

**GENERALIZED ULAM-VON NEUMANN TRANSFORMATIONS**

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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## Abstract

### GENERALIZED ULAM-VON NEUMANN TRANSFORMATIONS

by

Yunping Jiang

Adviser: Professor Dennis P. Sullivan

The first part of this thesis uses a singular change of metric on an interval to study mappings on the boundary of hyperbolicity. The change of metric makes it possible to apply the theory of expanding mappings. We classify these mappings up to smooth equivalence, by showing that all the eigenvalues at the periodic points, the type of power law at the critical point and a quantity which we call the asymmetry at the critical point form a complete set of invariants.

In the second part of this thesis, we study hyperbolic mappings depending on a parameter  $\varepsilon$ . Each of them has an invariant Cantor set. As  $\varepsilon$  tends to zero, the mapping approaches the boundary of hyperbolicity. We analyze the asymptotics of the gap geometry and the scaling function of the invariant Cantor set as  $\varepsilon$  goes to zero. For example,

in the quadratic case, we show that all the gaps close uniformly with speed  $\sqrt{\varepsilon}$ . There is a limiting scaling function of the limiting mapping and this scaling function has dense jump discontinuities because the limiting mapping is not expanding. Removing these discontinuities by continuous extension, we show that we obtain the scaling function of the limiting mapping with respect to the Ulam-von Neumann type metric.

A key technical result of this thesis is the uniform  $(\alpha + \gamma)$ -Koebe distortion lemma (Lemma 2.7). Its proof combines the ideas of the distortion lemmas of Denjoy and Koebe.

To my parent, to my wife Bin and my son Jeffrey

## Acknowledgements

This research owes much to the enthusiasm and insight of my advisor Dennis Sullivan. I wish to express my thanks to him, to Wellington de Melo and Sebastian van Strien and Richard Sacksteder for helpful conversations and to Charles Tresser and Frederick Gardiner for reading this paper.

I also wish to express my thanks to all members in the dynamical systems group at the Graduate Center of the City University of New York.

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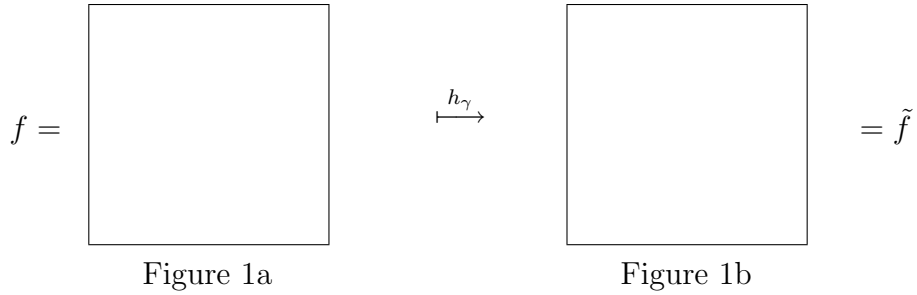
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## Introduction

Ulam and von Neumann studied the nonlinear self mapping  $q : x \mapsto 1 - 2x^2$  on the interval  $[-1, 1]$ . They observed that  $\rho_q = 1/(\pi\sqrt{1 - x^2})$  is the density function of a unique absolutely continuous  $q$ -invariant measure (we only consider probability measures in this thesis). In modern language, this observation follows from making the singular change of metric  $|dy| = (2|dx|)/(\pi\sqrt{1 - x^2})$ . If we let  $y = h(x)$  be the corresponding change of coordinate and  $\tilde{q} = h \circ q \circ h^{-1}$ , then  $q$  becomes  $\tilde{q}(y) = 1 - 2|y|$ , a piecewise linear mapping with expansion rate 2 on  $[-1, 1]$ . The dynamics of  $\tilde{q}$  is more easily understood.



The first part of this thesis analyzes certain smooth nonlinear self mappings like  $q$  (see Figure 1a). Under quite general conditions, a mapping whose graph looks like the graph shown in Figure 2 has hy-

perbolic properties. We may say that a mapping whose graph looks like the graph shown in Figure 1a is on the boundary of hyperbolicity (a more precise definition of what we call the boundary of hyperbolicity is given below).

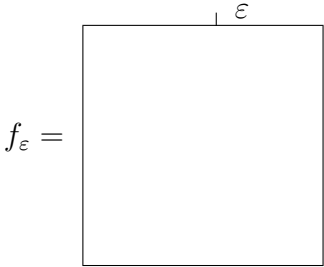


Figure 2

One knows that some mappings whose graph look like the one shown in Figure 3 have an iterate whose graph, when restricted to a subinterval, looks like the graph shown in Figure 1a. Some of our results apply to this wider class and to the more general setting of Markov mappings but to avoid confusion we leave aside further discussion of this extra generality in the introduction.

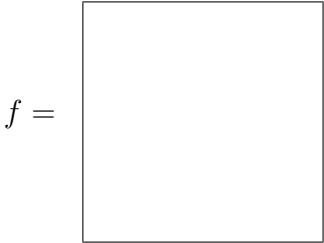


Figure 3

In order to study more general smooth self mappings of the interval with a unique power law critical point, we employ a change of metric similar to the one used by Ulam and von Neumann. The change of metric has singularities of same type at the two boundary points of the interval. It is universal in the sense that it does not depend on particular mapping  $f$ , but only on the power law  $|x|^\gamma$  at the critical point. Suppose  $y = h_\gamma(x)$  is the corresponding change of coordinate on the interval. After this change of coordinate,  $f$  becomes  $\tilde{f} = h_\gamma \circ f \circ h_\gamma^{-1}$  (see Figure 1b), which is smooth except at the critical point. The mapping  $\tilde{f}$  has nonzero derivative at every point except the critical point. At the critical point, the left and the right derivatives of  $\tilde{f}$  exist and are positive and negative, respectively.

A nice feature of the mapping  $q(x) = 1 - 2x^2$  is that  $\tilde{q}$  is expanding with Hölder continuous derivative, which implies that a certain binary tree of intervals associated with the dynamics of  $\tilde{q}$  has bounded geometry. The expanding property does not carry over to our more general setting but the bounded geometry does. An hierarchical structure of intervals has bounded geometry if the ratios of lengths in successive generations are bounded below. (In other contexts, one also requires

these ratios to be bounded away from one.)

Suppose  $E_0$  is the set consisting of the critical point of  $f$  and of the two boundary points of the interval. For every positive integer  $n$ , let  $E_n$  be the preimage of  $E_{n-1}$  under  $f$ . The  $n^{\text{th}}$ -partition  $\eta_n$  of the interval determined by  $f$  is the collection of all the subintervals bounded by consecutive points of  $E_n$ . Let  $\lambda_n$  be the maximum length of the intervals in the  $n^{\text{th}}$ -partition.

We say the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  decreases exponentially if  $\lambda_n$  decreases exponentially. We say the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  has bounded geometry if there is a positive constant  $C$  such that for any pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_{n+1}$  and  $I \in \eta_n$ , the ratio of the length of  $J$  to the length of  $I$  is bounded below by  $C$ .

Bounded geometry enables us to show that all mappings on the boundary of hyperbolicity are quasisymmetrically equivalent. (A mapping is quasisymmetric if it distorts symmetrically placed triples by a bounded amount). To prove bounded geometry for the sequence of nested partitions determined by a mapping whose graph looks like the

graph shown in Figure 1a, we develop a tool which we call the  $(\alpha + \gamma)$ -Koebe distortion lemma.

Suppose that  $f$  has the power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical point and that  $\tilde{f}$  is piecewise  $C^{1+\alpha}$  for some  $0 < \alpha \leq 1$ . The first partition  $\eta_1$  determined by  $f$  contains four intervals. For any pair  $x$  and  $y$  in one of these four intervals, let  $J_0$  be the interval bounded by  $x$  and  $y$  and  $\theta(x, y) = \{J_0, J_1, \dots\}$  be a sequence of backward images of  $J_0$  under  $f$ , that is, the restriction of  $f$  to  $J_n$  embeds  $J_n$  onto  $J_{n-1}$  for all positive integers  $n$ . Define  $g_n$  to be the inverse of the restriction of the  $n^{\text{th}}$  iterate of  $f$  to  $J_n$  and define the distortion of the  $n^{\text{th}}$  iterate of  $f$  along  $\theta(x, y)$  to be the ratio  $|g'_n(x)|/|g'_n(y)|$ . Let  $d_{xy}$  be the distance from  $\{x, y\}$  to the boundary of the interval where  $f$  is defined.

LEMMA 1.13 (the  $(\alpha + \gamma)$ -Koebe distortion lemma). *There are positive constants  $A$ ,  $B$  and  $C$  such that for any pair  $x$  and  $y$  in one of the four intervals in  $\eta_1$  and any sequence  $\theta(x, y) = \{J_0, J_1, \dots\}$  of backward images of  $J_0$  under  $f$ , the distortion of the  $n^{\text{th}}$  iterate of  $f$  along  $\theta(x, y)$  satisfies*

$$\frac{|g'_n(x)|}{|g'_n(y)|} \leq \exp\left(\left(A + B \sum_{i=0}^n |J_i| + \frac{C|J_0|}{d_{xy}}\right) \sum_{i=0}^n |J_i|^{\frac{\alpha}{\gamma}}\right)$$

for every positive integer  $n$ .

After having shown that all mappings on the boundary of hyperbolicity are quasisymmetrically equivalent, we go on to classify the smooth conjugacy equivalence classes of these mappings by also employing changes of metric. We then construct a unique smooth  $\tilde{f}$ -invariant measure which transports to a unique absolutely continuous  $f$ -invariant measure  $\mu_f$ . The corresponding density function of  $\mu_f$  is smooth on the interior of the interval and has the same singularity at both boundary points.

For all this to be true, we need

(\*)  $f$  to be a  $C^1$  self mapping of an interval with a unique power law,  $|x|^\gamma$  for some  $\gamma > 1$ , critical point and to map the critical point to the right endpoint of the interval and both endpoints of the interval to the left endpoint (see Figure 1a),

(\*\*) the derivative of  $\tilde{f}$  to be piecewise  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$  and

(\*\*\*) the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $f$  to decrease exponentially.

We prove this exponential decay of (\*\*\*) under either of the following two hypotheses.

(1) The mapping  $f$  is  $C^3$  with nonpositive Schwarzian derivative and expanding at both boundary points of the interval, that is, the absolute values of the derivatives of  $f$  at both boundary points are greater than one (see also [Mi]). The Schwarzian derivative of  $f$  is  $S(f) = f'''/f' - (3/2)(f''/f')^2$ .

(2) The mapping  $\tilde{f}$  is piecewise Lipschitz and all the periodic points of  $f$  are expanding, that is, the absolute values of the eigenvalues of  $f$  at all periodic points are greater than one (see also [Ma]). The eigenvalue of  $f$  at a periodic point  $p$  of period  $n$  of  $f$  is  $e_p = (f^{on})'(p)$ .

We call the set of mappings satisfying (\*), (\*\*), and (\*\*\*) the boundary of hyperbolicity,  $\mathcal{BH}$ . Now we can describe the results of the first part of this thesis more precisely.

**THEOREM 1.3.** *Any two mappings  $f$  and  $g$  on  $\mathcal{BH}$  are quasimetrically equivalent.*

Moreover, we classify the smooth equivalence classes on  $\mathcal{BH}$ . To

proceed, we need one more definition. The asymmetry of  $f$  at the critical point  $c$  is the limit of  $f'(-(x-c))/f'(x-c)$  as  $x$  decreases to the critical point  $c$ . Recall that the eigenvalue of a mapping  $f$  at a periodic point  $p$  of period  $n$  is  $e_f = (f^{cn})'(p)$ .

Now, assume that  $f$  and  $g$  are on  $\mathcal{BH}$ , that  $h$  gives the quasimetric equivalence between  $f$  and  $g$  and that the derivatives of  $\tilde{f}$  and  $\tilde{g}$  are piecewise  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$ .

**THEOREM 1.4.** *The conjugating mapping  $h$  is  $C^{1+\epsilon}$  for some  $0 < \epsilon \leq 1$  if and only if  $e_f(p) = e_g(h(p))$  at all the periodic points  $p$  of  $f$  and  $f$  and  $g$  have the same power law and the same asymmetry at the critical point.*

We say an object associated to  $f$  on  $\mathcal{BH}$  is a  $C^1$ -invariant if this object is the same for  $f$  as for  $h \circ f \circ h^{-1}$  whenever  $h$  is an orientation preserving  $C^1$ -diffeomorphism.

**COROLLARY 1.5.** *The eigenvalues at all periodic points, the power law and the asymmetry at the critical point form a complete set of  $C^1$ -invariants on the boundary of hyperbolicity.*

These invariants are optimal to form a complete set of  $C^1$ -invariants  
on the boundary of hyperbolicity (see Remark 1.5).

Suppose  $f$  is on  $\mathcal{BH}$  and has the power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical point and the derivative of  $\tilde{f}$  is piecewise  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$ . We extend the Ulam-von Neumann example in the following sense:

**THEOREM 1.6.** *The mapping  $f$  has a unique absolutely continuous  $f$ -invariant measure  $\mu_f = \int \rho_f(x)dx$ . Furthermore, if  $p_1$  and  $p_2$  are the boundary points of the interval, then  $\rho_f(x)(|x-p_1|)^{\frac{\gamma-1}{\gamma}}(|x-p_2|)^{\frac{\gamma-1}{\gamma}}$  is a positive  $\alpha$ -Hölder continuous function on the interval. In particular,  $\rho_f$  grows like  $|x-p|^{-\frac{\gamma-1}{\gamma}}$  at each boundary point  $p$  of the interval.*

The mappings  $f$  on  $\mathcal{BH}$  are limits of mappings  $f_\varepsilon$ , which do not keep the interval invariant, but keep invariant a Cantor set  $\Lambda_\varepsilon$  having bounded geometry (see Figure 2). We say the mappings like  $f_\varepsilon$  are hyperbolic (a more precise definition of what we call a hyperbolic mapping is given below). The space of hyperbolic mappings is the topic of the second part of this thesis, as well as the asymptotic behavior of these hyperbolic mappings as they approach the boundary of hyper-

bolicity. We use  $f_{\varepsilon,0}$  and  $f_{\varepsilon,1}$  to denote the left and right branches of  $f_\varepsilon$ , respectively. Let  $g_{\varepsilon,0}$  and  $g_{\varepsilon,1}$  be the inverses of  $f_{\varepsilon,0}$  and  $f_{\varepsilon,1}$ . For a finite string  $w = i_0 \cdots i_n$  of zeroes and ones, we use  $g_{\varepsilon,w}$  to denote the composition  $g_{\varepsilon,w} = g_{\varepsilon,i_0} \circ \cdots \circ g_{\varepsilon,i_n}$ . Define  $I_{\varepsilon,w}$  to be the image under  $g_{\varepsilon,w}$  of the interval where  $f_\varepsilon$  is defined (see Figure 4).

$$\begin{array}{c} \overline{I_{\varepsilon,w}} \\ \overline{I_{\varepsilon,w0}} \quad \overline{I_{\varepsilon,w1}} \end{array} \quad \text{or} \quad \begin{array}{c} \overline{I_{\varepsilon,w}} \\ \overline{I_{\varepsilon,w1}} \quad \overline{I_{\varepsilon,w0}} \end{array}$$

Figure 4

Suppose  $\eta_{n,\varepsilon}$  is the collection of  $I_{\varepsilon,w}$  for all finite strings  $w$  of zeroes and ones of length  $n + 1$ . We use  $\lambda_{n,\varepsilon}$  to denote the maximum length of the intervals in  $\eta_{n,\varepsilon}$ . Notice that the union of all the intervals in  $\eta_{n,\varepsilon}$  covers the maximal invariant set of  $f_\varepsilon$ .

$$\begin{array}{c} \overline{.w} \\ \overline{.w0} \quad \overline{.w1} \\ \vdots \\ \mathcal{C} \end{array}$$

Figure 5a

$$\begin{array}{c} \overline{w.} \\ \overline{w1.} \quad \overline{w0.} \\ \vdots \\ \mathcal{C}^* \end{array}$$

Figure 5b

There are two topologies on the set of all the labellings  $w$ . One topology is induced by reading the labellings  $w$  from left to right; the other by reading the labellings  $w$  from right to left. The limit set of the set of these labellings  $w$  in the topology induced by reading the

labellings  $w$  from right to left is the phase space of the dynamical system  $f_\varepsilon$ . We call it the topological Cantor set  $\mathcal{C}$  (see Figure 5a). Points in  $\mathcal{C}$  are one-sided infinite strings of zeroes and ones extending infinitely to the right. If we take the limit set of the labellings  $w$  in the topology induced by reading from right to left, we obtain the dual Cantor set  $\mathcal{C}^*$ . A point in  $\mathcal{C}^*$  is called a “dual point” which is one-sided infinite string of zeroes and ones extending infinitely to the left.

The scaling function of  $f_\varepsilon$ , when it is defined, is a function defined on  $\mathcal{C}^*$ . Assume  $a^* \in \mathcal{C}^*$ , so that  $a^*$  is a one-sided infinite string of zeroes and ones extending infinitely to the left. Suppose  $a^* = (\cdots wi.)$ , where  $w$  is a finite string of zeroes and ones and  $i$  is either zero or one. Note that  $I_{wi}$  is a subinterval of  $I_w$ . Let  $s(wi)$  equal the ratio of the lengths,  $|I_{wi}|/|I_w|$ . We let  $s(a^*)$  be the limit set of  $s(wi)$  as the length of  $w$  tends to infinity. If this limit set consists of just one number for every  $a^* \in \mathcal{C}$ , then we say that  $s(a^*)$  is the scale of  $f_\varepsilon$  at  $a^*$  and that  $s$  is the scaling function of  $f_\varepsilon$  defined on  $\mathcal{C}^*$ . Note that the scaling function  $s(a^*)$  of  $f_\varepsilon$  depends on  $\varepsilon$ . Sometimes we denote it by  $s_\varepsilon(a^*)$ . The same definition gives the scaling function  $s_0(a^*)$  of a mapping  $f_0$  on  $\mathcal{BH}$ . Since for the mappings  $f_\varepsilon$ ,  $\varepsilon \geq 0$ , the length of the interval  $I_w$  converges to zero

uniformly as the length of  $w$  approaches infinity, it is obvious that the scaling function is a  $C^1$ -invariant. Recall that a smooth invariant is an object associated to  $f_\varepsilon$  which is the same for  $f_\varepsilon$  as for  $h \circ f_\varepsilon \circ h^{-1}$  whenever  $h$  is an orientation preserving  $C^1$ -diffeomorphism.

To be sure that the limits defining the scaling function of  $f_\varepsilon$  actually exist, we need

- (i)  $f_\varepsilon$  to be a mapping from an interval to the real line which maps its unique critical point out of this interval and both endpoints of the interval to the left endpoint (see Figure 2),
- (ii)  $f_\varepsilon$  to be  $C^{1+\alpha}$  for some  $0 < \alpha \leq 1$  and
- (iii) the sequence of maximum lengths  $\lambda_{n,\varepsilon}$  determined by  $f_\varepsilon$  to decrease exponentially.

Parallel to what we do for the boundary of hyperbolicity, we prove the exponential decay of (iii) under either of the following two hypotheses.

- (1) The mapping  $f_\varepsilon$  is  $C^3$  with nonpositive Schwarzian derivative and expanding at both boundary points of the interval.

(2) The mapping  $f_\varepsilon$  is  $C^{1,1}$  and all the periodic points of  $f_\varepsilon$  are expanding.

We call the set of mappings  $f_\varepsilon$  satisfying (i), (ii) and (iii) the space of hyperbolic mappings,  $\mathcal{H}$ . Without loss of generality, we may assume that the interval where  $f_\varepsilon$  is defined is the interval  $[-1, 1]$  and that the critical point of  $f_\varepsilon$  is zero.

Sullivan [S2] showed that the scaling function is a complete invariant for  $C^1$ -conjugacy of mappings in  $\mathcal{H}$ . It plays the same role that eigenvalues play in the  $C^2$ -case (recall that Sullivan [S1] showed that the eigenvalues are complete invariants for  $C^1$ -conjugacy of  $C^2$  mappings of  $\mathcal{H}$ ). We examine the asymptotics of the scaling function of  $f_\varepsilon$  as  $\varepsilon$  decreases to zero.

