iii) \( f(D_{\varepsilon_i}(p_i)) \subset D_{\varepsilon_{i+1}}(p_{i+1}) \), \( i = 0, 1, \ldots, m-2 \) and \( f(D_{\varepsilon_{m-1}}(p_{m-1})) \subset D_\varepsilon(g) \), where \( D_{\varepsilon_i}(p_i) \) is the closure of \( D_{\varepsilon_i}(p_i) \) in \( \mathcal{M} \).

Denote \( E^*_r = \{ x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \| x \| < r \} \), \( 0 < r \leq 1 \).

We may take a coordinate function \( u_0 \) around \( g \), and coordinate functions \( u_i \) around \( p_i \), \( i = 1, \ldots, m-1 \), satisfying

(i) \( u_0(p_0) = 0 \), \( u_0(D_\varepsilon(g)) = E^*_\varepsilon \) and \( u_0(D_{\varepsilon_i}(p_i)) = E^*_\varepsilon \);

(ii) \( u_i(p_i) = 0 \), \( u_i(D_{\varepsilon_i}(p_i)) = E^*_\varepsilon \), \( i = 1, 2, \ldots, m-1 \).
\[ f_i = u_{i+1} \circ f \circ u_{i-1}^{-1} | E_i^1: E_i^* \rightarrow E_i^*, \quad i = 1, \cdots, m - 2, \]
\[ f_{m-1} = u_0 \circ f \circ u_{m-1}^{-1} | E_1^1: E_1^* \rightarrow E_1^*, \]
\[ f_0 = u_0 \circ f \circ u_0^{-1} | E_{12}^1: E_{12}^* \rightarrow E_1^*. \]

Take \( \lambda_i \in C^1([0, 1], [0, 1]) \) which satisfies
\begin{enumerate}
  \item \( \lambda_i(t) = 1 \) for \( 0 \leq t \leq 1/2, \)
  \item \( \lambda_i(t) = 0 \) for \( 2/3 \leq t \leq 1, \)
  \item \( 0 \leq \lambda_i(t) \leq 1 \) for \( t \in [0, 1]. \)
\end{enumerate}

Let
\[ \lambda_i(t) = 1 - \lambda_i(\hat{t}), \quad \hat{t} \in [0, 1]. \]

We define \( \hat{g}_i(x) = \lambda_i(\|x\|)x + \lambda_i(\|x\|)f_i(x), \quad x \in E_i^*, \quad i = 1, \cdots, m - 1, \)
\[ \hat{g}_0(x) = \lambda_i(\|x\|)x + \lambda_i(\|x\|)f_0(x), \quad x \in E_{12}^*. \]

Then \( \hat{g}_i \in C^1(E_i^*, E_i^*), \quad i = 1, \cdots, m - 1 \) and \( \hat{g}_0 \in C^1(E_{12}^*, E_{12}^*). \)

Let
\[ g_\varepsilon(\hat{q}) = \begin{cases} u_{i+1} \circ \hat{g}_i \circ u_i^{-1}(\hat{q}), & \hat{q} \in D_i(p_i), \quad i = 0, 1, \cdots (\text{mod} m), \\ f(\hat{q}), & \hat{q} \in M \setminus \bigcup_{i=0}^{m-1} D_i(p_i). \end{cases} \]

It is easy to verify that \( g_\varepsilon \) is well defined.

Denote
\[ L = M \setminus \left( \bigcup_{i=0}^{m-1} u_i^{-1}(E_{i+1}^*) \cup u_0^{-1}(E_{12}^*) \right), \]
\[ N = \bigcup_{i=0}^{m-1} D_i(p_i). \]

Then \( L \) and \( N \) are all open subset of \( M \) and \( M = \mathbb{R} \cup N. \) Since \( g_\varepsilon | L \) and \( g_\varepsilon | N \)
are all \( C^1 \) mappings, we have
\[ g_\varepsilon \in C^1(M, M). \]