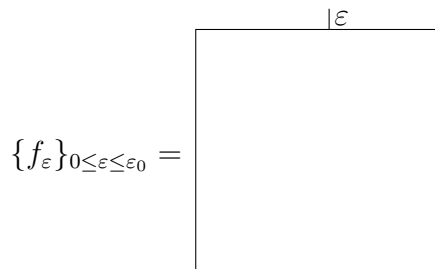


Figure 6

Hereafter, we say that a family  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  (see Figure 6) of mappings

in  $\mathcal{BH} \cup \mathcal{H}$  is a good family if it satisfies the following conditions:

1. the family  $f_\varepsilon(x)$  is  $C^1$  in both variables  $\varepsilon$  and  $x$ ,
2. each  $f_\varepsilon$  has the same power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical points and  $R^-(x, \varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$  defined on  $[-1, 0] \times [0, \varepsilon_0]$  and  $R^+(x, \varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$  defined on  $[0, 1] \times [0, \varepsilon_0]$  are continuous,
3. there are positive constants  $K'$  and  $\alpha' \leq 1$  such that  $f_\varepsilon$  is  $C^{1+\alpha'}$  and the  $\alpha'$ -Hölder constant of  $f'_\varepsilon$  is less than  $K'$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ .
4. there are positive constants  $K''$  and  $\alpha'' \leq 1$  such that  $f'_\varepsilon(x)/|x|^{\gamma-1}$  defined on  $[-1, 0]$  and  $f'_\varepsilon(x)/|x|^{\gamma-1}$  defined on  $[0, 1]$  are  $\alpha''$ -Hölder continuous and their  $\alpha''$ -Hölder constants are less than  $K''$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ ,
5. there are two positive constants  $C_0$  and  $\lambda < 1$  such that the maximum lengths  $\lambda_{n,\varepsilon}$  satisfy  $\lambda_{n,\varepsilon} \leq C_0\lambda^n$  for all positive integers  $n$  and all  $0 \leq \varepsilon \leq \varepsilon_0$ .

A function  $s$  defined on  $\mathcal{C}^*$  is called Hölder continuous if there are two positive constants  $C$  and  $\lambda < 1$  such that  $|s(a^*) - s(b^*)| \leq C\lambda^n$  for any  $a^*$  and  $b^*$  in  $\mathcal{C}^*$  with the same first  $n$  coordinates.

Let  $\mathcal{A}$  stand for the countable set of points in  $\mathcal{C}^*$  whose coordinates are eventually all zeroes and let  $\mathcal{B}$  stand for the complement of  $\mathcal{A}$  in  $\mathcal{C}^*$ .

We can now state the results of the second part of this thesis more precisely.

**THEOREM 2.1.** *Suppose  $\{f_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  is a good family. There is a family of Hölder continuous functions  $\{s_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  on the dual Cantor set  $\mathcal{C}^*$  such that  $s_\varepsilon$  is the scaling function of  $f_\varepsilon$  and*

(1) *for every  $0 < \varepsilon_1 \leq \varepsilon_0$ ,  $s_\varepsilon$  converges to  $s_{\varepsilon_1}$  uniformly on  $\mathcal{C}^*$  as  $\varepsilon$  tends to  $\varepsilon_1$ ,*

(2) *for every  $a^* \in \mathcal{C}^*$ , the limit  $s_0(a^*)$  of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  exists as  $\varepsilon$  tends to 0, the limiting function  $s_0(a^*)$  is the scaling function of  $f_0$  and satisfies:*

(2.1)  *$s_0$  has jump discontinuities at all points in  $\mathcal{A}$ ,*

(2.2)  *$s_0$  is continuous at all points in  $\mathcal{B}$  and the restriction of  $s_0$  to  $\mathcal{B}$  is a Hölder continuous function.*

Our proof of this theorem depends on a distortion lemma (Lemma

2.7) which we develop for a family of folding mappings with  $\alpha$ -Hölder continuous derivatives and with the power law  $|x|^\gamma$  for some  $\gamma > 1$  at the critical point. We call it the uniform  $(\alpha + \gamma)$ -Koebe distortion lemma. It is a generalization of Lemma 1.13.

More generally, every  $f$  on the boundary of hyperbolicity  $\mathcal{BH}$  has a scaling function (see Figure 7 and Figure 8). In fact, we show the following theorem.

**THEOREM 2.3.** *Suppose  $f$  is on  $\mathcal{BH}$  and  $\tilde{f}$  is again  $f$  viewed in the singular metric associated to  $f$ . There exist the scaling function  $s_f$  of  $f$  and the scaling function  $s_{\tilde{f}}$  of  $\tilde{f}$  and these scaling functions satisfy:*

- (a)  $s_{\tilde{f}}$  is Hölder continuous on  $\mathcal{C}^*$ ,
- (b)  $s_f$  has jump discontinuities at all points in  $\mathcal{A}$  and  $s_f$  is continuous at all points in  $\mathcal{B}$ ,
- (c) the restriction of  $s_f$  to  $\mathcal{B}$  equals the restriction of  $s_{\tilde{f}}$  to  $\mathcal{B}$ .

For any  $f$  on  $\mathcal{BH}$ , the eigenvalues of  $f$  at all the periodic points and the power law at the critical point of  $f$  can be calculated from  $s_f$ . We obtain the following consequence to Theorem 1.4.

COROLLARY 2.4. *The scaling function and the asymmetry at the critical point form a complete set of  $C^1$ -invariants on the boundary of hyperbolicity.*

These invariants are again optimal to form a complete set of  $C^1$ -invariant on the boundary of hyperbolicity (see Remark 1.5).

One example of a result about a scaling function for a mapping on the boundary of hyperbolicity is given in the following proposition (see Figure 7).

PROPOSITION 2.2. *Let  $q : x \mapsto 1 - 2x^2$ . Then  $s_q(a^*) = 1/2$  for all  $a^*$  in  $\mathcal{B}$  and  $s_q(a^*) \neq 1/2$  for all  $a^*$  in  $\mathcal{A}$ .*

Suppose  $f_\varepsilon$  is a mapping whose graph looks like the graph shown in Figure 2 and  $\{\eta_{n,\varepsilon}\}_{n=0}^\infty$  is the sequence determined by  $f_\varepsilon$ . We suppress the subscript  $\varepsilon$  when there can be no confusion. For every positive integer  $n$  and  $I_w$  in  $\eta_n$ , let  $I_{w0}$  and  $I_{w1}$  be the two intervals in  $\eta_{n+1}$  which are contained in  $I_w$ . We call the complement of  $I_{w0}$  and  $I_{w1}$  in  $I_w$  the gap on  $I_w$  and denote it by  $G_w$ . Let  $G$  be the complement of  $I_0$  and  $I_1$ . We call  $G$  the leading gap and the set of ratios  $\{|G_w|/|I_w|\}$  for all finite strings  $w$  of zeroes and ones the gap geometry of the maximal

invariant set of  $f_\varepsilon$ . We study the asymptotic dependence on  $\varepsilon$  of the gap geometry of the family of the maximal invariant sets of  $f_\varepsilon$  for  $0 \leq \varepsilon \leq \varepsilon_0$ .

Suppose  $\beta$  is a function defined on  $[0, 1]$ . We say  $\beta$  determines asymptotically the gap geometry of the maximal invariant sets of  $f_\varepsilon$  for  $0 \leq \varepsilon \leq \varepsilon_0$ , if there is a positive constant  $C$  such that for all  $0 \leq \varepsilon \leq \varepsilon_0$  and all finite strings  $w$  of zeroes and ones,

$$(1) C^{-1}\beta(\varepsilon) \leq |G_{\varepsilon,w}|/|I_{\varepsilon,w}| \leq C\beta(\varepsilon) \text{ and}$$

$$(2) |I_{\varepsilon,wi}|/|I_{\varepsilon,w}| \geq C^{-1}, \text{ where } i \text{ is either one or zero.}$$

The constant  $C$  is called a determining constant.

Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family. Then the sizes of the leading gaps  $G_\varepsilon$  is of order of  $\varepsilon^{\frac{1}{\gamma}}$ . Moreover, we prove the following theorem.

**THEOREM 2.2.** *The family of the maximal invariant sets of  $f_\varepsilon$  for  $0 < \varepsilon \leq \varepsilon_0$  is a family of Cantor sets. Furthermore, the function  $\varepsilon^{\frac{1}{\gamma}}$  determines asymptotically the gap geometry of the family of maximal invariant sets of  $f_\varepsilon$  for  $0 < \varepsilon \leq \varepsilon_0$ .*

Suppose  $HD(\varepsilon)$  is the Hausdorff dimension of the maximal invariant

set of  $f_\varepsilon$  and  $s_\varepsilon$  is the scaling function of  $f_\varepsilon$  for  $0 \leq \varepsilon \leq \varepsilon_0$ . Theorem 2.2 has the following two corollaries.

**COROLLARY 2.2.** *There is a positive constant  $C$  which does not depend on  $\varepsilon$  such that*

$$0 < HD(\varepsilon) \leq 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all  $0 \leq \varepsilon \leq \varepsilon_0$ .

**COROLLARY 2.3.** *There is a positive constant  $C$  which does not depend on  $\varepsilon$  such that*

$$1 - C^{-1}\varepsilon^{\frac{1}{\gamma}} < s_\varepsilon((a^*0.)) + s_\varepsilon((a^*1.)) < 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all  $a^* \in \mathcal{C}^*$  and all  $0 \leq \varepsilon \leq \varepsilon_0$ .

For the family  $\{1 + \varepsilon - (2 + \varepsilon)x^2\}_{0 \leq \varepsilon \leq 1}$  in  $\mathcal{H} \cup \mathcal{BH}$ , we can say even more:

**PROPOSITION 2.3.** *There is a constant  $C > 0$  which does not depend on  $\varepsilon$  such that*

$$1 - C^{-1}\sqrt{\varepsilon} \leq HD(\varepsilon) \leq 1 - C\sqrt{\varepsilon}$$

for all  $0 \leq \varepsilon \leq 1$ .

## **Part one. The geometry on the boundary of hyperbolicity.**

The example studied by Ulam and von Neumann [UN] in 1947 is the nonlinear transformation  $q : x \mapsto 1 - 2x^2$ . They discovered the density function  $\rho_q(x) = 1/(\pi\sqrt{1-x^2})$  of the unique absolutely continuous invariant measure for this mapping (see §1.3). From the metric point of view,  $dy = 2\rho_q(x)dx$  is a singular metric on the interval  $[-1, 1]$ . Under the corresponding change of coordinate  $y = h(x)$ ,  $q$  becomes a piecewise linear transformation  $\tilde{q} = h \circ q \circ h^{-1} : y \mapsto 1 - 2|y|$ . The point is that the dynamics of  $\tilde{q}$  is more easily understood. We will use this idea to study the space of certain nonlinear transformations like  $q$  in the first part of this thesis.

### **§1.1. The singular change of metric on the interval.**

Suppose  $f$  is a  $C^1$  self mapping of an interval with a unique critical point  $c$ . We always make the following assumptions: (1)  $f$  is a  $C^1$  self mapping of  $[-1, 1]$ , (2)  $f$  is increasing on  $[-1, c]$  and decreasing on  $[c, 1]$ , (3)  $f$  maps  $c$  to 1 and (4)  $f$  maps  $-1$  and  $1$  to  $-1$  (see Figure 1a). Without loss of generality, we always assume  $c$  equals 0.

Let  $r_f(x) = f'(x)/|x|^{\gamma-1}$  for  $x \neq 0$ . We say  $f$  has power law at

the critical point if there is some number  $\gamma > 1$  such that the limits of  $r_f(x)$  as  $x$  increases to zero and as  $x$  decreases to zero exist and equal nonzero numbers  $A$  and  $-B$ , respectively. For example,  $f$  has power law at the critical point if  $f(x) = 1 - (A/\gamma)|x|^\gamma$  for negative  $x$  close to zero and  $f(x) = 1 - (B/\gamma)|x|^\gamma$  for positive  $x$  close to zero. We call the ratio of  $A$  to  $-B$ , which is the limit of  $f'(-x)/f'(x)$  as  $x$  decreases to zero, the asymmetry of  $f$  at the critical point. We always assume that  $f$  has power law  $|x|^\gamma$  for some  $\gamma > 1$  at the critical point.

We define the singular metric associated to  $f$  to be

$$dy = \frac{dx}{(1-x^2)^{\frac{1}{\gamma^*}}}$$

on  $[-1, 1]$ , where  $\gamma^* = \gamma/(\gamma - 1)$  is the dual number of  $\gamma$  which means that  $1/\gamma + 1/\gamma^* = 1$ . The corresponding change of coordinate on  $[-1, 1]$  is  $y = h_\gamma(x)$ , where

$$h_\gamma(x) = -1 + b \int_{-1}^x \frac{dx}{(1-x^2)^{\frac{1}{\gamma^*}}}$$

with  $b = 2 / \int_{-1}^1 dx / (1-x^2)^{\frac{1}{\gamma^*}}$ . The representation of  $f$  under the singular metric associated to  $f$  is

$$\tilde{f} = h_\gamma \circ f \circ h_\gamma^{-1}.$$

Before we see some properties of  $f$  and  $\tilde{f}$ , we state some definitions.

DEFINITION 1.1. *Suppose  $I$  is an interval and  $g$  is a function on  $I$ .*

*We say that*

(a)  $g$  is an embedding if  $g$  is a homeomorphism from  $I$  to  $g(I)$ ,

(b)  $g$  is a  $C^1$ -embedding if  $g$  has a continuous derivative  $g'$  on  $I$  and the derivative of  $g$  at every point in  $I$  is not zero, as usual, the derivative of  $g$  at each boundary point of  $I$  is a one-sided limit,

(c)  $g$  is a  $C^{1+\alpha}$ -embedding for some  $0 < \alpha \leq 1$  if  $g$  is a  $C^1$ -embedding and the derivative  $g'$  on  $I$  is  $\alpha$ -Hölder continuous on  $I$ .

If  $\alpha = 1$ , we usually say  $g$  is a  $C^{1,1}$ -embedding in (c).

Suppose  $I$  is an interval and  $g$  is an  $\alpha$ -Hölder continuous function on  $I$  for some  $0 < \alpha \leq 1$ . There is a positive constant  $K$  such that  $|g(x) - g(y)| \leq K|x - y|^\alpha$  for all  $x$  and  $y$  in  $I$ . The smallest such  $K$  is called the  $\alpha$ -Hölder constant of  $g$ . If  $\alpha = 1$ , the smallest such  $K$  is usually called the Lipschitz constant of  $g$ .

LEMMA 1.1. *The mapping  $\tilde{f}$  is continuous on  $[-1, 1]$  and the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^1$ -embeddings.*

*Proof.* If  $y$  is not one of 0, 1 and  $-1$ , then  $\tilde{f}$  is differentiable at  $y$ .

Suppose  $x$  is the preimage of  $y$  under  $h_\gamma$ . By the chain rule,

$$\tilde{f}'(y) = f'(x)(1 - x^2)^{\frac{1}{\gamma^*}} / (1 - (f(x))^2)^{\frac{1}{\gamma^*}}. \quad (\text{EQ 1.1}).$$

Using this equation, we can get that  $\tilde{f}'(0-)$  and  $\tilde{f}'(0+)$  exist and equal nonzero numbers and that  $\tilde{f}'(-1) = (f'(-1))^{\frac{1}{\gamma}}$  and  $\tilde{f}'(1) = -(|f'(1)|)^{\frac{1}{\gamma}}$ . *QED.*

**REMARK 1.1.** The inverse of  $h_\gamma$  is  $C^1$ . If the restrictions of  $r_f$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$ , then the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are at least  $C^{1+\alpha}$  because of (EQ 1.1) (see [J3]).

**LEMMA 1.2.** *Suppose  $\tilde{f}$  is a continuous self mapping of  $[-1, 1]$ . Assume 0 is the unique turning point,  $f$  maps 0 to 1 and maps  $-1$  and 1 to  $-1$  and the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^1$ -embeddings. Then  $f = h_\gamma^{-1} \circ \tilde{f} \circ h_\gamma$  is a  $C^1$  mapping and has the power law  $|x|^\gamma$  at the critical point 0 for any  $\gamma > 1$ .*

*Proof.* If  $x$  is not one of 0, 1 and  $-1$ , then  $f$  is differentiable at  $x$ .

Suppose  $y = h_\gamma(x)$ . By the chain rule,

$$f'(x) = \tilde{f}'(y)(1 - (h_\gamma^{-1} \circ \tilde{f}(y))^2)^{\frac{1}{\gamma^*}} / (1 - (h_\gamma^{-1}(y))^2)^{\frac{1}{\gamma^*}}. \quad (\text{EQ 1.2}).$$

Using this equation,  $f'(-1) = (\tilde{f}'(-1))^\gamma$  and  $f'(1) = -|\tilde{f}'(1)|^\gamma$ , and the limits of  $r_f(x)$  as  $x$  increases to zero and as  $x$  decreases to zero exist and equal nonzero numbers. *QED.*

REMARK 1.2. The mapping  $h_\gamma$  is  $(1/\gamma)$ -Hölder continuous. If the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha}$  embeddings for some  $0 < \alpha \leq 1$ , then  $f$  is  $C^{1+\frac{\alpha}{\gamma}}$  and the restrictions of  $r_f$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\alpha/\gamma$ -Hölder continuous because of (EQ 1.2) (see [J3]).

### §1.2. The boundary of hyperbolicity, $\mathcal{BH}$ .

Let  $f_0$  and  $f_1$  be the restrictions of  $f$  to  $[-1, 0]$  and to  $[0, 1]$ , respectively. Then  $f_0$  and  $f_1$  are both embeddings. Let  $g_0$  and  $g_1$  be the inverse of  $f_0$  and  $f_1$ . For a finite string  $w = i_n \cdots i_0$  of zeroes and ones, let  $g_w$  be the composition,  $g_w = g_{i_0} \circ \cdots \circ g_{i_n}$ . Define  $I_w$  to be the image of  $[-1, 1]$  under  $g_w$ . The  $n^{\text{th}}$ -partition of  $[-1, 1]$  determined by  $f$  is the collection of  $I_w$  for all finite strings  $w$  of zeroes and ones of length  $n+1$ . We denote it by  $\eta_{n,f}$  or just by  $\eta_n$  when there is no possibility for confusion. We use  $\lambda_n$  to denote the maximum length of the intervals in  $\eta_n$ . The  $n^{\text{th}}$ -partition  $\eta_{n,\tilde{f}}$  of  $[-1, 1]$  determined by  $\tilde{f}$  and the maximum length  $\lambda_{n,\tilde{f}}$  of the intervals in  $\eta_{n,\tilde{f}}$  are defined similarly.

DEFINITION 1.2. *We say that the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $f$  decreases exponentially if there are two positive constants  $C$  and  $\lambda < 1$  such that  $\lambda_n \leq C\lambda^n$  for all positive integers  $n$ .*

LEMMA 1.3. *The sequence of nested partitions determined by  $f$  decreases exponentially if and only if the sequence of nested partitions determined by  $\tilde{f}$  decreases exponentially.*

*Proof:* Because  $h_\gamma$  is  $(1/\gamma)$ -Hölder continuous and the inverse of  $h_\gamma$  is  $C^1$ , we can easily see this lemma. *QED.*

DEFINITION 1.3. *The nonlinear mapping  $f$  is on the boundary of hyperbolicity,  $\mathcal{BH}$ , if*

(a) *the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha}$  embeddings for some  $0 < \alpha \leq 1$ ,*

(b) *the sequence of nested partitions  $\{\eta_n\}$  determined by  $f$  decreases exponentially.*

Next lemma follows from Remark 1.1 and Remark 1.2.

LEMMA 1.4. *The nonlinear mapping  $f$  is on  $\mathcal{BH}$  if and only if*

- (i)  $f$  is  $C^{1+\alpha}$  for some  $0 < \alpha \leq 1$  and the restrictions of  $r_f$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\beta$ -Hölder continuous for some  $0 < \beta \leq 1$  and
- (ii) the sequence of nested partitions determined by  $f$  decreases exponentially.

The condition (i) in Lemma 1.4 and  $r_f(0-) = r_f(0+)$  are equivalent to the statement that  $f(x) = F(-|x|^\gamma)$  where  $F$  is a  $C^{1+\alpha}$  diffeomorphism from  $[-1, 0]$  to  $[-1, 1]$  (see [J3] for more details).

We give two examples of mappings on  $\mathcal{BH}$ .

EXAMPLE 1.1. *Mappings  $f$  such that (1)  $f$  is  $C^3$  with nonpositive Schwarzian derivative, (2)  $f$  is expanding at both boundary points of  $[-1, 1]$ , that is,  $f'(-1)$  and  $|f'(1)|$  are greater than one, and (3) the restrictions of  $r_f$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$ .*

The Schwarzian derivative  $S(f)$  of  $f$  is  $S(f) = f'''/f' - (3/2)(f''/f')^2$ .

EXAMPLE 1.2. *Mappings  $f$  such that (1) the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1,1}$  embeddings and (2) all the periodic points of  $f$  are expanding, that is, the absolute values of the eigenvalues of  $f$  at all*

*periodic points are greater than one.*

The eigenvalue of  $f$  at a periodic point  $p$  of period  $n$  of  $f$  is  $e_f(p) = (f^{\circ n})'(p)$ .

The facts that Examples 1.1 and 1.2 are on  $\mathcal{BH}$  follow directly from Theorems 1.1 and 1.2, respectively. To see these facts, we introduce a geometric concept, the bounded geometry, for the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$ . The concept of bounded geometry was first defined in [S2] for Cantor sets on the line.

DEFINITION 1.4. *The sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  has bounded geometry if there is a positive constant  $C$  such that for every positive integer  $n$  and every pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_n$  and  $I \in \eta_{n-1}$ , the ratio  $|J|/|I| \geq C$ . The largest such  $C$  is denoted as  $C_f$  and called the bounded geometry constant of  $\{\eta_n\}_{n=0}^{\infty}$ .*

REMARK 1.3. Then the ratio  $|J|/|I| \leq 1 - C$  because  $I$  is the union of the two intervals in  $\eta_{n+1}$  and the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  decreases exponentially.

For completeness, we prove two distortion lemmas (see [D], [GS])

and [S3]). One of them is the naive distortion lemma. We call the other the  $C^3$ -Koebe distortion lemma.

First, let us prove the naive distortion lemma. Suppose  $g$  is a  $C^{1+\alpha}$  self mapping of an interval for some  $0 < \alpha \leq 1$ . Let  $K$  be the  $\alpha$ -Hölder constant of the derivative  $g'$  on this interval. Suppose  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  are two sequences of points in this interval. We call the product of ratios  $\prod_{i=1}^n |g'(x_i)|/|g'(y_i)|$  the distortion of  $g$  at  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$ .

LEMMA 1.5 (the naive distortion lemma). *Suppose  $J_1, \dots, J_n$  are subintervals in the interval. Let  $c$  be the minimum value of  $|g'|$  on  $\cup_{i=1}^n J_i$ . For any two sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  with  $x_i$  and  $y_i$  in  $J_i$ , the distortion of  $g$  at  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  satisfies*

$$\exp\left(-\frac{K}{c} \sum_{i=1}^n |J_i|^\alpha\right) \leq \prod_{i=1}^n \frac{|g'(x_i)|}{|g'(y_i)|} \leq \exp\left(\frac{K}{c} \sum_{i=1}^n |J_i|^\alpha\right).$$

*Proof.* Take the function  $\log x$  at  $\prod_{i=1}^n |g'(x_i)|/|g'(y_i)|$ . Then

$$\log\left(\prod_{i=1}^n \frac{|g'(x_i)|}{|g'(y_i)|}\right) = \sum_{i=1}^n (\log |g'(x_i)| - \log |g'(y_i)|).$$

Because  $\log x$  is Lipschitz continuous with the Lipschitz constant  $1/c$

on  $[c, +\infty)$  and the  $\alpha$ -Hölder constant of  $g'$  on the interval is  $K$ , we have that

$$\left| \sum_{i=1}^n (\log |g'(x_i)| - \log |g'(y_i)|) \right| \leq \frac{K}{c} \sum_{i=1}^n |x_i - y_i|^\alpha$$

which is bounded above by  $(K/c) \sum_{i=1}^n |J_i|^\alpha$ . *QED.*

Next let us prove the  $C^3$ -Koebe distortion lemma. Suppose  $I$  and  $J$  are two intervals and  $g$  is a  $C^3$  diffeomorphism from  $I$  to  $J$ . A measure of the nonlinearity of  $g$  is the function  $n(g) = g''/g'$ . If the absolute value of  $n(g)$  on  $I$  is bounded above by a positive constant  $C$ , then the distortion  $|g'(x)|/|g'(y)|$  of  $g$  at any pair  $x$  and  $y$  in  $I$  is bounded above by  $\exp(C|x - y|)$ . Suppose  $d_I(x)$  is the distance from  $x$  to the boundary of  $I$ .

**LEMMA 1.6** (the  $C^3$  Koebe distortion lemma). *Suppose  $g$  has nonnegative Schwarzian derivative. Then  $n(g)(x)$  is bounded above by  $2/d_I(x)$  for any  $x$  in  $I$ .*

*Proof.* The relation between the Schwarzian derivative and the measure of nonlinearity of  $g$  is  $n(g)' = (n(g))^2/2 + S(g)$ . For any  $x$  in  $I$ , consider

the ordinary differential equation

$$\begin{cases} y'(t) = (y(t))^2/2, \\ y(0) = n(g)(x). \end{cases}$$

The unique solution of this ordinary differential equation is the function  $y_0(t) = 2/(-t + 2/y(0))$ . Suppose  $t_1$  is the unique pole of this solution.

Then  $n(g)(x) = 2/t_1$ . Suppose  $t = y - x$  for  $y$  in  $I$ . The function  $y_1(t) = n(g)(t + x)$  is a solution of the ordinary differential inequality

$$\begin{cases} y'(t) \geq (y(t))^2/2, \\ y(0) = n(g)(x). \end{cases}$$

If  $n(g)(x)$  is greater than or equal to zero, then  $y_1(t)$  is greater than or equal to  $y_0(t)$  for  $t \geq 0$  because of the comparison theorem from ordinary differential equations (see, for example, [A]). Suppose  $I = [a, b]$ , then  $y_0(t)$  is a finite number for  $t \leq b - x$ . Hence  $t_1$  has to be greater than  $b - x$ , and furthermore,  $n(g)(x)$  is less than  $2/(b - x)$  which is less than  $2/d_I(x)$ . If  $n(g)(x)$  is negative, similar arguments show that  $-n(g)(x) \leq 2/d_I(x)$ . *QED.*

**THEOREM 1.1.** *Suppose  $f$  satisfies the conditions (1) and (2) in Example 1.1. Then the sequence of partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  has bounded geometry .*

Before proving this theorem, we state some basic properties of mappings with nonpositive Schwarzian derivative. Suppose  $h$  and  $g$  are two  $C^3$  diffeomorphisms. Assume the domain of  $h$  is contained in the range of  $g$ . Cayley identity says  $S(h \circ g) = S(g) + (g')^2 \cdot S(h) \circ g$ . Applying Cayley identity with  $h = g^{-1}$ , one get  $S(g) = -(g')^2 \cdot S(g^{-1}) \circ g$ . As a consequence, we have the following corollary.

**PROPOSITION 1.1.** *The composition of mappings which have nonpositive Schwarzian derivatives has nonpositive Schwarzian derivative. The inverse of a mapping which has nonpositive Schwarzian derivative has nonnegative Schwarzian derivative.*

Next lemma is a special property of mappings with nonpositive Schwarzian derivative. The reader may refer to [VS, pp32-34] for the proof.

**LEMMA 1.7.** *Suppose  $T = [a, b]$  and  $f : T \mapsto f(T)$  is a  $C^3$  injective mapping with nonpositive Schwarzian derivative. Suppose  $x$  is an interior point in  $T$ . Let  $L$  and  $R$  be two connected components of  $T \setminus \{x\}$ .*

*Then*

$$(a) |f'(x)||f(T)|/|T| \geq (|f(R)|/|R|)(|f(L)|/|L|),$$

(b)  $|f'(x)| \geq \min\{|f'(a)|, |f'(b)|\}$ .

*Proof of Theorem 1.1.* Suppose  $T$  is either  $[-1, 0]$  or  $[0, 1]$  and  $x$  is either the right endpoint of  $I_{00}$  or the left endpoint of  $I_{10}$ . Note that  $I_{00}$  and  $I_{10}$  are the two intervals in  $\eta_1$  adjacent to 1 and  $-1$ , respectively. By (a) of Lemma 1.7, the derivative of  $f$  at  $x$  is greater than 2. Let  $c$  be the minimum of 2,  $|f'(1)|$  and  $f'(-1)$ . Then (b) of Lemma 1.7 says the minimum value of  $|f'|$  on  $I_{00} \cup I_{10}$  is greater than or equal to  $c$ .

Suppose  $C_1$  is the minimum of  $\{|J|/|I|\}$  for all pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_1$  and  $I \in \eta_0$  and all pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_2$  and  $I \in \eta_1$ . Let  $K$  be the maximum of the Lipschitz constants of the restrictions of  $f'$  to  $I_{00}$  and to  $I_{10}$ .

For any integer  $n \geq 2$  and any pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_{n+1}$  and  $I \in \eta_n$ . Let  $J_i$  and  $I_i$  be the images of  $J$  and  $I$  under the  $i^{th}$  iterate of  $f$  for  $0 \leq i \leq n-1$ . Note that  $I_{n-1}$  is in  $\eta_1$  and  $J_{n-1}$  is in  $\eta_2$ . There is a unique finite string  $w$  of zeroes and ones of length  $n-1$  such that the restriction of  $g_w$  to  $I_{n-1}$  is the inverse of the restriction of the  $(n-1)^{th}$  iterate of  $f$  to  $I$ .

Remember that  $\eta_1$  contains four intervals  $I_{00}$ ,  $I_{01}$ ,  $I_{11}$  and  $I_{10}$ . Let

us call  $I_{00}$  the left interval,  $I_{01}$  and  $I_{11}$  the two middle intervals and  $I_{10}$  the right interval.

We divide the possible itineraries of the sequence of intervals  $\{I_i\}_{i=0}^{n-1}$  into two cases. The first case is that there is no  $I_i$  in the union of the middle intervals. The second case is that some  $I_k$  is in one of the middle intervals.

In the first case, because the restrictions of  $f$  to the left interval and to the right interval are expanding mappings with expanding constant  $c$ , the length of  $I_{n-i}$  is bounded above by  $c^{-i}$  for  $0 < i \leq n$ . We use the naive distortion lemma to show that the distortion  $|g'_w(x)|/|g'_w(y)|$  at any  $x$  and  $y$  in  $I_{n-1}$  is bounded below by a constant  $C_2$ , where  $C_2 = \exp(-K/(c-1))$ . Because  $|J|/|I| = (|g'_w(x)|/|g'_w(y)|)(|J_{n-1}|/|I_{n-1}|)$  for some  $x$  and  $y$  in  $I_{n-1}$ , we have that  $|J|/|I| \geq C_1 C_2$ .

In the second case, assume  $m$  is the largest positive integer such that  $I_m$  is in one of the middle intervals. We could divide this case into two subcases according to  $m$ . One is that  $m$  is  $n-1$ . The other is that  $m$  is less than  $n-1$ .

If  $m = n-1$ , then  $I_{n-1}$  is one of the middle intervals. From Propo-

sition 1.1,  $g_w$  has nonnegative Schwarzian derivative. The  $C^3$  Koebe distortion lemma says  $n(g_w)(x) \leq 2/d_{[-1,1]}(x)$  for any  $x$  in  $[-1, 1]$ . Moreover,  $n(g_w)(x)$  is less than  $2/\min(|I_{00}|, |I_{10}|)$  for any  $x$  in  $I_{n-1}$ . Suppose  $C_3 = \exp(-2/\min(|I_{00}|, |I_{10}|))$ . The distortion  $|g'_w(x)|/|g'_w(y)|$  of  $g$  at any  $x$  and  $y$  in  $I_{n-1}$  is greater than or equal to  $C_3$ . We get that  $|J|/|I| \geq C_1 C_3$  because  $|J|/|I| = (|g'_w(x)|/|g'_w(y)|)(|J_{n-1}|/|I_{n-1}|)$  for some  $x$  and  $y$  in  $I_{n-1}$ .

If  $m < n-1$ , then  $I_i$  are in one of the left interval and right intervals for  $m < i \leq n-1$ . Because  $I_{n-1}$  is either the left interval or the right interval,  $I_{m+1}$  has the critical value 1 as a boundary point and  $I_m$  has the critical point 0 of  $f$  as a boundary point. For the sequence  $I_{m+1}, \dots, I_{n-1}$ , no one of them is in the union of the middle intervals. The same arguments as that in the first case say  $|J_{m+1}|/|I_{m+1}| \geq C_1 C_2$ . Suppose  $J'_{m+1}$  is the complement of the interior of  $J_{m+1}$  in  $I_{m+1}$ . Then  $J'_{m+1}$  is also an interval in  $\eta_{n-m}$  which is contained in  $I_{m+1}$ . The same arguments say  $|J'_{m+1}|/|I_{m+1}| \geq C_1 C_2$ . Because  $I_{m+1} = J_{m+1} \cup J'_{m+1}$ , we get  $|J_{m+1}|/|I_{m+1}| \leq 1 - C_1 C_2$ . For the sequence  $I_0, \dots, I_m$ , the last interval  $I_m$  is in one of the middle intervals. Similar arguments to that in the subcase  $m = n-1$  say  $|J|/|I| \geq C_3 |J_m|/|I_m|$ . Because

the restriction of  $f$  to  $I_m$  is comparable with the power law mapping  $x \mapsto |x|^\gamma$ , we have a positive constant  $C_4$  such that  $|J_m|/|I_m|$  is greater than both of  $C_4(|J_{m+1}|/|I_{m+1}|)^{1/\gamma}$  and  $C_4(1 - (1 - |J'_{m+1}|/|I_{m+1}|)^{1/\gamma})$ . Let  $C_5$  be the minimum of  $C_4(C_1C_2)^{1/\gamma}$  and  $C_4(1 - (1 - C_1C_2)^{1/\gamma})$ . Then  $|J|/|I| \geq C_5$ .

Let  $C$  be the minimum of the constants  $C_1C_2$ ,  $C_1C_3$  and  $C_5$ . The constant  $C$  satisfies the requirement in the definition of bounded geometry. *QED*.

**COROLLARY 1.1.** Example 1.1 is on  $\mathcal{BH}$ .

*Proof.* The proof follows from Theorem 1.1, Remark 1.1 and Remark 1.3.

Suppose  $M$  is  $[-1, 1]$  or the circle  $S^1$  which is  $[-1, 1]$  glued at its boundary points. A partition of  $M$  is a collection of intervals with pairwise disjoint interiors such that the union of all intervals is  $M$ . We say a piecewise monotone mapping  $F$  from  $M$  into itself is a Markov mapping if there is a partition  $\eta_F$  of  $M$  such that for every interval in the partition, the continuous extension of the restriction of  $F$  to the interval is an embedding and the image of this interval under the

continuous extension of this restriction is a union of some intervals in the partition. We call  $\eta_F$  a Markov partition associated to  $F$ . We note that when we talk the value of  $F$  at a boundary point  $a$  of an interval in  $\eta_F$  we means either  $F(a-)$  or  $F(a+)$  according to different situations.

Suppose  $F : M \mapsto M$  is a Markov mapping and  $\eta_F = \{I_0, \dots, I_k\}$  is a Markov partition associated to  $F$ . From the definition of a Markov mapping, all the turning points and the jump points of  $F$  are boundary points of some intervals in  $\eta_F$ . We call the turning points and the jump points of  $F$  the singular points of  $F$ . For every  $I_i$  in  $\eta_F$ , let  $G_i$  be the inverse of the restriction of  $F$  to  $I_i$ . The domain of  $G_i$  is  $F(I_i)$ .

Suppose  $w = i_0 \dots i_n$  is a finite string of 0's,  $\dots$ ,  $k$ 's of length  $n + 1$ . We say  $w$  is an admissible if the image of  $I_{i_l}$  under  $F$  contains  $I_{i_{l+1}}$  for all integers  $0 \leq l < n$ . In the other words,  $w$  is an admissible if and only if the composition  $G_w = G_{i_0} \circ \dots \circ G_{i_n}$  is well-defined. For an admissible finite string  $w$ , define  $I_w$  to be the image of  $F(I_{i_n})$  under  $G_w$ . We call the collection of the intervals  $I_w$  for all the admissible finite strings  $w$  of zeroes and ones of length  $n + 1$  the  $n^{\text{th}}$ -partition of  $M$  determined by  $(F, \eta_F)$  or just by  $F$  if there is no possibility for

confusion. We denote this partition by  $\eta_{n,F}$  or just by  $\eta_n$ . Suppose  $\lambda_n$  is the supreme of lengths of the intervals in  $\eta_n$ .

We call  $(F, \eta_F)$  or just  $F$  a finite Markov mapping if  $\eta_F$  contains only finite intervals. Otherwise, we call  $F$  an infinite Markov mapping.

We extend Definition 1.4 to Markov mappings.

**DEFINITION 1.5.** *Suppose  $F$  is a finite Markov mapping. The sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $F$  has bounded geometry if there is a positive constant  $C$  such that for every positive integer  $n$  and every pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_n$  and  $I \in \eta_{n-1}$ , the ratio  $|J|/|I| \geq C$ . The largest such  $C$  is denoted as  $C_F$  and called the bounded geometry constant of  $\{\eta_n\}_{n=0}^\infty$ .*

**REMARK 1.4.** The ratio  $|J|/|I|$  could be one.

We say a periodic point  $p$  of  $F$  is topologically expanding if there is a number  $\delta > 0$  and an integer  $n > 0$  such that for any interval satisfying that  $p$  is a endpoint of this interval and the length of this interval is less than  $\delta$ , the image of this interval under the  $n^{\text{th}}$  iterate of  $F$  contains this interval.

THEOREM 1.2. *Suppose  $F$  is a piecewise  $C^{1,1}$ , finite Markov mapping. Moreover, suppose all the periodic points of  $F$  are topologically expanding. Then the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $F$  has bounded geometry.*

We prove this theorem through several lemmas. To state the first of these lemmas, we define the notion of homterval. A homterval is an interval of  $M$  such that all the iterates of  $F$  restricted to this interval are embeddings.

LEMMA 1.8. *The mapping  $F$  does not have any homterval.*

*Proof.* Suppose  $F$  does have a homterval and  $J$  is a maximal such interval. Here maximal means that there is not a different homterval containing this homterval. We use  $J_n$  to denote the image of  $J$  under the  $n^{\text{th}}$  iterate of  $F$  for all  $n \geq 0$ . Let  $K_i$  be the Lipschitz constant of the restriction of  $F'$  to  $I_i$  in  $\eta_F$  and  $K$  be the maximum of all  $K_i$ . Let  $c$  be the minimum value of  $|F'|$  on  $M$ .

For the sequence  $\{J_n\}_{n=0}^\infty$ , either they have pairwise disjoint interiors or some of them are overlap.

Suppose  $\{J_n\}_{n=0}^\infty$  have pairwise disjoint interiors. Then  $\sum_{i=0}^\infty |J_i| \leq$

2. By the naive distortion lemma,  $|(F^{on})'(x)|/|(F^{on})'(y)| \leq C_1$  for any  $x$  and  $y$  in  $J$  and all  $n \geq 0$  where  $C_1 = \exp(2K/c)$ . This implies that  $|(F^{on})'(x)| \leq C_1|F^{on}(J)|/|J|$  for all  $x$  in  $J$  and all  $n \geq 0$ . We claim that there are two positive constants  $\epsilon$  and  $C_2$  such that for any interval  $T \supset J$ , if the restrictions of the  $i^{th}$  iterate of  $F$  to  $T$  are embeddings for all  $0 < i \leq n+1$  and the lengths of the two connected components of  $T \setminus J$  are less than  $\epsilon$ , then

$$(i) \sum_{i=0}^n |F^{oi}(T)| \leq 4,$$

$$(ii) |(F^{on+1})'(y)|/|(F^{on+1})'(x)| \leq C_2 \text{ for all } x \text{ and } y \text{ in } T.$$

We use the induction to prove (i) and (ii). For  $n$  is zero, (i) and (ii) hold by choosing a constant  $C_2 > C_1$  and positive constant  $\epsilon$  small enough.