Since \( 0 < \varepsilon_i < \varepsilon < \varepsilon/2, \quad i = 0, 1, \cdots, m - 1, \) we have \( g_\varepsilon \in B_\varepsilon(f) \). Thus \( g_\varepsilon \in C^1(M, M) \cap B_\varepsilon(f). \)

By the representation of \( g_\varepsilon, \) we observe that \( g_\varepsilon \) satisfies
\[ g_\varepsilon(u_i^{-1}(E_{i+1}^*)) = u_{i+1}^{-1}(E_{i+1}^*), \quad i = 0, 1, \cdots (\text{mod} m) \]
and
\[ u_{i+1} \circ g_\varepsilon \circ u_i^{-1} | E_i = id E_{i+1}^*. \]

Let \( U_i = u_i^{-1}(E_{i+1}^*), \quad i = 0, 1, \cdots, m - 1. \) We still denote \( u_i \) for \( u_i | U_i. \) Then \( (U_i, u_i) \) is a local coordinate system around \( p_i, \quad i = 0, 1, \cdots, m - 1. \)

Now the above \( g_\varepsilon \) and \( (U_i, u_i), \quad i = 0, 1, \cdots, m - 1, \) satisfy the requirement
of Lemma 4.

According to Lemma 2, we see further that \( h(\varphi_\ast) < +\infty \).

III. EXAMPLE

Let \( I = [-1, 1], I^n = I^{n-1} \times I, n \geq 2. \)

Let \( \varphi \in C^0(I^{n-1}, I^{n-1}) \) be such that \( h(\varphi) = +\infty \) and \( \varphi|_{\partial I^{n-1}} = \text{id}_{\partial I^{n-1}} \) (for a concrete construction of \( \varphi \) we refer the reader to Lemma 1.)

Let

\[
\varphi(x, t) = (1-t)\varphi(x) + tx, \quad \forall (x, t) \in I^{n-1} \times [0, 1].
\]

Then \( \varphi \) is a continuous mapping of \( I^{n-1} \times [0, 1] \) to \( I^{n-1} \) satisfying

(a) \( \varphi(x, 0) = \varphi(x), \varphi(x, 1) = x, \forall x \in I^{n-1}, \)

(b) \( \varphi(\cdot, t)|_{\partial I^{n-1}} = \text{id}_{\partial I^{n-1}}, \forall t \in [0, 1]. \)

Define a mapping

\[
\alpha(x, s) = \begin{cases} 
(\varphi(x, s), s), & 0 \leq s \leq 1, x \in I^{n-1}, \\
(\varphi(x, -s), s), & -1 \leq s \leq 0, x \in I^{n-1}.
\end{cases}
\]

It is easy to verify \( \alpha \in C^0(I^n, I^n) \) and \( \alpha|_{\partial I^n} = \text{id}_{\partial I^n} \). Since \( \alpha(x, 0) = (\varphi(x), 0) \) and \( I^{n-1} \times \{0\} \) is an invariant set of \( \alpha \), we have

\[
h(\alpha) \geq h(\alpha|_{I^{n-1}} \times \{0\}) = h(\varphi).
\]

Hence

\[
h(\alpha) = +\infty.
\]

Let \( \beta(t) \) be a continuous real-value function defined on \([0, 1]\) satisfying

(i) \( \beta(t) > 0 \) for \( t \in (-1, 1) \) and \( \beta(\pm 1) = 0, \)

(ii) \( \beta(-t) = \beta(t) \) for \( t \in [0, 1], \)

(iii) \( \beta(t) \leq 1 - t \) for \( t \in [0, 1]. \)

(See Fig. 1)

(Fig. 1)
For any \( s, 0 < s < 1 \), suppose
\[
\alpha_s(s) = s + \varepsilon \beta(s), \quad \forall \varepsilon \in I.
\]

Then \( \alpha_s(s) \) is a continuous mapping of \( I \) to \( I \).