Suppose (i) and (ii) hold for all  $0 \leq n < N$ . Because the restriction of the  $i^{th}$  iterate of  $F$  to  $T$  is embedding for every  $0 < i \leq N$ , then  $F^{oi}(T)$  is the union of three intervals  $F^{oi}(L)$ ,  $F^{oi}(J)$  and  $F^{oi}(R)$  for every  $0 < i \leq N$ . The length of  $F^{oi}(T)$  equals  $|F^{oi}(L)| + |F^{oi}(J)| + |F^{oi}(R)|$  which is less than  $|F^{oi}(J)|(1 + 2\epsilon/|J|)$  because of (ii) in the

claim. Let  $\epsilon$  be a positive number and less than  $|J|/(2C_2)$ , then  $\sum_{i=0}^N |F^{\circ i}(T)|$  is less than 4. We proved (i) in the claim for  $n = N$ . Using the naive distortion lemma, we derive (ii) in the claim for  $n = N$ .

Suppose  $T_n$  is the maximal interval containing  $J$  such that the restriction of the  $n^{\text{th}}$  iterate of  $F$  to  $T_n$  is an embedding. Let  $L_n$  and  $R_n$  be the two connected components of  $T_n \setminus J$ . The lengths of  $R_n$  and  $L_n$  (one of them may be empty set) go to zero as  $n$  goes to infinity because  $J$  is a maximal homterval. Moreover, the lengths of  $F^{\circ n}(R_n)$  and  $F^{\circ n}(L_n)$  go to zero as  $n$  goes to infinity because of (ii) in the claim. This implies that the length of the image of  $J \cup L_n$  under the  $n^{\text{th}}$  iterate of  $F$  goes to zero as  $n$  increases to infinity.

One of the boundary points of  $F^{\circ n}(J \cup L_n) = J_n \cup F^{\circ n}(L_n)$  is either a singular point of  $F$  or a boundary point of  $M$ . Remember that every singular point of  $F$  is a boundary point of an interval in  $\eta_F$ . Because  $F$  permutes the set of the boundary points of the intervals in  $\eta_F$ ,  $F$  maps eventually any boundary point of any interval in  $\eta_F$  to a periodic point of  $F$ . We can find a subsequence  $\{n_i\}$  of the integers such that  $J_{n_i} \cup F^{\circ n_i}(L_{n_i})$  goes eventually to a periodic point of  $F$ . This

periodic point cannot be topologically expanding. This contradicts the assumptions on  $F$ . Hence some  $J_i$  have to be overlap.

Suppose  $J_m$  and  $J_n$  are overlap for some integers  $m > n \geq 0$ , that is, the intersection of  $J_m$  and  $J_n$  is a nontrivial interval. Let  $k = m - n$ . Because the restriction of the  $k^{th}$  iterate of  $F$  to  $J_k \cup J_0$  is an embedding,  $J_k$  and  $J_0$  are overlap. By the same reason,  $J_{lk}$  and  $J_{(l-1)k}$  are overlap for all integers  $l > 0$ . Let  $P$  be the union of  $J_{(l-1)k}$  where  $l$  runs over all positive integers. Then  $P$  is a nontrivial interval. The restriction of the  $k^{th}$  iterate of  $F$  to  $P$  is an embedding and the image of  $P$  under  $F$  is in  $P$ . The  $k^{th}$  iterate of  $F$  must have a nonexpanding fixed point in  $P$ . This contradicts the assumptions on  $F$ . *QED*.

**COROLLARY 1.2.** *The maximum length of the intervals in  $\eta_n$  goes to zero as  $n$  goes to infinity .*

*Proof.* If not, there is a positive constant  $\epsilon_0$  with the following property: for every positive integer  $n$ , there is an interval  $J_n$  in  $\eta_n$  such that the length of  $J_n$  is greater than  $\epsilon_0$ . Because  $M$  is a compact space, we can find a subsequence  $\{J_{n_i}\}$  such that it tends to an interval  $\tilde{J}$ . The length of  $\tilde{J}$  is bigger than or equal to  $\epsilon_0$ . There is a little small interval in

$\tilde{J}$  such that  $J_{n_i}$  contains this interval for  $i$  large enough. This implies that all the iterates of  $F$  restricted to this interval are embeddings. This contradicts the previous lemma. *QED.*

Using similar arguments to the proof of Lemma 1.8, we can still prove the following property for an infinite Markov mapping.

**LEMMA 1.9.** *Suppose  $F$  is a piecewise  $C^{1,1}$  mapping and an infinite Markov mapping. Moreover, suppose all the periodic points of  $F$  are topologically expanding and  $F$  maps eventually every point, which is either a singular point of  $F$  or a boundary points of  $M$ , to a periodic point of  $F$ . Then  $F$  does not have any homterval.*

Let us go on to consider the finite Markov mapping  $F$  in Theorem 1.2. Suppose  $J$  is an interval in  $\eta_n$ . Let  $J_{n-i}$  be the image of  $J$  under the  $i^{th}$  iterate of  $F$  for  $0 \leq i \leq n$ . Notice that  $J_0$  is in  $\eta_F$  and  $J_n$  is  $J$ . There is a unique admissible finite string  $w$  of  $0$ 's,  $\dots$ ,  $k$ 's of length  $n$  such that the inverse of the restriction of the  $n^{th}$  iterate of  $F$  to  $J$  is the restriction of  $G_w$  to  $J_0$ . Next lemma actually means that for any periodic point of  $F$  with large period  $n$ , the inverse of the  $n^{th}$  iterate of  $F$  is a contracting mapping on a neighborhood of this periodic point

and the neighborhood has definite size.

LEMMA 1.10. *There is an  $n_1 > 0$  such that for any integer  $n > n_1$  and  $J$  in  $\eta_n$ , if  $J_0, J_1, \dots, J_n$  have pairwise disjoint interiors, then the maximum value of the restriction of  $|G'_w|$  to  $J_0$  is less than  $1/2$ .*

*Proof.* Suppose  $C_1$  is the minimum length of the intervals in  $\eta_F$ . Let  $K$  be the maximum of the Lipschitz constants of the restrictions of  $F$  to  $I_i$  in  $\eta_F$  for  $0 \leq i \leq k$  and  $c$  be the minimum value of  $|F'|$  on  $M$ . By the naive distortion lemma,  $|G'_w(x)| \leq \exp((K/c) \sum_{i=0}^n |J_i|) |J_n|/|J_0|$  for any  $x$  in  $J_0$  because  $J_n = G_w(J_0)$ . Choosing  $n_1$  is a positive integer such that  $\lambda_n$  is less than  $C_1/(2 \exp(2K/c))$  for all  $n > n_1$  since  $\lambda_n \leq |J_n|$  and  $|J_0| \geq C_1$ . The maximum value of the restriction of  $|G'_w|$  to  $J_0$  is less than  $1/2$ . *QED.*

Let  $\tau_1$  be the inverse of the minimum of the absolute values of the eigenvalues of  $F$  at the periodic points whose periods are less than  $n_1 + 1$ . The number  $\tau_1$  is less than one because there are only finite many such periodic points. Suppose  $n$  and  $n_0$  are two positive integers and  $J$  is an interval in  $\eta_{(n+n_0)}$ . Let  $J_{n-i}$  be the image of  $J$  under the  $i^{\text{th}}$  iterate of  $F$  for  $0 \leq i \leq n$ . The interval  $J_0$  is in  $\eta_{n_0}$  and  $J_n$  is  $J$ . There

is a unique admissible finite string  $w$  of  $0$ 's,  $\dots$ ,  $k$ 's of length  $n$  such that the inverse of the restriction of the  $n^{\text{th}}$  iterate of  $F$  to  $J$  is the restriction of  $G_w$  to  $J_0$ . Suppose  $\tau$  is a constant which is greater than both of  $1/2$  and  $\tau_1$  and less than one. Next lemma extends Lemma 1.10 to all the periodic points.

LEMMA 1.11. *There is an integer  $n_0 > 0$  such that for any periodic point  $p$  of period  $n$  of  $F$  and the interval  $J$  in  $\eta_{(n+n_0)}$  which contains  $p$ , the maximum value of the restriction of  $|G'_w|$  to  $J_0$  is less than  $\tau$ .*

*Proof.* If  $n > n_1$ , then the maximum value of the restriction of  $|G'_w|$  to  $J_0$  is less than  $1/2$  for all  $n_0 > 0$  because Lemma 1.10.

Suppose  $n \leq n_1$ . Because  $|G'_w(p)|$  is less than  $\tau$  and  $G'_w$  is a continuous function, we can find an positive integer  $n_p$  such that for the interval  $J$  in  $\eta_{n+n_p}$  which contains  $p$ , the maximum value of the restriction of  $|G'_w|$  to  $J_0$  is less than  $\tau$ .

Let  $n_0$  be the maximum of  $n_1$  and  $\{n_p\}$  for all periodic points  $p$  whose periods are less than  $n_1 + 1$ . It satisfies this lemma. *QED.*

We say a sequence of intervals  $\{J_0, J_1, \dots\}$  is a sequence of backward

images of  $J_0$  under  $F$  if the restriction of the  $n^{\text{th}}$  iterate of  $F$  to  $J_n$  embeds  $J_n$  onto  $J_{n-1}$  for any  $n > 0$ . Let  $G_n$  be the inverse of the restriction of the  $n^{\text{th}}$  iterate  $F$  to  $J_n$ . We note that for any  $n > 0$ ,  $G_n$  is the restriction of  $G_w$  to  $J_0$  for some admissible finite string of  $0$ 's,  $\dots$ ,  $k$ 's of length  $n$ .

LEMMA 1.12. *There is a positive constant  $C$  such that for any  $J_0$  in  $\eta_{m_0}$  and any sequence of backward images  $\{J_0, J_1, \dots\}$  of  $J_0$  under  $F$ ,*

(a)  $|G'_n(x)|/|G'_n(y)| \leq C$  for all  $x$  and  $y$  in  $J_0$  and  $n > 0$ , moreover,

(b)  $\sum_{n=1}^{\infty} |G'_n(x)| \leq C$  for all  $x$  in  $J_0$ .

*Proof.* The naive distortion lemma says  $|G'_n(x)|/|G'_n(y)|$  is bounded above by  $\exp((K/c) \sum_{i=0}^{n-1} |J_i|)$  for any  $x$  and  $y$  in  $J_0$ .

For the sequence of backward images  $\{J_0, J_1, \dots\}$  of  $J_0$  under  $F$ , either they have pairwise disjoint interiors or some of them are subintervals of others.

If  $\{J_0, J_1, \dots\}$  have pairwise disjoint interiors, then  $\sum_{i=0}^{n-1} |J_i| \leq 2$ . There is a positive constant  $C_1 = \exp(2K/c)$  such that the ratio  $|G'_n(x)|/|G'_n(y)| \leq C_1$  for all  $x$  and  $y$  in  $J_0$  and  $n > 0$ .

Suppose some of  $\{J_0, J_1, \dots\}$  are subintervals of others. Let  $m$  be the positive integer such that  $J_0, \dots, J_{m-1}$  have pairwise disjoint interiors and  $J_m$  is a subinterval of  $J_k$  for some  $0 \leq k < m$ . Let  $l = m - k$ . There is a period point of period  $l$  of  $F$  in  $J_m$ . From Lemma 1.11,  $|J_m| \leq \tau |J_k|$ . The same reason implies that  $|J_{j+l+i+m}| \leq \tau |J_{(j-1)l+i+m}|$  for all  $j > 0$  and  $0 < i \leq l - 1$ . Now we can get that  $\sum_{i=0}^{\infty} |J_i| \leq 2/(1 - \tau) + 2$ . Let  $C_2 = \exp((K/c)((2/(1 - \tau)) + 2))$ . Then  $|G'_n(x)|/|G'_n(y)| \leq C_2$  for all  $x$  and  $y$  in  $J_0$  and  $n > 0$ .

Suppose  $C_3$  is the minimum length of the intervals in  $\eta_{n_0}$ . Then  $|G'_n(x)| \leq (C_2/C_3)|J_n|$  from the naive distortion lemma. Moreover,  $\sum_{n=1}^{\infty} |G'_n(x)| \leq C = (C_2/C_3)(2/(1 - \tau))$ . *QED.*

*Proof of Theorem 1.2.* Suppose  $\mu_n$  is the minimum length of the intervals in  $\eta_n$ . Let  $C_1$  be the minimum of  $\mu_{n+1}/\lambda_n$  for all  $0 \leq n \leq n_0$ .

For any integer  $n > 0$  and any pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_{n+1}$  and  $I \in \eta_n$ . If  $n \leq n_0$ , then  $|J|/|I| \geq C_1$ . Suppose  $n > n_0$ . Let  $J_0$  and  $I_0$  be the images of  $J$  and  $I$  under the  $(n - n_0)^{th}$  iterate of  $F$  and  $G_n$  be the inverse of the restriction of the  $(n - n_0)^{th}$  iterate of  $F$  to  $I$ . The intervals  $J_0$  and  $I_0$  are in  $\eta_{n_0+1}$  and  $\eta_{n_0}$ , respectively.

From Lemma 1.12, we get that  $|J|/|I| \geq C|J_0|/|I_0| \geq CC_1$  because  $|J|/|I| = (|G_n(x)|/|G_n(y)|)|J_n|/|I_n|$  for some  $x$  and  $y$  in  $I_0$ . *QED.*

**COROLLARY 1.3.** *Example 1.2 is on  $\mathcal{BH}$ .*

*Proof.* Because the corresponding change of coordinate  $h_\gamma$  is smooth at the interior points of  $[-1, 1]$ , all the periodic points of  $\tilde{f}$  in the interior of  $[-1, 1]$  are expanding. Lemma 1.1 says  $-1$  is an expanding fixed point of  $\tilde{f}$ . From Theorem 1.2, the sequence of nested partitions determined by  $\tilde{f}$  decreases exponentially because every interval in the  $n^{\text{th}}$ -partition contains the two intervals in the  $(n+1)^{\text{th}}$ -partition. Then from Lemma 1.3, the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $f$  decreases exponentially. *QED.*

In Theorem 1.2, the condition that  $F$  is  $C^{1,1}$  can not be weakened to the condition that it be  $C^{1+\alpha}$  for  $0 \leq \alpha < 1$ . It is enough to weaken the  $C^{1,1}$  condition of Example 1.2 to construct a counterexample similar to the Denjoy counterexample in the theory of circle diffeomorphisms (see [D] or [H]).

#### CONSTRUCTION OF THE COUNTEREXAMPLE

Suppose  $\tilde{q} : y \rightarrow 1 - 2|y|$  is the piecewise linear mapping. Let  $x_0$  be

a point in  $[-1, 1]$  such that the forward images of  $x_0$  under  $\tilde{q}$  is dense on  $[-1, 1]$ . Let  $x_m$  be the image of  $x_0$  under the  $m^{\text{th}}$  iterate of  $\tilde{q}$  for any positive integer  $m$ . Suppose  $g_0$  and  $g_1$  are the inverses of the restrictions of  $\tilde{q}$  to  $[-1, 0]$  and to  $[0, 1]$ , respectively. Suppose  $w = i_0 \cdots i_n$  is a finite string of zeros and ones. Let  $g_w$  be the composition,  $g_w = g_{i_0} \circ \cdots \circ g_{i_n}$ . Let  $x_m(w)$  be the image of  $x_m$  under  $g_w$  for every nonnegative integer  $m$  and every finite strings  $w$  of zeros and ones.

Suppose  $\alpha$  is a nonnegative number and less than one. For all integers  $m \geq 0$  and finite strings  $w$ , we choose a nontrivial intervals  $I_m(w)$  containing  $x_m(w)$  such that  $\{I_m(w)\}_{m,w}$  are pairwise disjoint intervals,  $\sum_{m,w} |I_m(w)| = 2$  and  $\sum_{m,w} |I_m(w)|^\alpha = +\infty$ . For a pair  $x_m(w)$  and  $x_{m'}(w')$ , if  $\tilde{q}(x_m(w)) = x_{m'}(w')$ , then we construct a  $C^{1+\alpha}$  diffeomorphism  $\tilde{f}_{m,w}$  from  $I_m(w)$  to  $I_{m'}(w')$ . The mapping  $\tilde{f}_{m,w}$  satisfies that (1) the  $\alpha$ -Hölder constant of its derivative is bounded above by a fixed constant, (2) its derivative is piecewise monotone, (3) the maximum value of its derivative is less than a fixed constant and (4) the derivatives at both boundary points of  $I_{m,w}$  are greater than one. We construct such  $\tilde{f}_{m,w}$  such that its derivative  $\tilde{f}'_{m,w}$  has the same sign as that of  $\tilde{q}'(x_{m,w})$ .

The mapping  $\tilde{f}_0$  from  $\cup_{n,w} I_{n,w}$  to itself is defined as  $\tilde{f}_0(x) = \tilde{f}_{n,w}(x)$  for  $x$  in  $I_{n,w}$ . Let  $\tilde{f}$  be the unique continuous extension of  $\tilde{f}_0$  to  $[-1, 1]$ . From the construction of  $\tilde{f}_0$ , we can make that 0 is the unique turning point of  $\tilde{f}$  and the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha}$  embeddings.

Let  $f = h_\gamma^{-1} \circ \tilde{f} \circ h_\gamma$  with  $\gamma > 1$ . It is a  $C^{1+\frac{\alpha}{\gamma}}$  mapping and has the power law  $|x|^\gamma$  at the critical point. Moreover, all the periodic points of  $f$  are expanding. Because  $f$  has a homterval, it is not on the boundary of hyperbolicity. *QED.*

There is only one topological conjugacy class on the boundary of hyperbolicity. This means that for any pair  $f$  and  $g$  on  $\mathcal{BH}$ , there is a unique homeomorphism  $h$  from the interval  $[-1, 1]$  to itself such that  $g = h \circ f \circ h^{-1}$ . The homeomorphism  $h$  can be constructed from the sequences of nested partitions  $\{\eta_{n,f}\}_{n=0}^{+\infty}$  and  $\{\eta_{n,g}\}_{n=0}^{+\infty}$  determined by  $f$  and  $g$ . In the other words,  $h$  is determined by the following rules: it maps the intervals in  $\eta_{n,f}$  to the intervals in  $\eta_{n,g}$  preserving the order of intervals. The following sections are devoted to study some geometric properties of the mappings  $f$ ,  $g$  and  $h$ .

### §1.3. Quasisymmetric classification of $\mathcal{BH}$ .

A homeomorphism  $H$  from an interval to itself is quasisymmetric if there is a positive constant  $C$  such that for any pair  $x$  and  $y$  in the interval,  $|H(x) - H(z)|/|H(y) - H(z)|$  is bounded above by  $C$  where  $z$  is the middle point of  $x$  and  $y$ .

**THEOREM 1.3.** *For any pair  $f$  and  $g$  on  $\mathcal{BH}$ , the conjugating mapping  $h$  between  $f$  and  $g$  is quasisymmetric.*

To prove this theorem, we develop a tool which we call the  $(\alpha + \gamma)$ -Koebe distortion lemma. This lemma has a generalization, Lemma 2.7, which is for a family of mappings and which we call the uniform  $(\alpha + \gamma)$ -Koebe distortion lemma.

Suppose  $f$  is a self mapping of  $[-1, 1]$  with unique critical point. Assume that  $f$  has the power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical point 0,  $f$  maps 0 to 1 and maps  $-1$  and  $1$  to  $-1$ . Suppose the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha}$  embeddings for some  $0 < \alpha \leq 1$ . Remember that  $\eta_1$  is the first partition determined by  $f$  and contains four intervals  $I_{00}$ ,  $I_{01}$ ,  $I_{11}$  and  $I_{10}$ . Suppose  $x$  and  $y$  are two points in one of these four intervals. Let  $J_0$  be the interval bounded by  $x$  and

$y$  and  $\theta(x, y) = \{J_0, J_1, \dots\}$  be a sequence of backward images of  $J_0$  under  $f$ , that is, the restriction of  $f$  to  $J_n$  embeds  $J_n$  onto  $J_{n-1}$  for all  $n > 0$ . Let  $g_n$  be the inverse of the restriction of the  $n^{\text{th}}$  iterate of  $f$  to  $J_n$ . Note that for each  $n$ , there is a finite string  $w$  of zeroes and ones of length  $n$  such that the restriction of  $g_w$  to  $J_0$  is  $g_n$ . We define the distortion of the  $n^{\text{th}}$  iterate of  $f$  along  $\theta(x, y)$  to be  $|g'_n(x)|/|g'_n(y)|$ . Let  $d_{xy}$  be the distance from  $\{x, y\}$  to  $\{-1, 1\}$ .

LEMMA 1.13 (the  $(\alpha + \gamma)$ -Koebe distortion lemma). *There are positive constants  $A$ ,  $B$  and  $C$  such that for any pair  $x$  and  $y$  in one of the intervals in  $\eta_1$  and any sequence  $\theta(x, y) = \{J_0, J_1, \dots\}$  of backward images of  $J_0$  under  $f$ , the distortion of the  $n^{\text{th}}$  iterate of  $f$  along  $\theta(x, y)$  satisfies*

$$\frac{|g'_n(x)|}{|g'_n(y)|} \leq \exp\left(\left(A + B \sum_{i=0}^n |J_i| + \frac{C|J_0|}{d_{xy}}\right) \sum_{i=0}^n |J_i|^{\frac{\alpha}{\gamma}}\right)$$

for all positive integers  $n$ .

*Proof.* The second partition  $\eta_2$  determined by  $f$  contains eight intervals. The two of them adjacent to 0 are  $I_{010}$  and  $I_{110}$ . We call these two intervals the middle intervals. The union of other intervals in  $\eta_2$  consists of two intervals. We call them the left interval and the right

interval (see Figure 9).

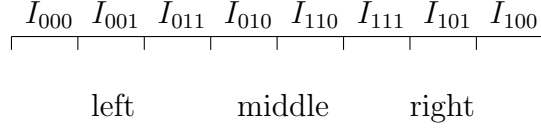


Figure 9

Since we are assuming that  $\tilde{f}$  is  $C^{1+\alpha}$ , the restrictions of  $f$  to the left interval and to the right interval are  $C^{1+\frac{\alpha}{\gamma}}$  embeddings because of Remark 1.2. Let  $K_1$  be the maximum of the  $(\alpha/\gamma)$ -Hölder constants of the derivatives of these restrictions, that is,  $|f'(x) - f'(y)| \leq K_1|x - y|^{\alpha/\gamma}$  for all pair  $x$  and  $y$  in the left interval and for all pair  $x$  and  $y$  in the right interval. Let  $c_1$  be the minimum value of  $|f'|$  on the union the left and right intervals.

The restrictions of  $\tilde{f}$  to the images of the middle intervals under  $h_\gamma$  are  $C^{1+\alpha}$  embeddings by the assumption. Let  $K_2$  be the maximum of the  $\alpha$ -Hölder constants of the derivatives of these restrictions. Let  $c_2$  be the minimum value of  $|\tilde{f}'|$  on the image of the union of the middle intervals under  $h_\gamma$ . Remember that at the turning point,  $\tilde{f}$  has two one-sided derivatives.

The restriction of  $h_\gamma$  to the union of the middle intervals is  $C^{1,1}$ . Let

$K_3$  be the Lipschitz constant of the derivative of this restriction and  $c_3$  be the minimum value of  $h'_\gamma$  on the union of the middle intervals.

Let  $x_i$  and  $y_i$  be the images of  $x$  and  $y$  under  $g_i$ . Then  $x_i$  and  $y_i$  lie in the same interval in  $\eta_{i+1}$  for  $i \geq 0$ . For every integer  $n > 0$ ,

$$g'_n(x)/g'_n(y) = (f^{\circ n})'(y_n)/(f^{\circ n})'(x_n).$$

By the chain rule, the ratio  $(f^{\circ n})'(y_n)/(f^{\circ n})'(x_n)$  equals the product of ratios  $f'(y_{n-i})/f'(x_{n-i})$  where  $i$  runs from 0 to  $n-1$ . This product can be factored into two products,

$$\prod_{x_i, y_i \in LR} f'(y_i)/f'(x_i) \quad \text{and} \quad \prod_{x_i, y_i \in M} f'(y_i)/f'(x_i).$$

Here  $LR$  stands for the union of the left and right intervals and  $M$  stands for the union of the two middle intervals. Note that the subscript  $i$  in the products are positive integers and less than  $n+1$ .

Using the naive distortion lemma, we can show that the absolute value of the first product  $\prod_{x_i, y_i \in LR} f'(y_i)/f'(x_i)$  is less than or equal to  $d_1 = \exp((K_1/c_1) \sum_{i=1}^n |J_i|^{\frac{\alpha}{\gamma}})$ .

The second product  $\prod_{x_i, y_i \in M} f'(y_i)/f'(x_i)$  can be factored into three

factors,

$$\prod_{x_i, y_i \in M} \frac{f'(y_i)}{f'(x_i)} = \prod_{x_i, y_i \in M} \frac{h'_\gamma(y_i)}{h'_\gamma(x_i)} \cdot \prod_{x_i, y_i \in M} \frac{\tilde{f}'(h_\gamma(y_i))}{\tilde{f}'(h_\gamma(x_i))} \cdot \prod_{x_i, y_i \in M} \frac{h'_\gamma(f(x_i))}{h'_\gamma(f(y_i))}.$$

By the naive distortion lemma again, we can show that the absolute value of the first factor is bounded above by  $d_2 = \exp((K_3/c_3) \sum_{i=1}^n |J_i|)$  and the absolute value of the second factor is bounded above by  $d_3 = \exp(((K_3^\alpha K_2)/c_2) \sum_{i=1}^n |J_i|^\alpha)$ .

The third factor can be factored again into two products,

$$\prod_{x_i, y_i \in M} \frac{(1 + f(y_i))^{\frac{1}{\gamma^*}}}{(1 + f(x_i))^{\frac{1}{\gamma^*}}} \quad \text{and} \quad \prod_{x_i, y_i \in M} \frac{(1 - f(y_i))^{\frac{1}{\gamma^*}}}{(1 - f(x_i))^{\frac{1}{\gamma^*}}}.$$

Because  $x_i$  and  $y_i$  are in one of the two middle intervals, then  $f(x_i) = x_{i-1}$  and  $f(y_i) = y_{i-1}$  are in  $I_{10}$  which is the interval in  $\eta_1$  adjacent to 1. This implies that the first of these product is bounded above by  $d_4 = \exp(1/\gamma^* \sum_{i=0}^{n-1} |J_i|)$ .

To control the second of these products, we write

$$\frac{1 - f(y_i)}{1 - f(x_i)} = 1 + \frac{f(x_i) - f(y_i)}{1 - f(x_i)}$$

for all pair  $x_i$  and  $y_i$  in  $M$ .

Suppose  $k_0$  is the smallest positive integer such that  $x_{k_0}$  and  $y_{k_0}$  are

in  $M$ . We may consider  $k_0$  in two cases. The first case is that  $k_0 = 1$  and the second case is that  $k_0 > 1$ .

In the first case, the images of  $x_{k_0}$  and  $y_{k_0}$  under  $f$  are  $x$  and  $y$ . We have that

$$\frac{|f(x_{k_0}) - f(y_{k_0})|}{|1 - f(x_{k_0})|} = \frac{|x - y|}{|1 - x|} \leq \frac{|J_0|}{d_{xy}}.$$

In the second case, the intervals  $J_i$  for  $0 < i < k_0$  are in the union of the left and right intervals. The interval  $J_0$ , which is bounded by  $x$  and  $y$ , and the interval bounded by  $-1$  and  $x$  are the images of the interval  $J_{k_0-1}$ , which is bounded by  $f(x_{k_0})$  and  $f(y_{k_0})$ , and the interval bounded by  $1$  and  $f(x_{k_0})$  under the  $(k_0 - 1)^{th}$  iterate of  $f$ , respectively. We get that  $|x - y|/|f(x_{k_0}) - f(y_{k_0})| = (f^{\circ(k_0-1)})'(\theta)$  for some  $\theta$  in  $J_{k_0-1}$  and  $|1 + x|/|1 - f(x_{k_0})| = (f^{\circ(k_0-1)})'(\eta)$  for some  $\eta$  in the interval bounded by  $1$  and  $f(x_{k_0})$ . By the naive distortion lemma,  $|(f^{\circ(k_0-1)})'(\eta)|/|(f^{\circ(k_0-1)})'(\theta)|$  is bounded by  $d_1$ . This implies that

$$\frac{|f(x_{k_0}) - f(y_{k_0})|}{|1 - f(x_{k_0})|} \leq d_1 \frac{|x - y|}{|1 + x|} \leq d_1 \frac{|J_0|}{d_{xy}}.$$

For any  $k > k_0$  with  $x_k$  and  $y_k$  in  $M$ , let  $l$  be the smallest positive integer such that  $x_{k-l}$  and  $y_{k-l}$  are in  $M$ . The interval  $J_{k-l}$ , which is

bounded by  $x_{k-l}$  and  $y_{k-l}$ , and the interval bounded  $-1$  and  $x_{k-l}$  are the images of the interval  $J_{k-1}$ , which is bounded by  $f(x_k)$  and  $f(y_k)$ , and the interval bounded by  $1$  and  $f(x_k)$  under the  $(l-1)^{th}$  iterate of  $f$ , respectively. We get that  $|x_{k-l} - y_{k-l}|/|f(x_k) - f(y_k)| = (f^{\circ(l-1)})'(\theta)$  for some  $\theta$  in  $J_{k-1}$  and  $|1 + x_{k-l}|/|1 - f(x_k)| = (f^{\circ(l-1)})'(\eta)$  for some  $\eta$  in the interval bounded by  $f(x_k)$  and  $1$ . By the naive distortion lemma,  $|(f^{\circ(l-1)})'(\eta)|/|(f^{\circ(l-1)})'(\theta)|$  is less than or equal to  $d_1$ . This implies that

$$\frac{|f(x_k) - f(y_k)|}{|1 - f(x_k)|} \leq d_1 \frac{|x_{k-l} - y_{k-l}|}{|1 + x_{k-l}|}.$$

Because  $x_{k-l}$  is in one of the middle intervals,  $1 + x_{k-l}$  is greater than the length of  $I_{00}$  which is the interval in  $\eta_1$  adjacent to  $-1$ . Then

$$\frac{|f(x_k) - f(y_k)|}{|1 - f(x_k)|} \leq \frac{d_1}{|I_{00}|} |J_{k-l}|.$$

Now we have that

$$\prod_{x_i, y_i \in M} \frac{(1 - f(y_i))^{\frac{1}{\gamma^*}}}{(1 - f(x_i))^{\frac{1}{\gamma^*}}} \leq \exp\left(\frac{d_1 |J_0|}{\gamma^* d_{xy}} + \frac{d_1}{\gamma^* |I_{00}|} \sum_{i=1}^n |J_i|\right)$$

We put all the estimations together. Because  $\sum_{i=1}^n |J_i|^{\frac{\alpha}{\gamma}}$ ,  $\sum_{i=1}^n |J_i|^\alpha$ ,  $\sum_{i=0}^{n-1} |J_i|$  and  $\sum_{i=1}^n |J_i|$  are less than or equal to  $\sum_{i=0}^n |J_i|^{\frac{\alpha}{\gamma}}$ , we proved this lemma and  $A = K_1/c_1 + (K_3^\alpha K_2)/c_2 + K_3/c_3 + 1/\gamma^*$ ,  $B = 1/(\gamma^* |I_{00}|)$  and  $C = 1/\gamma^*$ . *QED.*

COROLLARY 1.4. *Suppose  $f$  is on  $\mathcal{BH}$ . Then there are positive constants  $D$  and  $E$  such that for any  $x$  and  $y$  in one of the intervals in  $\eta_1$  and any sequence  $\theta(x, y) = \{J_0, J_1, \dots\}$  of backward images of  $J_0$  under  $f$ , the distortion of the  $n^{\text{th}}$  iterate of  $f$  along  $\theta(x, y)$  satisfies*

$$\frac{|g'_n(x)|}{|g'_n(y)|} \leq \exp\left(D + \frac{E}{d_{xy}}\right)$$

*for all positive integers  $n$ .*

*Proof.* Because  $f$  is on  $\mathcal{BH}$ , there are two positive numbers  $C_1$  and  $\lambda < 1$  such that  $|J_i| \leq C_1 \lambda^i$  for all integers  $i > 0$ . Then  $\sum_{i=0}^n |J_i|$  is less than  $C_2 = C_1/(1 - \lambda)$  and  $\sum_{i=0}^n |J_i|^{\frac{\alpha}{\gamma}}$  is less than  $C_3 = C_1/(1 - \lambda^{\frac{\alpha}{\gamma}})$ . We use  $D$  to denote  $(A + BC_2)C_3$  and  $E$  to denote  $CC_3$ , where  $A$ ,  $B$  and  $C$  are the constants in the previous lemma. Now it is easy to see this corollary. *QED.*

LEMMA 1.14. *Suppose  $f$  is a mapping on  $\mathcal{BH}$ . Then the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  has bounded geometry.*

*Proof.* The proof of this lemma is similar to the proof of Theorem 1.1. The condition that the sequence of nested partitions  $\{\eta_n\}_{n=0}^{\infty}$  determined by  $f$  decreases exponentially plays the same role as the expanding condition of the restrictions of  $f$  to the left interval and to the

right interval plays in the proof of Theorem 1.1. Corollary 1.4 plays the same role as that  $C^3$ -Koebe distortion lemma plays in the proof of Theorem 1.1. *QED.*

*Proof of Theorem 1.3.* Suppose  $C_f$  and  $C_g$  are the bounded geometry constants for the sequences  $\{\eta_{n,f}\}_{n=0}^\infty$  and  $\{\eta_{n,g}\}_{n=0}^\infty$ . For any pair  $x$  and  $y$  in  $[-1, 1]$ , let  $n$  be the largest positive integer such that there is the interval  $I_w$  in  $\eta_{n,f}$  which contains both  $x$  and  $y$ . Suppose  $I_{w00}$ ,  $I_{w01}$ ,  $I_{w11}$  and  $I_{w10}$  are the intervals in  $\eta_{n+2,f}$  which are contained in  $I_w$  (see Figure 10). We call  $I_{w01}$  and  $I_{w11}$  the middle intervals and  $I_{w00}$  and  $I_{w10}$  the left and right intervals.

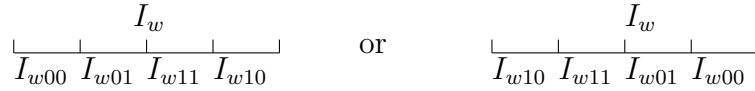


Figure 10

We may assume  $x < y$  and consider  $x$  and  $y$  in two cases according to the positions of  $x$  and  $y$  in  $I_w$ . The first case is that one of them is in one of the left and right intervals. The second case is that both of them in the interiors of the union of the middle intervals.

In the first case, the interval bounded by  $x$  and  $y$  contains one of the middle intervals. This implies that the ratio of  $|x - y|$  to  $|I_w|$  is greater

than or equal to  $C_f^2$  by the property of bounded geometry. We can find a positive integer  $k$  which only depends on  $C_f$  such that in  $\eta_{n+k,f}$ , there are two intervals  $J_1 \subset [x, z]$  and  $J_2 \subset [z, y]$ . Then  $h(J_1)$  and  $h(J_2)$  are in  $\eta_{n+k,g}$  and  $h(J_1) \subset [h(x), h(z)]$  and  $h(J_2) \subset [h(z), h(y)]$ . Because the interval bounded by  $h(x)$  and  $h(y)$  is also contained in  $h(I_w)$ , we can find a positive constant  $C_1$  which only depends on  $C_f$  and  $C_g$  such that  $C_1^{-1} \leq |h(z) - h(x)|/|h(z) - h(y)| \leq C_1$ .

In the second case, let  $p$  be the common end point of the two middle intervals. For any  $i > 0$ , let  $L_i$  be the interval in  $\eta_{(n+i),f}$  which is adjacent to  $p$  and in  $I_{w01}$  and  $R_i$  be the interval in  $\eta_{(n+i),f}$  which is adjacent to  $p$  and in  $I_{w11}$ . Suppose  $l$  is the largest positive integer such that  $L_l$  contains one of  $x$  and  $y$  and  $m$  is the largest positive integer such that  $R_m$  contains one of  $x$  and  $y$ . Assume  $l \leq m$  (if  $l > m$ , the proof is same). We call  $L_l$  and  $R_l$  the central intervals. Under the  $(n+1)^{th}$  iterate of  $f$ ,  $I_w$  is mapped to  $[-1, 1]$ ,  $p$  is mapped to 0 and  $L_l$  and  $R_l$  are mapped to the two intervals in  $\eta_{(l-1),f}$  adjacent to 0. Let  $L_c$  and  $R_c$  be the images of the central intervals under the  $(n+1)^{th}$  iterate of  $f$ . Note that  $l$  is greater than one in this case and  $L_c$  and  $R_c$  are in the union of the middle intervals  $I_{01}$  and  $I_{11}$  in  $\eta_{1,f}$ . The restriction of

$f$  to the union of  $L_c$  and  $R_c$  and the restriction of  $g$  to the image of the union of  $L_c$  and  $R_c$  under  $h$  are comparable with the power law mapping  $|x|^\gamma$ ,  $|x|^{\gamma'}$ , respectively. We can find a positive constant  $C_2$  such that  $C_2^{-1} \leq |L_c|/|R_c| \leq C_2$  and  $C_2^{-1} \leq |h(L_c)|/|h(R_c)| \leq C_2$  because the images of  $L_c$  and  $R_c$  under  $f$  are same and the images of  $h(L_c)$  and  $h(R_c)$  under  $g$  are same. The union of  $L_c$  and  $R_c$  and the image of this union under  $h$  are far to the boundary of  $[-1, 1]$ , that is, the distances from this union and the image of this union under  $h$  to  $\{-1, 1\}$  are bounded below by  $|I_{10}|$  and  $|h(I_{10})|$ , respectively. By Corollary 1.4, we can find a positive constant  $C_3$  such that  $C_3^{-1} \leq |L_l|/|R_l| \leq C_3$  and  $C_3^{-1} \leq |h(L_l)|/|h(R_l)| \leq C_3$ . Moreover, we can find a positive constant  $C_4$  such that  $|R_i| \leq C_4|L_i|$  and  $|h(R_i)| \leq C_4|h(L_i)|$  for all  $i \geq l$  by using the property of bounded geometry. This implies that there is a positive constant  $C_5$  such that  $C_5^{-1}|L_l| \leq |x - y| \leq C_5|L_l|$  and  $C_5^{-1}|h(L_l)| \leq |h(x) - h(y)| \leq C_5|h(L_l)|$ . Using the property of bounded geometry again, we can find a positive integer  $k$  which only depends on  $C_f$  such that in  $\eta_{l+k,f}$ , there are  $J_1 \subset [x, z]$  and  $J_2 \subset [z, y]$ . The intervals  $h(J_1)$  and  $h(J_2)$  are in  $\eta_{l+k,g}$  and  $h(J_1) \subset [h(x), h(z)]$  and  $h(J_2) \subset [h(z), h(y)]$ . Finally, we can find a positive constant  $C_6$  which only depends on  $C_f$

and  $C_g$  such that  $C_6^{-1} \leq |h(z) - h(x)|/|h(z) - h(y)| \leq C_6$ . QED.

**§1.4.  $C^{1+\epsilon}$  classification of  $\mathcal{BH}$ .**

The conjugating mapping  $h$  between  $f$  and  $g$  is usually not a diffeomorphism because  $f$  has a lot  $C^1$ -invariants, for example, the eigenvalues of  $f$  at the periodic points, the power law and the asymmetry of  $f$  at the critical point. A question is that do all these  $C^1$ -invariants form a complete set of  $C^1$ -invariants for dynamical systems on the boundary of hyperbolicity? Next theorem gives an affirmative answer to this question.

Recall that the asymmetry of  $f$  at the critical point is the limit of  $f'(-x)/f'(x)$  as  $x$  decreases to zero and the eigenvalue of  $f$  at a periodic point  $p$  of period  $n$  is  $e_f(p) = (f^{on})'(p)$ .