Let
\[
\alpha_s(x, s) = \begin{cases} 
\phi(x, s), & 0 < s < 1, x \in I^{-1}, \\
\phi(x, -s), & -1 < s < 0, x \in I^{-1}.
\end{cases}
\]

Then \( \alpha_s \in C^o(I^+, I^+) \), \( \alpha_s | I^{-1} \times \{-1, 1\} = \text{id}_{I^{-1} \times \{-1, 1\}} \) and
\[
\rho(a, \alpha_s) = \sup_{(x, s) \in I^+} \|a(x, s) - \alpha_s(x, s)\| < s.
\]

Let \( s \in I \) with \( s \neq \pm 1 \). Since \( 0 < \beta(s) \leq 1 \), there holds \( r_s(s) > s \). Let \( V \) be a connected open neighborhood of \( r_s(s) \) in \( I \) such that \( s \in V \) (the closure of \( V \) in \( I \)) and let \( U \) be a connected open neighborhood of \( s \) in \( I \) such that \( r_s(U) \subseteq V \) and \( U \cap V = \emptyset \). Let also \( t_0 \in I \) be such that \( U \subseteq [-1, t_0] \) and \( V \subseteq [t_0, 1] \). Since
\[
r_s^m(V) \subseteq [t_m, 1] \text{ for } m = 0, 1, \ldots,
\]
we have
\[
\alpha_s^m(I^{-1} \times U) \subseteq I^{-1} \times V \subseteq I^{-1} \times [t_m, 1] \text{ for } m = 1, 2, \ldots.
\]

Thus
\[
\alpha_s^m(I^{-1} \times U) \cap I^{-1} \times U = \emptyset, \text{ for } m = 1, 2, \ldots.
\]

This means that if \( s \in I \), \( s \neq \pm 1 \) and \( x \in I^{-1} \), then \( (x, s) \) is a wandering point of \( a \). Hence \( \mathcal{Q}(a_s) \subseteq I^{-1} \times \{-1, 1\} \).

By
\[
\alpha_s | I^{-1} \times \{-1, 1\} = \text{id}_{I^{-1} \times \{-1, 1\}},
\]
we have
\[
\mathcal{Q}(a_s) = I^{-1} \times \{-1, 1\}.
\]

By
\[
k(a_s) = k(a_s | \mathcal{Q}(a_s))
\]
and
\[
k(\text{id}_{I^{-1} \times \{-1, 1\}}) = 0,
\]
we have
\[
k(a_s) = 0.
\]

Since \( \rho(a_s, a) \to 0 \) as \( s \to 0^+ \), we may conclude that for any open neighborhood \( \mathcal{U}(a) \) of \( a \) in \( C^o(I^+, I^+) \), there exists an \( \alpha_s \in C^o(I^+, I^+) \cap \mathcal{U}(a) \) such that
\[
k(\alpha_s) = 0.
\]

We may modify the above example such that
\[
\alpha_s | \partial I^+ = \text{id}_{\partial I^+}
\]
and
\[
k(a_s) = 0.
\]

In fact, we may take a continuous self-mapping \( \varphi \) of \([\frac{-1}{2}, \frac{1}{2}] \) with \( k(\varphi) = \).
Take
\[ \psi(x, t) = (1 - t)x + tx, \forall (x, t) \in \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times [0, 1]. \]

Let
\[ \alpha_s(x, s) = \begin{cases} (\psi(x, s), s), & 0 \leq s \leq 1, \ x \in \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1}, \\ (\psi(x, -s), s), & -1 \leq s \leq 0, \ x \in \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1}. \end{cases} \]

Since \( \alpha_0 \mid \partial \left( \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I \right) = \text{id}_{\partial \left( \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I \right)} \), there exists a continuous extension \( \alpha \) of \( \alpha_0 \) over \( I \) such that \( \alpha \left( \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I \right) = \alpha_0 \) and
\[ \alpha \mid I^* \left( \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I \right) = \text{id}. \]

Since \( \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I \) is an invariant set of \( \alpha \) we have
\[ h(\alpha) \geq h(\alpha_0) = +\infty. \]

For \( x = (x', \cdots, x^{n-1}) \), denote \( |x| = \max_{1 \leq i \leq n} \{|x^i|\} \).