Suppose  $f$  and  $g$  are on  $\mathcal{BH}$  and the restrictions of  $\tilde{f}$  and  $\tilde{g}$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha}$  embeddings for some  $0 < \alpha \leq 1$ . Let  $h$  give the quasisymmetric equivalence between  $f$  and  $g$ .

**THEOREM 1.4.** *The conjugating mapping  $h$  is a  $C^{1+\epsilon}$ -diffeomorphism for some  $0 < \epsilon \leq 1$  if and only if  $e_f(p) = e_g(h(p))$  for all the periodic*

points  $p$  of  $f$  and  $f$  and  $g$  have the same power law and the same asymmetry at the critical point.

A  $C^1$ -invariant on  $\mathcal{BH}$  is an object associated to  $f$  on  $\mathcal{BH}$  which is the same for  $f$  as for  $h \circ f \circ h^{-1}$  whenever  $h$  is an orientation preserving  $C^1$ -diffeomorphism.

**COROLLARY 1.5.** *The eigenvalues at all periodic points, the power law and the asymmetry at the critical point form a complete set of  $C^1$ -invariants on the boundary of hyperbolicity .*

**REMARK 1.5.** *These invariants are optimal to form a complete set of  $C^1$ -invariants on the boundary of hyperbolicity. In fact, we can find two mappings  $f$  and  $g$  on  $\mathcal{BH}$  such that (1)  $f$  and  $g$  have the same eigenvalues at the corresponding periodic points and the same power law at the critical point and (2)  $f$  and  $g$  have different asymmetries at the critical point. We also can find two mappings  $f$  and  $g$  on  $\mathcal{BH}$  such that  $f$  and  $g$  have the same eigenvalues at the corresponding periodic points and the same asymmetry at the critical point and (2)  $f$  and  $g$  have different power laws at the critical point.*

*In order to construct these counterexamples  $f$  and  $g$ , which satisfy*

(1) and (2), respectively, we let  $\tilde{f}(y) = 1 - 2|y|$  on  $[-1, 1]$ . Suppose  $\tilde{h}$  is a function on  $[-1, 1]$  satisfying that  $\tilde{h}(y) = \frac{3}{2}y + \frac{1}{2}$  for  $-1 \leq y \leq 0$  and  $\tilde{h}(y) = \frac{1}{2}y + \frac{1}{2}$  for  $0 \leq y \leq 1$  in the first case and satisfying that  $\tilde{h}$  is  $C^1$  on  $[-1, 1]$  and  $\tilde{h}(y) = (1 + y)^{\frac{2}{\gamma}}$  for  $1+y$  small enough in the second case. Let  $\tilde{g} = \tilde{h} \circ \tilde{f} \circ \tilde{h}^{-1}$  and  $g = h_\gamma \circ \tilde{g} \circ h_\gamma^{-1}$ . Let  $f(x) = 1 - 2x^2$ . Then  $f$  and  $g$  for  $\gamma = 2$  satisfy (1) and (2) in the first case and  $f$  and  $g$  for  $\gamma \neq 2$  satisfy (1) and (2) in the second case.

Let  $h$  gives the quasimetric equivalence between  $f$  and  $g$  in both cases. Then  $h$  is not  $C^1$ -diffeomorphism on  $[-1, 1]$ . In the second part of this thesis, we also show that the scaling function plays the same role as the eigenvalues and the power law at the critical point play together here (see Corollary 2.4). There the scaling function and the asymmetry at the critical point are optimal to form a complete set of  $C^1$  invariants on the boundary of hyperbolicity too.

Theorem 1.4 is a corollary of the more general Theorem 1.5 which states for Markov mappings.

Just as in §1.2,  $M$  is the interval  $[-1, 1]$  or the circle  $S^1$ ,  $F$  is a Markov mapping from  $M$  to itself and  $\eta_F = \{I_0, \dots, I_k\}$  is a Markov

partition associated to  $F$ . Suppose the restriction of  $F$  to each  $I_i$  is a  $C^{1+\alpha}$  embedding and  $F$  is an expanding mapping, that is, under an appropriate smooth metric on  $M$ , the minimum value of  $|F'|$  on  $M$  is greater than 1. We call such  $F$  a  $C^{1+\alpha}$  expanding Markov mapping. Recall that we call the turning points and the jump points of  $F$  the singular points of  $F$ . For each singular point  $a$  of  $F$ , the ratio  $F'(a-)/F'(a+)$  is a  $C^1$ -invariant. Here a  $C^1$ -invariant means again an object associated to  $F$  which is the same for  $F$  as for  $H \circ F \circ H^{-1}$  whenever  $H$  is an orientation preserving  $C^1$ -diffeomorphism. We call the ratio  $F'(a-)/F'(a+)$  the asymmetry of  $F$  at  $a$  and denote it by  $sv_F(a)$ . Recall that the eigenvalue of  $F$  at a periodic point  $p$  of period  $n$  is  $e_F(p) = (F^{on})'(p)$ . Suppose  $G$  is also an  $C^{1+\alpha}$  expanding Markov mapping and topologically conjugate to  $F$  by an orientation preserving homeomorphism  $H$  from  $M$  to itself, that is,  $G = H \circ F \circ H^{-1}$ . We call  $H$  the conjugating mapping between  $F$  and  $G$ . If  $F$  and  $G$  are also finite Markov mappings, then we can prove that  $H$  is quasisymmetric by similar arguments to the proof of Theorem 1.3 because the sequences of nested partitions determined by  $F$  and  $G$  have bounded geometry (compare to Theorem 1.2). Moreover, we prove the following theorem.

THEOREM 1.5. *Suppose  $F$  and  $G$  are finite,  $C^{1+\alpha}$  expanding Markov mappings and the restriction of  $F$  to every interval in  $\eta_F$  embeds this interval onto  $M$ . The conjugating mapping  $H$  is a  $C^{1+\alpha}$  diffeomorphism from  $M$  to itself if and only if  $e_F(p) = e_G(H(p))$  for all periodic points  $p$  of  $F$  and  $sv_F(a) = sv_G(H(a))$  for all singular points  $a$  of  $F$ .*

First, we prove a lemma. Suppose  $F$  and  $G$  are  $C^{1+\alpha}$  expanding Markov mappings (allowed to be infinite Markov mappings) and  $H$  is the conjugating mapping between  $F$  and  $G$ . We say  $H$  is an absolutely continuous conjugacy between  $F$  and  $G$  if it is non-singular with respect to the Lebesgue measure  $m$ , that is,  $m(X) = 0$  if and only if  $m(H(X)) = 0$ . For example, if  $H$  and  $H^{-1}$  are Lipschitz continuous, then  $H$  is an absolutely continuous conjugacy.

LEMMA 1.14. *Suppose  $H$  is an absolutely continuous conjugacy between  $F$  and  $G$ . If  $H$  is differentiable at an interior point of  $M$  which is not singular point of  $F$  and at which the derivative of  $F$  is nonzero, then the restriction of  $H$  to each interval of  $\eta_F$  is a  $C^{1+\alpha}$  embedding.*

*Proof.* Suppose  $\bar{x}$  is the interior point of  $M$  at which  $F$  is differentiable and the derivative of  $F$  is nonzero. Because  $\bar{x}$  is not a singular point

of  $F$ ,  $H(\bar{x})$  is not a singular point of  $G$  too.

Using the equation,  $H \circ F = G \circ H$ , we can show that  $H$  is differentiable at any point in the grand preimage of  $\bar{x}$  under  $F$  which is the set  $\cup_{n=0}^{\infty} F^{-n}(\bar{x})$ . For every  $z \neq \bar{x}$  in the grand preimage of  $\bar{x}$  under  $F$ , by the chain rule, we have that

$$H'(F(z))F'(z) = G'(H(z))H'(z),$$

and furthermore, the derivative of  $H$  at  $z$  is nonzero.

Consider an interval in  $\eta_F$ . Let  $x$  and  $y$  be any two points in the intersection of this interval and the grand preimage of  $F$  at  $\bar{x}$  and  $J_0$  be the interval with boundary points  $x$  and  $y$ . Suppose  $\{J_0, J_1, \dots\}$  is a sequence of backward images of  $J_0$  under  $F$ , that is, the restriction of  $F$  to  $J_n$  embeds  $J_n$  onto  $J_{n-1}$  for all positive integers  $n$ . We use  $x_n$  and  $y_n$  to denote the boundary points of  $J_n$ . Then  $H$  is differentiable at  $x_n$  and  $y_n$  because they are in the grand preimage of  $\bar{x}$  under  $F$ .

For any integer  $n > 0$  and any boundary point  $z$  of  $J_n$ , applying the function  $\log x$  to  $H'(F(z))/H'(z) = G'(H(z))/F'(z)$ , we get

$$\log |H'(F(z))| - \log |H'(z)| = \log |G'(H(z))| - \log |F'(z)|.$$

We add the last equation over all integers  $n$  and all boundary points  $z$  of  $J_n$  to get that

$$|\log |H'(x_0)| - \log |H'(y_0)|| =$$

$$\left| \sum_{n=0}^{\infty} (\log |G'(H(x_n))| - \log |G'(H(y_n))|) + (\log |F'(x_n)| - \log |F'(y_n)|) \right|.$$

Because the derivatives of  $F$  and  $G$  are  $\alpha$ -Hölder continuous and the minimum value of  $|F'|$  and  $|G'|$  on  $M$  is nonzero, we can find a positive constant  $C_1$  such that

$$|\log |H'(x_0)| - \log |H'(y_0)|| \leq C_1 \sum_{n=0}^{\infty} (|H(x_n) - H(y_n)|^\alpha + |x_n - y_n|^\alpha).$$

The sequences  $\{J_0, J_1, \dots\}$  and  $\{H(J_0), H(J_1), \dots\}$  are sequences of backward images of  $J_0$  and  $H(J_0)$  under  $F$  and  $G$ , respectively. By the expanding property of  $F$  and  $G$ , there are two positive constants  $C_2$  and  $\lambda < 1$  such that  $|H(J_i)| \leq C_2 \lambda^i |H(J_0)|$  and  $|J_i| \leq C_2 \lambda^i |J_0|$ . We can find a positive constant  $C_3$  such that

$$|\log |H'(x)| - \log |H'(y)|| \leq C_3 (|H(J_0)|^\alpha + |J_0|^\alpha).$$

Because  $H$  is uniformly continuous on  $M$ , from last inequality,  $\log |H'|$  is uniformly continuous on the intersection of this interval and the grand preimage of  $\bar{x}$  under  $F$ . It can be extended to a continuous

function on this interval. Moreover, the derivatives of  $H$  on the intersection of this interval and the grand preimage of  $\bar{x}$  under  $F$  can be extended to a continuous function on this interval. This continuous extension is the derivative of the restriction of  $H$  to this interval because  $H$  is an absolutely continuous conjugacy. Hence the restriction of  $H$  to this interval is  $C^1$ .

Now the inequality,  $|\log |H'(x)| - \log |H'(y)|| \leq C_3(|H(J_0)|^\alpha + |J_0|^\alpha)$ , says the restriction of  $H$  to this interval is a  $C^{1+\alpha}$  embedding. *QED*.

*Proof of Theorem 1.5.* The “only if” part is obvious. We give a proof for the “if” part.

For any positive integer  $n$  and any interval  $I$  in  $\eta_{n,F}$ , the restriction of the  $(n+1)^{th}$  iterate of  $F$  to  $I$  embeds  $I$  onto  $M$ . There is a periodic point  $p$  of  $F$  in  $I$ . The image of  $p$  under  $H$  is also a periodic point of  $G$  in the image of  $I$  under  $H$ .

Consider the ratio  $|H(I)|/|I|$ . It equals  $|(G^{\circ(n+1)})'(\zeta)|/|(F^{\circ(n+1)})'(\xi)|$  for some  $\xi$  in  $I$  and some  $\zeta$  in  $H(I)$  because both images of  $I$  and  $H(I)$  under the  $(n+1)^{th}$  iterate of  $F$  and  $G$  are  $M$ . The naive distortion lemma says there is a positive constant  $C$  such that both

$|((F^{\circ(n+1)})'(p))|/|((F^{\circ(n+1)})'(\xi))|$  and  $|((G^{\circ(n+1)})'(\zeta))|/|((G^{\circ(n+1)})'(H(p)))|$  are bounded above and below by  $C$  and  $C^{-1}$ , respectively. Because  $e_F(p) = e_G(H(p))$ ,  $|H(I)|/|I|$  is bounded above and below by  $C^2$  and  $C^{-2}$ .

For any  $x$  and  $y$  in  $M$ , the interval bounded by  $x$  and  $y$  can be written in a union of intervals  $\{I_i\}$  where each  $I_i$  is an interval in  $\eta_{n_i, F}$  for some positive integer  $n_i$ . The set  $\{I_i\}$  may consist of either finite intervals or infinite intervals. The ratio  $|H(x) - H(y)|/|x - y|$  is bounded above and below by  $C^2$  and  $C^{-2}$  because each ratio  $|H(I_i)|/|I_i|$  is bounded above and below by  $C^2$  and  $C^{-2}$ . Hence  $H$  and  $H^{-1}$  are Lipschitz continuous. Moreover, there is an interior point of  $M$  which is not a singular point of  $F$  and at which  $F$  is differentiable with nonzero derivative. From the previous lemma, the restriction of  $H$  to each interval in  $\eta_F$  is a  $C^{1+\alpha}$  embedding.

Suppose  $I$  is an interval in  $\eta_F$  and  $a$  is a boundary point of  $I$ . Assume  $a$  is not a boundary point of  $M$ . Let  $a_0$  be one of the preimage of  $a$  under  $F$ . Then  $a_0$  is an interior point of an interval in  $\eta_F$ . By the chain rule,  $H'(a+) = H'(a_0)G'(H(a+))/F'(a+)$  and  $H'(a-) = H'(a_0)G'(H(a-))/F'(a-)$ . If  $a$  is not a singular point of  $F$ , then it is

easy to see that  $H'(a+) = H'(a-)$ . Suppose  $a$  is a singular point of  $F$ . The condition  $sv_F(a) = sv_G(H(a))$  implies that  $G(H(a+)/F'(a+) = G(H(a-)/F'(a-)$ . Moreover,  $H'(a+) = H'(a-)$  and  $H$  is a  $C^{1+\alpha}$  diffeomorphism from  $M$  to itself because  $F$  has only finite singular points. *QED.*

**COROLLARY 1.6.** *Suppose  $F$  and  $G$  are  $C^{1,1}$  expanding Markov mappings and the restriction of  $F$  to every interval of  $\eta_F$  embeds this interval onto  $M$ . The conjugating mapping  $H$  is a  $C^{1,1}$  diffeomorphism if and only if  $e_F(p) = e_G(H(p))$  for all periodic points of  $F$  and  $sv_F(a) = sv_G(H(a))$  for all singular points of  $F$  regardless of  $F$  and  $G$  are finite or infinite Markov mappings.*

**REMARK 1.6.** The condition, the restriction of  $F$  to every interval in  $\eta_F$  embeds this interval onto  $M$ , guarantee that the periodic points of  $F$  are dense on  $M$ . For a finite,  $C^{1+\alpha}$  expanding Markov mapping whose periodic points are not dense on  $M$ , we still can use a new geometric invariant, the scaling function for this Markov mapping (see [J2] or the second part of this thesis), to character the smooth conjugacy class of this Markov mapping. Actually, by using similar arguments to the

proof of theorem 1.5, we can prove that for any finite,  $C^{1+\alpha}$  expanding Markov mappings  $F$  and  $G$ , the conjugating mapping  $H$  is a  $C^{1+\alpha}$  diffeomorphism if and only if they have the same scaling function and the same asymmetry at any singular point.

*Proof of Theorem 1.4.* The proof of the “only if” part is easy. We give the proof of the “if” part.

Suppose  $F = \tilde{f}$ ,  $G = \tilde{g}$  and  $H = h_\gamma \circ h \circ h_\gamma^{-1}$  is the topological conjugacy between  $F$  and  $G$ . Using similar arguments to the proof of Theorem 1.5, we can prove that  $H$  is a  $C^{1+\alpha}$  diffeomorphism from  $[-1, 1]$  to itself.

For any  $x$  in  $[-1, 1]$ , let  $y$  be the image of  $x$  under  $h_\gamma$ . Applying the chain rule to  $H = h_\gamma \circ h \circ h_\gamma^{-1}$ , we get that

$$h'(x) = H'(y)(1 - (h_\gamma^{-1}(H(y)))^2)^{\frac{1}{\gamma^*}} / (1 - (h_\gamma^{-1}(y))^2)^{\frac{1}{\gamma^*}}.$$

From this equation, we can get that that  $h'$  is  $(\alpha/\gamma)$ -Hölder continuous on  $[-1, 1]$ . In the other words,  $h$  is a  $C^{1+\epsilon}$  diffeomorphism from  $[-1, 1]$  to itself where  $\epsilon \geq \alpha/\gamma$ . *QED.*

**REMARK.1.7.** By some discussions on the relation between the smoothness of  $f$  and  $g$  and the smoothness of  $\tilde{f}$  and  $\tilde{g}$  (see Lemma 1.1, Lemma

1.2, Remark 1.1, Remark 1.2, (EQ 1.1) and (EQ 1.2)), we may show that if  $f$  and  $g$  on  $\mathcal{BH}$  satisfy that (1)  $f$  and  $g$  have the same power law  $|x|^\gamma$  with  $\gamma > 1$  and are  $C^{1+\alpha'}$  for some  $0 < \alpha' \leq 1$  and (2) there is a neighborhood  $U$  of 0 such that the restrictions of  $r_f = f'(x)/|x|^{\gamma-1}$  and  $r_g = g'(x)/|x|^{\gamma-1}$  to  $U \cap [-1, 0)$  and to  $U \cap (0, 1]$  are  $\alpha'$ -Hölder continuous for same  $\alpha'$ , then  $\epsilon = \alpha'$  in Theorem 1.4.

### §1.5. Invariant measures for mappings on $\mathcal{BH}$ .

Suppose  $\mu$  is a measure on  $[-1, 1]$  and  $f$  is a mapping defined on  $[-1, 1]$ . We say  $\mu$  is an  $f$ -invariant if  $\mu(f^{-1}(A)) = \mu(A)$  for all the Lebesgue measurable subsets in  $[-1, 1]$ . We say  $\mu$  is absolutely continuous with respect to the Lebesgue measure if  $\mu = \int \rho(x)dx$  for a nonnegative Lebesgue measurable function  $\rho$ . We call  $\rho$  the density function of  $\mu$ . We say  $\mu$  is a probability measure if  $\mu([-1, 1]) = 1$ . If a probability measure  $\mu$  is absolutely continuous with respect to the Lebesgue measure and  $f$ -invariant, then we call it an absolutely continuous  $f$ -invariant measure.

Suppose  $f$  is on  $\mathcal{BH}$ , the derivative of  $\tilde{f}$  is piecewise  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$  and  $f$  has the power law  $|x|^\gamma$  with  $\gamma > 1$

at the critical point. Remember that  $\gamma^*$  is the dual number of  $\gamma$ .

**THEOREM 1.6.** *The mapping  $f$  has a unique absolutely continuous  $f$ -invariant measure  $\mu_f = \int \rho_f(x)dx$ . Moreover,  $\rho_f(x)(|1 - x^2|)^{\frac{1}{\gamma^*}}$  is a positive  $\alpha$ -Hölder continuous function on  $[-1, 1]$ . In particular,  $\rho_f$  grows like  $|x - p|^{-\frac{1}{\gamma^*}}$  at each boundary point  $p$  of  $[-1, 1]$ .*

**LEMMA 1.15.** *The mapping  $\tilde{f}$  has a unique absolutely continuous  $\tilde{f}$ -invariant measure  $\mu_{\tilde{f}} = \int \rho_{\tilde{f}}dx$ . Moreover,  $\rho_{\tilde{f}}$  is a positive,  $\alpha$ -Hölder continuous function on  $[-1, 1]$ .*

This lemma follows from a more general theorem proved by Sacksteder (see [Sa]). We write down a proof here for the sake of completeness.

*Proof of Lemma 1.15.* Let  $g_0$  and  $g_1$  be the inverses of the restrictions of  $\tilde{f}$  to  $[-1, 0]$  and to  $[0, 1]$ . For a continuous function  $\rho$  on  $[-1, 1]$ , the measure  $\mu = \int \rho dx$  is  $\tilde{f}$ -invariant if and only if  $\rho$  satisfies that

$$(\rho \circ g_0)(g'_0 \circ g_0) + (\rho \circ g_1)(g'_1 \circ g_1) = \rho$$

on  $[-1, 1]$ .

Let  $C_+^\alpha$  be the space of nonnegative,  $\alpha$ -Hölder continuous functions on  $[-1, 1]$ . For any  $\rho$  in  $C_+^\alpha$ , we use  $\|\rho\|_0$  to denote the maximum value

of  $\rho$  on  $[-1, 1]$ . For any  $x$  in  $[-1, 1]$ , let  $s_\alpha(\rho)(x) = \sup_{x \neq y \in [-1, 1]} (|\rho(x) - \rho(y)|/|x - y|^\alpha)$  and  $\|\rho\|_\alpha$  be the maximum value of  $s_\alpha(\rho)$  on  $[-1, 1]$ . The space  $C_+^\alpha$  equipped with the norm  $\|\rho\| = \|\rho\|_0 + \|\rho\|_\alpha$  is a Banach space. We define an operator  $\mathcal{L}$  from  $C_+^\alpha$  into itself by

$$\mathcal{L}(\rho) = (\rho \circ g_0)(g'_0 \circ g_0) + (\rho \circ g_1)(g'_1 \circ g_1).$$

It is a bounded linear operator.

Let  $\rho_0$  be the constant function 1 in  $C_+^\alpha$ . Suppose  $\rho_n$  is the image of  $\rho_0$  under the  $n^{\text{th}}$  iterate of  $\mathcal{L}$ , that is,  $\rho_n = \mathcal{L}^{\circ n}(\rho_0)$ . The function  $\rho_n$  has the form  $\sum_w g'_w$ , where the sum is over all finite strings  $w$  of zeroes and ones of length  $n + 1$ .

Using the naive distortion lemma, we can show that there is a positive constant  $C_1$  such that  $s_\alpha(g'_w) \leq C_1|g'_w|$  on  $[-1, 1]$ , and furthermore,  $s_\alpha(\rho_n) \leq C_1\rho_n$  on  $[-1, 1]$ . Because  $\rho_n$  is a positive function on  $[-1, 1]$  and  $\int_{-1}^1 \rho_n(x)dx = 1$ , there is  $x_n$  in  $[-1, 1]$  such that  $\rho_n(x_n) = 1$ . By  $s_\alpha(\log \rho_n) \leq s_\alpha(\rho_n)/\rho_n$ , we can get that  $\|\log \rho_n\|_\alpha \leq C_1$ . This implies that

$$|\log \rho_n(x) - \log \rho_n(y)| \leq C_1|x - y|^\alpha$$

for all  $x$  and  $y$  in  $[-1, 1]$ . For  $y = x_n$ , last inequality says  $|\log \rho_n(x)| \leq$

$C_1|x - x_n|^\alpha \leq 2^\alpha C_1$  for all  $x$  in  $[-1, 1]$  and all positive integers  $n$ . Moreover,  $\exp(-2^\alpha C_1) \leq \rho_n \leq \exp(2^\alpha C_1)$  on  $[-1, 1]$  for all positive integers  $n$ . Using the naive distortion lemma again, we can show that there is a positive constant  $C_2$  such that  $1 - C_2|x - y|^\alpha \leq |g_w(x)'|/|g_w'(y)| \leq 1 + C_2|x - y|^\alpha$ , and furthermore,

$$1 - C_2|x - y|^\alpha \leq \rho_n(x)/\rho_n(y) \leq 1 + C_2|x - y|^\alpha.$$

Combining the fact,  $\rho_n \geq \exp(-2^\alpha C_1)$ , we have that  $|\rho_n(x) - \rho_n(y)| \leq C_3|x - y|^\alpha$  for all  $x$  and  $y$  in  $[-1, 1]$  and all positive integers  $n$ . This implies that  $\{\rho_n\}_{n=0}^{+\infty}$  is a uniformly bounded and equicontinuous family in  $C_+^\alpha$ .

Let  $X$  be the closure of convex hull of  $\{\rho_n\}_{n=0}^{+\infty}$  in  $C_+^\alpha$ . Then  $X$  is a compact subspace of  $C_+^\alpha$ . The operator  $\mathcal{L}$  maps  $X$  into itself. The Schauder theorem says  $\mathcal{L}$  has a fixed point  $\rho_{\tilde{f}}$  in  $X$ . Because  $\tilde{f}$  is ergodic on  $[-1, 1]$ ,  $\rho_{\tilde{f}}$  is a unique fixed point of  $\mathcal{L}$  in  $C_+^\alpha$ . The measure  $\mu_{\tilde{f}} = \int \rho_{\tilde{f}}(x)dx$  is a unique absolutely continuous  $\tilde{f}$ -invariant measure. (see [Sa] for the proof of the uniqueness).

*Proof of Theorem 1.6.* Let  $\rho_f(x) = \rho_{\tilde{f}}(1-x^2)^{-\frac{1}{\gamma^*}}$ . Then  $\mu_f = \int \rho_f(x)dx$  is a unique absolutely continuous  $f$ -invariant measure. *QED.*

## Part two. The asymptotic geometry on the space of hyperbolic mappings

A type of perturbation of  $q : x \mapsto 1 - 2x^2$  is a mapping  $q_\varepsilon : x \mapsto 1 + \varepsilon - (2 + \varepsilon)x^2$  for a positive number  $\varepsilon$ . The mapping  $q_\varepsilon$  maps the critical point out of  $[-1, 1]$  and it does not keep  $[-1, 1]$  invariant but invariant a Cantor set which has bounded geometry. In the second part of this thesis, we will study the asymptotic behavior of certain mappings like  $q_\varepsilon$  as they approach the boundary of hyperbolicity (see Figure 6).

### §2.1. The definition of the scaling function.

Suppose  $\varepsilon$  is a positive number and  $f_\varepsilon$  is a  $C^1$  mapping from  $[-1, 1]$  to the real line with a unique critical point  $c$ . We always make the following assumptions: (1)  $f_\varepsilon$  is increasing on  $[-1, c]$  and decreasing on  $[c, 1]$ , (2)  $f_\varepsilon(c) = 1 + \varepsilon$  and (3)  $f_\varepsilon$  maps 1 and  $-1$  to  $-1$  (see Figure 2). Without loss of generality, we always assume  $c$  equals 0.

Let  $f_{\varepsilon,0}$  and  $f_{\varepsilon,1}$  be the restrictions of  $f_\varepsilon$  to  $[-1, 0]$  and to  $[0, 1]$ . They are two embeddings. Let  $g_{\varepsilon,0}$  and  $g_{\varepsilon,1}$  be the inverses of  $f_{\varepsilon,0}$  and  $f_{\varepsilon,1}$ . For a finite string  $w = i_0 \cdots i_n$  of zeroes and ones, let  $g_{\varepsilon,w}$  be the

composition  $g_{\varepsilon,w} = g_{\varepsilon,i_0} \circ \cdots \circ g_{\varepsilon,i_n}$  and  $I_{\varepsilon,w}$  be the image of  $[-1, 1]$  under  $g_{\varepsilon,w}$ . Suppose  $\eta_{n,\varepsilon}$  is the collection of  $I_{\varepsilon,w}$  for all finite strings  $w$  of zeroes and ones of length  $n + 1$  and  $\lambda_{n,\varepsilon}$  is the maximum length of the intervals in  $\eta_{n,\varepsilon}$ . The union of the intervals in  $\eta_{n,\varepsilon}$  covers the maximal invariant set of  $f_\varepsilon$ . We always assume that  $\lambda_{n,\varepsilon}$  goes to zero as  $n$  increases to infinity.

For every interval in  $\eta_{n,\varepsilon}$ , there is the labelling  $w$  where  $w$  is the finite string of zeroes and ones such that this interval is the image  $I_w$  of  $[-1, 1]$  under  $g_{\varepsilon,w}$ . There are two topologies on the set of all the labellings  $w$ . One topology is induced by reading the labellings  $w$  from left to right; the other topology is induced by reading the labellings  $w$  from right to left.

Suppose we read all the labellings  $w$  from left to right and  $\mathcal{C}_n = \{w_n \mid w_n = (.i_0i_1 \cdots i_n), \text{ where } i_k \text{ is either } 0 \text{ or } 1 \text{ for } k \geq 0 \text{ and } n \geq 0\}$ . Let  $\mathcal{C}_n$  have the product topology. The continuous mapping  $\sigma_n : \mathcal{C}_{n+1} \mapsto \mathcal{C}_n$  is defined by  $\sigma_n((.i_0i_1 \cdots i_n)) = (.i_1 \cdots i_n)$  for  $n \geq 0$ . The pairs  $\{(\mathcal{C}_n, \sigma_n)\}_{n=0}^\infty$  form an inverse limit set. Let  $\mathcal{C}$  be the inverse limit of this inverse limit set and  $\sigma$  be the induced mapping on  $\mathcal{C}$ . We call  $\mathcal{C}$

the topological Cantor set. For any  $a$  in  $\mathcal{C}$ , it is an infinite string of zeroes and ones extending to the right, that is,  $a = (.i_0i_1\cdots)$  where  $i_k$  is either zero or one for  $k \geq 0$ . The mapping  $\sigma$  is the shift mapping on  $\mathcal{C}$ , that is,  $\sigma$  maps  $(.i_0i_1\cdots)$  to  $(.i_1\cdots)$ . We call  $(\mathcal{C}, \sigma)$  the symbolic dynamical system of  $f_\varepsilon$  because of the following lemma.

LEMMA 2.1. *Suppose  $\Lambda_\varepsilon$  is the maximum invariant set of  $f_\varepsilon$ . There is a homeomorphism  $h_\varepsilon$  from  $\mathcal{C}$  to  $\Lambda_\varepsilon$  such that  $h_\varepsilon \circ \sigma = f_\varepsilon \circ h_\varepsilon$ . In the other words,  $(\Lambda_\varepsilon, f_\varepsilon)$  and  $(\mathcal{C}, \sigma)$  are conjugate.*

*Proof.* Suppose  $a = (.i_0i_1\cdots)$  is any point in  $\mathcal{C}$ . Let  $w_n = (.i_0\cdots i_n)$  be the first  $n + 1$  coordinates of  $a$ . The intersection of nested intervals  $\{I_{w_n}\}_{n=0}^\infty$  is nonempty and contains only one point  $x(a)$  because the length of  $I_{w_n}$  goes to zero as  $n$  increases to infinity. Define  $h_\varepsilon(a) = x(a)$ . Then  $h_\varepsilon$  is a homeomorphism from  $\mathcal{C}$  to  $\Lambda_\varepsilon$  and  $h_\varepsilon \circ \sigma = f_\varepsilon \circ h_\varepsilon$ . *QED.*

Suppose we read all the labellings  $w$  from right to left and  $\mathcal{C}_n^* = \{w_n^* | w_n^* = (i_n \cdots i_1 i_0.)\}$ , where  $i_k$  is either zero or one and  $n \geq 0\}$ . Let  $\mathcal{C}_n^*$  have the product topology. The continuous mapping  $\sigma_n^* : \mathcal{C}_{n+1}^* \mapsto \mathcal{C}_n^*$  is defined by  $\sigma_n^*((i_n \cdots i_1 i_0.)) = (i_n \cdots i_1.)$  for  $n \geq 0$ . The pairs  $\{(\mathcal{C}_n^*, \sigma_n^*)\}_{n=0}^\infty$  also form an inverse limit set. Let  $\mathcal{C}^*$  be the inverse limit

of this inverse limit set and  $\sigma^*$  be the induced mapping on  $\mathcal{C}^*$ . We call  $\mathcal{C}^*$  the dual Cantor set. Any  $a^*$  in  $\mathcal{C}^*$  is an infinite string of zeroes and ones extending to the left, that is,  $a^* = (\cdots i_1 i_0 \cdot)$  where  $i_k$  is either zero or one for  $k \geq 0$ . We call  $a^*$  a “dual point” of  $f_\varepsilon$ . The mapping  $\sigma^*$  is the shift mapping on  $\mathcal{C}^*$ , that is,  $\sigma^*$  maps  $(\cdots i_1 i_0 \cdot)$  to  $(\cdots i_1 \cdot)$ . We call  $(\mathcal{C}^*, \sigma^*)$  the dual symbolic dynamical system of  $f_\varepsilon$ .

A sequence  $\{x_n\}_{n=0}^\infty$  in the maximum invariant set of  $f_\varepsilon$  is a sequence of backward images of  $x_0$  under  $f_\varepsilon$  if  $f_\varepsilon(x_n) = x_{n-1}$  for all positive integers  $n$ . The dual Cantor set will not represent the maximal invariant set of  $f_\varepsilon$ , but there is a one-to-one corresponding from the dual Cantor set to the set of sequences of backward images of  $x_0$  under  $f_\varepsilon$  for all points  $x_0$  in the maximal invariant set of  $f_\varepsilon$ . The scaling function of  $f_\varepsilon$  is defined on the dual Cantor set  $\mathcal{C}^*$  if it exists.

Suppose  $a^*$  is in  $\mathcal{C}^*$ , so that  $a^*$  is an infinite string of zeroes and ones extending to the left. Assume  $a^* = (\cdots wi \cdot)$  where  $w$  is a finite string of zeroes and ones and  $i$  is either zero or one. Note that  $I_{wi}$  is a subinterval of  $I_w$ . Let  $s(wi)$  equal the ratio of the lengths,  $|I_{wi}|/|I_w|$ . We let  $s(a^*)$  be the limit set of  $s(wi)$  as the length of  $w$  tends to infinity.

The following definition is due to Sullivan [S2].

**DEFINITION 2.1.** *Suppose  $f_\varepsilon$  is a mappings in  $\mathcal{H}$ . If the limit set  $s(a^*)$  consists of only one number for every  $a^*$  in  $\mathcal{C}^*$ , then we say  $s(a^*)$  is the scale of  $f_\varepsilon$  at  $a^*$  and  $s$  is the scaling function of  $f_\varepsilon$  defined on  $\mathcal{C}^*$ . Note that the scaling function  $s(a^*)$  of  $f_\varepsilon$  depending on  $\varepsilon$ . Sometimes we denote it by  $s_\varepsilon(a^*)$ .*

**REMARK 2.1.** For a finite Markov mapping  $F$  (see §1.2). Using similar arguments, we can also define the scaling function of  $F$  on the dual symbolic representation of  $F$  by using the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $F$  (see [J2]). For example, let  $f$  be on  $\mathcal{BH}$ , we can use the the same arguments as Definition 2.1 to define the scaling function  $s_f$  of  $f$  on  $\mathcal{C}^*$  by the sequence of nested partitions  $\{\eta_n\}_{n=0}^\infty$  determined by  $f$ .

## **§2.2. The space of hyperbolic mappings, $\mathcal{H}$ .**

Suppose  $\{\eta_{\varepsilon,n}\}_{n=0}^\infty$  is the sequence determined by  $f_\varepsilon$ . Just as in definition 1.3, we say the sequence  $\{\eta_{n,\varepsilon}\}_{n=0}^\infty$  determined by  $f_\varepsilon$  decreases exponentially if  $\lambda_{n,\varepsilon}$  decreases exponentially. Using the same statements as Definition 1.2, one can define the concept of bounded geome-

try of  $\{\eta_{\varepsilon,n}\}_{n=0}^{\infty}$  (see [S2]). If  $\{\eta_{\varepsilon,n}\}_{n=0}^{\infty}$  has bounded geometry, then it decreases exponentially.

DEFINITION 2.2. The nonlinear mapping  $f_{\varepsilon}$  is in the space of hyperbolic mappings,  $\mathcal{H}$ , if

- (a)  $f_{\varepsilon}$  is  $C^{1+\alpha}$  for some  $0 < \alpha \leq 1$  and
- (b) the sequence  $\{\eta_{n,\varepsilon}\}_{n=0}^{\infty}$  determined by  $f_{\varepsilon}$  decreases exponentially.

We give two examples of mappings in  $\mathcal{H}$ . They are similar to Example 1.1 and Example 1.2 in §1.2.

EXAMPLE 2.1. *Mapping  $f_{\varepsilon}$  such that (1)  $f_{\varepsilon}$  is a  $C^3$  mapping on  $[-1, 1]$  with nonpositive Schwarzian derivative and (2)  $f$  is expanding at both boundary points of  $[-1, 1]$ , that is,  $f'_{\varepsilon}(-1)$  and  $|f'_{\varepsilon}(1)|$  are greater than one.*

EXAMPLE 2.2. *Mapping  $f_{\varepsilon}$  such that (1)  $f_{\varepsilon}$  is  $C^{1,1}$  and (2) all the periodic points of  $f_{\varepsilon}$  are expanding, that is, the absolute values of all eigenvalues of  $f_{\varepsilon}$  at periodic points are greater than one.*

The proofs, that Example 2.1 and Example 2.2 are in  $\mathcal{H}$ , are similar to the proofs of Example 1.1 and 1,2. In fact, we can extend Theo-

rem 1.1 to the mappings satisfying the conditions in Example 2.1 and Theorem 1.2 to the mappings satisfying the conditions in Example 2.2.

**DEFINITION 2.3.** *A function  $s$  defined on dual Cantor set  $C^*$  is Hölder continuous if there are two positive constants  $C$  and  $\lambda < 1$  such that  $|s(a^*) - s(b^*)| \leq C\lambda^n$  for any  $a^*$  and  $b^*$  in  $C^*$  with the same first  $n$  coordinates. We call  $C$  a Hölder constant of  $s$ .*

**LEMMA 2.2.** *Suppose  $f_\varepsilon$  is in  $\mathcal{H}$ . There exists a Hölder continuous scaling function  $s_\varepsilon$  of  $f_\varepsilon$ .*

*Proof.* Let  $d_\varepsilon$  be the minimum value of the restriction of  $f_\varepsilon$  to the union of  $I_{\varepsilon,0}$  and  $I_{\varepsilon,1}$ . We suppress  $\varepsilon$  if there can be no confusion. Note that  $d_\varepsilon$  goes to zero as  $\varepsilon$  decreases to zero. For any  $a^*$  in  $C^*$ , we use  $w_n i$  to denote the first  $(n+1)$  coordinates of  $a^*$  and  $s(w_n i)$  to denote the ratio,  $|I_{w_n i}|/|I_{w_n}|$ . By (b) of Definition 2.1, we have two positive constants  $C_0$  and  $\lambda < 1$  such that  $\lambda_{n,\varepsilon} \leq C_0\lambda^n$ . Let  $K$  be the Hölder constant of  $f'_\varepsilon$  on  $[-1, 1]$ . Because  $s(w_m i) = (|(f^{\circ(m-n)})'(x)|/|(f^{\circ(m-n)})'(y)|)s(w_n i)$  for some  $x$  and  $y$  in  $I_{w_m}$ , by the naive distortion lemma, there is a constant  $C_\varepsilon$  which equals  $C_0K/(d_\varepsilon(1 - \lambda^\alpha))$  such that for any  $m > n > 0$ ,

$$|s(w_m i) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha.$$

The last inequality implies that the limit of sequence  $\{s(w_n i)\}_{n=0}^{+\infty}$  exists as the length of  $w_n i$  increases to infinity. We denote this limit by  $s(a^*)$  or  $s_\varepsilon(a^*)$  if we need to indicate dependence on  $\varepsilon$ . Let  $m$  tend to infinity, then  $|s(a^*) - s(w_n i)| \leq C_\varepsilon |I_{\varepsilon, w_n}|^\alpha$  for all positive integers  $n$ .

Suppose  $a^*$  and  $b^*$  are in  $\mathcal{C}^*$  with the same first  $(n+1)$  coordinates, that is,  $a^* = (\cdots w_n i.)$  and  $b^* = (\cdots w_n i.)$ . Because  $|s(a^*) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha$  and  $|s(b^*) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha$ , we have that  $|s(a^*) - s(b^*)| \leq 2C_\varepsilon |I_{w_n}|^\alpha \leq 2C_0 C_\varepsilon \lambda^n$ . In other words,  $s_\varepsilon$  is Hölder continuous on  $\mathcal{C}^*$  with a Hölder constant  $2C_0 C_\varepsilon$ . *QED*.