Take \( \beta(t) \) as above. Let
\[ \alpha_s(x, s) = \begin{cases} (\psi(x, s), s + \varepsilon \beta(s)), & x \in \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I, \ 0 \leq s \leq 1, \\ (\psi(x, -s), s + \varepsilon \beta(s)), & x \in \left[ \frac{-1}{2}, \frac{1}{2} \right]^{n-1} \times I, \ -1 \leq s \leq 0, \\ (x, s + 2\varepsilon \beta(s)(1 - |x|)), & x \in \left( \frac{-1}{2}, \frac{1}{2} \right)^{n-1} \times I, \ s \in I. \end{cases} \]

Then we have \( Q(\alpha_s) = \partial I^* \) and \( \alpha_s \mid \partial I^* = \text{id} \).

Hence
\[ h(\alpha_s) = 0. \]

Since \( \rho(\alpha_s, \alpha) \leq \varepsilon \), we have
\[ \rho(\alpha_s, \alpha) \to 0 \text{ as } \varepsilon \to 0^+. \]

IV. Proof of Theorem 1

Let \( \varphi \in C^0(M, M) \) and \( B_\varepsilon(\varphi) = \{ f \in C^0(M, M) \mid \rho(f, \varphi) < \varepsilon \} \), \( \varepsilon > 0 \). In the proof we shall follow the same notations as Lemma 4.

By Lemmas 1—4, for \( B_\varepsilon(\varphi) \) there exist \( g = g_{\varepsilon/2} \in C^1(M, M) \cap B_{\varepsilon/2}(\varphi) \) and a
positive integer \( m \) and a point \( p = p_{i/2} \in M \) such that \( p \in P(g) \) and the period of \( g \) at \( p \) is \( m \), and \( g, m \) and \( p \) satisfy the conditions of Lemma 4.

Denote \( n = \dim M \geq 2 \), \( I_i^r = \{ x = (x^1, \cdots, x^n) \in \mathbb{R}^n : |x^i| < r, \ i = 1, \cdots, n \} \), \( 0 < r \leq 1 \). Since \( I_i^r \) and \( E_i^r \) are diffeomorphic, we may assume \( E_i^r = I_i^r \).

For \( 0 < r \leq \frac{1}{4} \), denote \( U_i(r) = u_i^{-1}(I_i^r), i = 0, 1, \cdots, m - 1 \). Then \( g \) satisfies

(a) \( g(U_i(r)) = U_{i+1}(r), i = 0, 1, \cdots (\text{mod } m) \).

(b) \( \tilde{g}_i = u_{i+1} \circ g \circ u_i^{-1} | I_i^r = \text{id}_{I_i^r}, i = 0, 1, \cdots (\text{mod } m) \).

For any \( \delta \in (0, 1) \), let \( \alpha \) and \( \alpha_0 \) be the mappings in the end of Sec II. Let

\[
\varphi(x) = r\alpha\left(\frac{1}{r} x\right), \ x \in I_i^r,
\]

\[
\varphi(x) = r\alpha\left(\frac{1}{r} x\right), \ x \in I_i^r.
\]

Then we have

(i) \( \varphi, \varphi_t \in C^0(I_i^r, \mathbb{R}) \),

(ii) \( \varphi(0) = +\infty, \varphi(\varphi_t) = 0 \),

(iii) \( \varphi_t | \partial I_i^r = \text{id}_{I_i^r}, \varphi_t | \partial I_i^r = \text{id}_{0^+} \).

We define

\[
G_i(x) = \begin{cases} u_i \circ \varphi \circ u_i(x), x \in \overline{U_i(r)} = U_i(r), i = 0, 1, \cdots (\text{mod } m), \\ g(x), x \in M \setminus \bigcup_{i=0}^{m-1} U_i(r). \end{cases}
\]

By the previous lemma, we may verify \( G_i \in C^0(M, M) \).

Since \( \rho(G_i, g) = \sup_{x \in U_i(x)} \{ d(G_i(x), g(x)) \} \)

we have \( \rho(G_i, g) \to 0 \) as \( r \to 0^+ \). Hence there exists an \( r_s, 0 < r_s \leq \frac{1}{4} \) such that

\( G_{r_s} \in B(r) \).

Now let \( 0 < r \leq \frac{1}{4} \). Because \( \overline{U_i(r)} \) is an invariant subset of \( G_i^r \), there holds

\( \rho(G_i^r) = h(G_i^r \mid \overline{U_i(r)}) \).

However, \( G_i^r \mid \overline{U_i(r)} = u_i^{-1} \circ \varphi \circ u_i \mid \overline{U_i(r)} \) and \( h \) is a topological conjugate invariant; we have

\[
h(G_i^r \mid \overline{U_i(r)}) = h(\varphi_t).
\]

On account of \( h(\varphi_t) = m h(\varphi_r) \), we admit
\[ h(G_r^*) = +\infty. \]

Hence

\[ h(G_r) = \frac{1}{m} h(G_r^*) = +\infty. \]

For \( 0 < r \leq \frac{1}{4} \) and \( \delta \in (0, 1) \), we define

\[ G_{\delta,r}(x) = \begin{cases} w_{\delta} \circ w_{\delta} \circ \cdots \circ w_{\delta}(x), & x \in \overline{U}_i(r)i = 0, 1, \cdots (\text{mod } m), \\ g(x), & x \in M \setminus \left( \bigcup_{i=0}^{m-1} U_i(r) \right). \end{cases} \]

By the previous lemma, we may verify \( G_{\delta,r} \in C(M, M) \).

Denote

\[ F_{\delta,r} = \bigcup_{k=0}^{\infty} \left( G_{\delta,r}^k \right) \left( \bigcup_{i=0}^{m-1} U_i(r) \right) \subset M. \]

Then we have

\[ \mathcal{Q}(G_{\delta,r}^\infty) \subset M | F_{\delta,r}. \quad (\ast) \]

Hence

\[ G_{\delta,r}^\infty | \mathcal{Q}(G_{\delta,r}^\infty) = g^m | \mathcal{Q}(G_{\delta,r}^\infty), \]

and \( \mathcal{Q}(G_{\delta,r}^\infty) \) is an invariant subset of \( g^m \). Then by \((\ast)\) we get

\[ \mathcal{Q}(G_{\delta,r}^\infty) \subset \mathcal{Q}(g^m). \]

By the formulas

\[ h(G_{\delta,r}^\infty) = h(G_{\delta,r}^\infty | \mathcal{Q}(G_{\delta,r}^\infty)) \]

and

\[ h(g^m | \mathcal{Q}(G_{\delta,r}^\infty)) \leq h(g^m | \mathcal{Q}(g^m)) = h(g^m), \]

we have

\[ h(G_{\delta,r}^\infty) \leq h(g^m). \]

By the formulas

\[ h(G_{\delta,r}^\infty) = mh(G_{\delta,r}), \quad h(g^m) = mh(g), \]

we have

\[ h(G_{\delta,r}) \leq h(g). \]

Fix \( r \in \left(0, \frac{1}{4}\right] \). Since \( \rho(w_r, w_{\delta, r}) \to 0 \) as \( \delta \to 0^+ \), we have

\[ \rho(G_{\delta,r}, G_r) \to 0 \text{ as } \delta \to 0^+. \]

It follows that \( G_r \in U_m(M) \).

Summing up the above results, we have obtained a \( G_{\delta,r} \in U_m(M) \cap B_\varepsilon(q) \). This proves that \( U_m(M) \) is everywhere dense in \( C^2(M, M) \). Q.E.D.
The authors thank Profs. Liao Shantao and Qian Min for their help and encouragement, and the authors also wish to thank Zhou Zou-ling for suggestion of the paper written by Koichi Yano.

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