### §2.3. Good families of mappings in $\mathcal{BH} \cup \mathcal{H}$ .

Suppose  $f_\varepsilon$  is in  $\mathcal{H}$ . Let  $r_\varepsilon(x) = f'_\varepsilon(x)/|x|^{\gamma-1}$  for nonzero  $x$  in  $[-1, 1]$ . We say that  $f_\varepsilon$  has power law at the critical point if there is some  $\gamma > 1$  such that the limits of  $r_\varepsilon(x)$  as  $x$  increases to zero and as  $x$  decreases to zero exist and equal nonzero numbers  $A_\varepsilon$  and  $-B_\varepsilon$ , respectively.

We define the smooth metric associated to  $f_\varepsilon$  to be

$$dy = \frac{dx}{((1+\varepsilon)^2 - x^2)^{\frac{1}{\gamma^*}}}$$

on  $[-1, 1]$ , where  $\gamma^* = \gamma/(\gamma-1)$  is the dual number of  $\gamma$  (see §1.1).

The corresponding change of coordinate is  $y = h_{\gamma,\varepsilon}$  where

$$h_{\gamma,\varepsilon}(x) = -1 + b_\varepsilon \int_{-1}^x \frac{dx}{((1+\varepsilon)^2 - x^2)^{\frac{1}{\gamma^*}}}$$

with  $b_\varepsilon = 2 / \int_{-1}^1 dx / ((1+\varepsilon)^2 - x^2)^{\frac{1}{\gamma^*}}$ . The representation of  $f_\varepsilon$  under the smooth metric associated to  $f_\varepsilon$  is

$$\tilde{f}_\varepsilon = h_{\gamma,\varepsilon} \circ f_\varepsilon \circ h_{\gamma,\varepsilon}^{-1}.$$

LEMMA 2.3. *If  $f_\varepsilon$  has power law  $|x|^\gamma$  with  $\gamma > 1$ , then the mapping  $\tilde{f}_\varepsilon$  is continuous on  $[-1, 1]$  and the restrictions of  $\tilde{f}_\varepsilon$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^1$  embeddings.*

REMARK 2.2. The mapping  $h_{\gamma,\varepsilon}$  is a  $C^\infty$  diffeomorphism from  $[-1, 1]$  to itself. If the derivative  $f'_\varepsilon$  and the restrictions of  $r_\varepsilon$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\alpha$ -Hölder continuous for some  $0 < \alpha \leq 1$ , then the derivative  $\tilde{f}'_\varepsilon$  is  $\alpha$ -Hölder continuous. The  $\alpha$ -Hölder constant of  $\tilde{f}_\varepsilon$  depends on  $\varepsilon$  and may go to infinity as  $\varepsilon$  goes to zero.

LEMMA 2.4. *Suppose  $\tilde{f}_\varepsilon$  is a continuous mapping from  $[-1, 1]$  to the real line with a unique turning point 0. Suppose  $\tilde{f}_\varepsilon$  maps 1 and  $-1$  to  $-1$  and  $\tilde{f}_\varepsilon(0) = 1 + \varepsilon$ . If the restrictions of  $\tilde{f}_\varepsilon$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^1$  embeddings, then  $f_\varepsilon = h_{\gamma,\varepsilon}^{-1} \circ \tilde{f}_\varepsilon \circ h_{\gamma,\varepsilon}$  for any  $\gamma > 1$  is a  $C^1$*

mapping from  $[-1, 1]$  to the real line and has the power law  $|x|^\gamma$  at the critical point.

The proofs of Lemma 2.3 and Lemma 2.4 are the same as those of Lemma 1.1 and 1.2.

LEMMA 2.5. *Suppose  $f_\varepsilon$  has the power law  $|x|^\gamma$  with  $\gamma > 1$ . The restrictions of  $\tilde{f}_\varepsilon$  to  $[-1, 0]$  and  $[0, 1]$  are  $C^{1+\alpha}$  embeddings for some  $0 < \alpha \leq 1$  if and only if the restrictions of  $r_\varepsilon$  to  $[-1, 0]$  and to  $(0, 1]$  are  $\alpha'$ -Hölder continuous for some  $0 < \alpha' \leq 1$ .*

The proof of Lemma 2.5 is similar to Remark 1.1 and Remark 1.2.

Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a family in  $\mathcal{H} \cup \mathcal{BH}$  where  $f_\varepsilon(0) = 1 + \varepsilon$ .

DEFINITION 2.4. *The family  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family if it satisfies the following conditions:*

1. *the mapping  $F(x, \varepsilon) = f_\varepsilon(x)$  is  $C^1$  in both variables  $x$  in  $[-1, 1]$  and  $\varepsilon$  in  $[0, \varepsilon_0]$ ,*
2. *each  $f_\varepsilon$  has the same power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical points and the functions  $R^-(x, \varepsilon) = r_\varepsilon(x)$  defined on  $[-1, 0] \times [0, \varepsilon_0]$  and  $R^+(x, \varepsilon) = r_\varepsilon(x)$  defined on  $[0, 1] \times [0, \varepsilon_0]$  are continu-*

ous,

3. there are positive constants  $K'$  and  $\alpha' \leq 1$  such that  $f_\varepsilon$  is  $C^{1+\alpha'}$  and the  $\alpha'$ -Hölder constant of  $f'_\varepsilon$  is less than  $K'$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ ,
4. there are positive constants  $K''$  and  $\alpha'' \leq 1$  such that the restrictions of  $r_\varepsilon$  to  $[-1, 0)$  and to  $(0, 1]$  are  $\alpha''$ -Hölder continuous and the  $\alpha''$ -Hölder constants of these restrictions are less than  $K''$  for any  $0 \leq \varepsilon \leq 1$ ,
5. there are two positive constants  $C_0$  and  $\lambda < 1$  such that  $\lambda_{n,\varepsilon} \leq C_0 \lambda^n$  for all positive integers  $n$  and  $0 \leq \varepsilon \leq \varepsilon_0$ .

Let  $\alpha$  be the minimum of  $\alpha'$  and  $\alpha''$ .

An example of a good family in  $\mathcal{BH} \cup \mathcal{H}$  follows the following proposition.

**PROPOSITION 2.1.** *Assume that (a)  $\{F_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a family of  $C^3$  embeddings on  $[-1, 0]$  with nonpositive Schwarzian derivatives, (b)  $F_\varepsilon$  fixes  $-1$  and maps  $0$  to  $1 + \varepsilon$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ , (c) the derivative  $F'_\varepsilon(-1)$  of  $F_\varepsilon$  at  $-1$  is greater than  $1/\gamma$  and (d)  $G(x, \varepsilon) = F_\varepsilon(x)$  is  $C^2$  on*

$[-1, 1] \times [0, \varepsilon_0]$ . If  $f_\varepsilon(x) = F_\varepsilon(-|x|^\gamma)$ , then the family  $\{F_\varepsilon(-|x|^\gamma)\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family.

To prove this proposition, we only need to check the condition (5) in Definition 2.4. The condition (5) is a direct consequence of the following lemma. Suppose  $\{\eta_{n,\varepsilon}\}_{n=0}^\infty$  is the sequence determined by  $f_\varepsilon$  for every  $0 \leq \varepsilon \leq \varepsilon_0$ .

**LEMMA 2.6.** *Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is the family in Proposition 2.1. There is a positive constant  $C$  which does not depend on parameter  $\varepsilon$  such that for any  $0 \leq \varepsilon \leq \varepsilon_0$  and any pair  $(J, I)$  with  $J \subset I$ ,  $J \in \eta_{m+1,\varepsilon}$  and  $I \in \eta_{n,\varepsilon}$ ,  $|J|/|I| \geq C$ .*

*Proof.* We suppress  $\varepsilon$  if there can be no confusion. For any  $0 \leq \varepsilon \leq \varepsilon_0$ , the first partition  $\eta_1$  contains four intervals  $I_{00}$ ,  $I_{01}$ ,  $I_{11}$  and  $I_{10}$ . There is a positive constant  $C_1$  which does not depend on  $\varepsilon$  such that the lengths of the left interval  $I_{00}$  and the right interval  $I_{10}$  are greater than  $C_1$ . The  $C^3$ -Koebe distortion lemma says  $n(g_w)(x) \leq 2/d_{[-1,1]}(x)$  for any finite string  $w$  of zeroes and ones. Moreover,  $n(g_w)(x) \leq 2/C_1$  if  $x$  is in the union of two middle intervals  $I_{01}$  and  $I_{11}$ . We also can find a constant  $\tau > 1$  which does not depend on  $\varepsilon$  such that  $|f'_\varepsilon(x)| \geq \tau$  for

all  $x$  in the union of the left interval  $I_{00}$  and the right interval  $I_{10}$ . Now the proof just follows the proof of Theorem 1.1. *QED*.

Let  $\mathcal{A}$  stand the countable set of points in  $\mathcal{C}^*$  whose coordinates are eventually all zeroes and let  $\mathcal{B}$  stand the complement of  $\mathcal{A}$  in  $\mathcal{C}^*$ .

**THEOREM 2.1.** *Suppose  $\{f_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  is a good family. There is a family of Hölder continuous functions  $\{s_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  on the dual Cantor set  $\mathcal{C}^*$  such that  $s_\varepsilon$  is the scaling function of  $f_\varepsilon$  for any  $0 < \varepsilon_0 \leq \varepsilon_0$ , and*

1. *for every  $0 < \varepsilon_1 \leq \varepsilon_0$ ,  $s_\varepsilon$  converges to  $s_{\varepsilon_1}$  uniformly on  $\mathcal{C}^*$  as  $\varepsilon$  tends to  $\varepsilon_1$ ,*
2. *for every  $a^* \in \mathcal{C}^*$ , the limit  $s_0(a^*)$  of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  as  $\varepsilon$  decreases to zero exists, the limiting function  $s_0(a^*)$  is the scaling function of  $f_0$  and satisfies:*
  - 2.1.  *$s_0$  has jump discontinuities at all points in  $\mathcal{A}$ ,*
  - 2.2.  *$s_0$  is continuous at all points in  $\mathcal{B}$  and the restriction of  $s_0$  to  $\mathcal{B}$  is a Hölder continuous function.*

We will prove this theorem through several lemmas. The first lemma is a generalization of Lemma 1.13. We call it the uniform  $(\alpha + \gamma)$ -Koebe

distortion lemma. It is one of the key lemmas in this thesis.

Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a family of mappings in  $\mathcal{H} \cup \mathcal{BH}$  and satisfies the conditions (1)–(4) in Definition 2.4. We suppress  $\varepsilon$  when there can be no confusion. For each  $0 \leq \varepsilon \leq \varepsilon_0$ ,  $\eta_1$  contains four intervals  $I_{00}$ ,  $I_{01}$ ,  $I_{11}$  and  $I_{10}$ . Suppose  $x$  and  $y$  are in one of these four intervals and  $J_0$  is the interval bounded by  $x$  and  $y$ . Let  $\theta(x, y) = \{J_0, J_1, \dots\}$  be a sequence of backward images of  $J_0$  under  $f_\varepsilon$ , that means, the restriction of  $f_\varepsilon$  to  $J_n$  embeds  $J_n$  onto  $J_{n-1}$  for any positive integer  $n$ . Let  $g_n$  be the inverse of the restriction of the  $n^{\text{th}}$  iterate of  $f_\varepsilon$  to  $J_n$ . Let  $d_{xy}$  be the distance from  $\{x, y\}$  to  $\{-1, 1\}$ . Define the distortion of the  $n^{\text{th}}$  iterate of  $f_\varepsilon$  at  $x$  and  $y$  along  $\theta(x, y)$  to be the ratio  $|g'_n(x)|/|g'_n(y)|$ .

LEMMA 2.7 (the uniform  $(\alpha + \gamma)$ -Koebe distortion lemma). *There are positive constants  $A$ ,  $B$  and  $C$  such that for any  $0 \leq \varepsilon \leq \varepsilon_0$ , any  $x$  and  $y$  in one of the intervals in  $\eta_{\varepsilon,1}$  and any sequence of backward images  $\theta_\varepsilon(x, y) = \{J_{\varepsilon,0}, J_{\varepsilon,1}, \dots\}$  of  $J_{\varepsilon,0}$  under  $f_\varepsilon$ , the distortion of the  $n^{\text{th}}$  iterate of  $f_\varepsilon$  at  $x$  and  $y$  along  $\theta_\varepsilon(x, y)$  satisfies*

$$\frac{|g'_{\varepsilon,n}(x)|}{|g'_{\varepsilon,n}(y)|} \leq \exp\left(\left(A + B \sum_{i=1}^n |J_{\varepsilon,i}| + \frac{C|J_{\varepsilon,0}|}{d_{xy}}\right) \sum_{i=1}^n |J_{\varepsilon,i}|^\alpha\right)$$

*for every positive integer  $n$ .*

*Proof.* For every  $0 \leq \varepsilon \leq \varepsilon_0$ ,  $\eta_2$  contains eight intervals. The two of them which close to 0 are  $I_{010}$  and  $I_{110}$ . Suppose  $I_{010} = [a, b]$  and  $I_{110} = [c, d]$ . We call  $[a, 0]$  and  $[0, d]$  the middle intervals,  $[-1, a]$  the left interval and  $[d, 1]$  the right interval (see Figure 11).

$$\begin{array}{ccccccc} \underline{I_{\varepsilon,000}} & \underline{I_{\varepsilon,001}} & \underline{I_{\varepsilon,011}} & \underline{I_{\varepsilon,010}} & \underline{I_{\varepsilon,110}} & \underline{I_{\varepsilon,111}} & \underline{I_{\varepsilon,101}} & \underline{I_{\varepsilon,100}} \\ & & & 0 & & & & \\ \text{left} & & \text{middle} & & \text{right} & & & \end{array}$$

Figure 11

By the condition (1) in Definition 2.4, there is a positive constant  $C_1$  which does not depend on  $\varepsilon$  such that the lengths of the left interval and the right interval are greater than  $C_1$ .

By the conditions (1) and (3) in Definition 2.4, there are positives constants  $c_1$  and  $K_1$  which do not depend on  $\varepsilon$  such that the minimum value of  $|f'_\varepsilon|$  on the union of the left and right intervals is greater than  $c_1$  and the  $\alpha'$ -Holder constants of the restrictions of  $f'_\varepsilon$  to the left interval and to the right interval are less than  $K_1$ .

Suppose  $y$  is not one of 0, 1 and  $-1$  and  $x$  is the preimage of  $y$  under  $h_{\gamma,\varepsilon}$ . By the chain rule,

$$\tilde{f}'_\varepsilon(y) = \frac{f'_\varepsilon(x)((1+\varepsilon)^2 - x^2)^{\frac{1}{\gamma^*}}}{((1+\varepsilon)^2 - (f_\varepsilon(x))^2)^{\frac{1}{\gamma^*}}}.$$

This equation and the conditions (1) and (4) in Definition 2.4 imply the restrictions of  $\tilde{f}_\varepsilon$  to  $[-1, 0]$  and to  $[0, 1]$  are  $C^{1+\alpha''}$  embeddings. Moreover, there are constants  $c_2$  and  $K_2$  which do not depend on  $\varepsilon$  such that the minimum value of  $|\tilde{f}'_\varepsilon|$  on the image of every one of the middle intervals under  $h_{\gamma,\varepsilon}$  is greater than  $c_2$  and the  $\alpha''$ -Hölder constant of the restriction of  $\tilde{f}'_\varepsilon$  to the image of every one of the middle intervals is less than  $K_2$ .

The restriction of  $h'_{\gamma,\varepsilon}$  to the union of the middle intervals is Lipschitz continuous. There are positive constants  $c_3$  and  $K_3$  which do not depend on  $\varepsilon$  such that the minimum value of restriction of  $h'_{\gamma,\varepsilon}$  to the union of middle intervals is greater than  $c_3$  and the Lipschitz constant of such restriction is less than  $K_3$ .

Let  $x_i$  and  $y_i$  be the images of  $x$  and  $y$  under  $g_{\varepsilon,i}$ . Notice that this implies that  $x_i$  and  $y_i$  lie in the same interval of  $\eta_{i+1}$  for  $i \geq 0$ . For every integer  $n > 0$ ,  $g'_{\varepsilon,n}(x)/g'_{\varepsilon,n}(y)$  equals  $(f_\varepsilon^{\circ n})'(y_n)/(f_\varepsilon^{\circ n})'(x_n)$ . By the chain rule, the ratio  $(f_\varepsilon^{\circ n})'(y_n)/(f_\varepsilon^{\circ n})'(x_n)$  equals the product of ratios

$f'_\varepsilon(y_{n-i})/f'_\varepsilon(x_{n-i})$  where  $i$  runs from 0 to  $n - 1$ . This product can be factored into two products,

$$\prod_{x_i, y_i \in LR} f'_\varepsilon(y_i)/f'_\varepsilon(x_i) \quad \text{and} \quad \prod_{x_i, y_i \in M} f'_\varepsilon(y_i)/f'_\varepsilon(x_i).$$

Here  $LR$  stands for the union of the left and right intervals and  $M$  stands for the union of the two middle intervals. We factor the product  $\prod_{x_i, y_i \in M} f'_\varepsilon(y_i)/f'_\varepsilon(x_i)$  into three factors,

$$\prod_{x_i, y_i \in M} \frac{f'_\varepsilon(y_i)}{f'_\varepsilon(x_i)} = \prod_{x_i, y_i \in M} \frac{h'_{\gamma, \varepsilon}(y_i)}{h'_{\gamma, \varepsilon}(x_i)} \cdot \prod_{x_i, y_i \in M} \frac{\tilde{f}'_\varepsilon(h_{\gamma, \varepsilon}(y_i))}{\tilde{f}'_\varepsilon(h_{\gamma, \varepsilon}(x_i))} \cdot \prod_{x_i, y_i \in M} \frac{h'_{\gamma_i, \varepsilon}(f(x_i))}{h'_{\gamma_i, \varepsilon}(f(y_i))}.$$

The third factor of them can be factored again into two products,

$$\prod_{x_i, y_i \in M} \frac{(1 + f_\varepsilon(y_i))^{\frac{1}{\gamma^*}}}{(1 + f_\varepsilon(x_i))^{\frac{1}{\gamma^*}}} \quad \text{and} \quad \prod_{x_i, y_i \in M} \frac{(1 - f_\varepsilon(y_i))^{\frac{1}{\gamma^*}}}{(1 - f_\varepsilon(x_i))^{\frac{1}{\gamma^*}}}.$$

Now just following the arguments in the proof of Lemma 1.13, we can estimate every factors. We then put all estimations together to get  $A = K_1/c_1 + (K_3^\alpha K_2)/c_2 + K_3/c_3 + 1/\gamma^*$ ,  $B = 1/(\gamma^* C_1)$  and  $C = 1/\gamma^*$ . The constants  $A$ ,  $B$  and  $C$  do not depend on the parameter  $\varepsilon$ . *QED*.

**COROLLARY 2.1.** *If the family  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family, then there are positive constants  $D$  and  $E$  such that for any  $0 \leq \varepsilon \leq \varepsilon_0$ , any  $x$  and  $y$  in one of the intervals in  $\eta_{1, \varepsilon}$  and any sequence  $\theta_\varepsilon(x, y) =$*

$\{J_{\varepsilon,0}, J_{\varepsilon,1}, \dots\}$  of backward images of  $J_{\varepsilon,0}$  under  $f_\varepsilon$ , the distortion of the  $n^{\text{th}}$  iterate of  $f_\varepsilon$  along  $\theta_\varepsilon(x, y)$  satisfies

$$\frac{|g'_{\varepsilon,n}(x)|}{|g'_{\varepsilon,n}(y)|} \leq \exp\left(D + \frac{E}{d_{xy}}\right) |J_{\varepsilon,0}|^\alpha$$

for every positive integer  $n$ .

*Proof.* The condition (5) in Definition 2.4 implies  $\sum_{i=0}^n |J_{\varepsilon,i}|$  is less than  $C_2 = 2C_0/(1-\lambda)$  and  $\sum_{i=0}^n |J_{\varepsilon,i}|^\alpha$  is less than  $P = 2^\alpha |J_{\varepsilon,0}|^\alpha C_0/(1-\lambda^\alpha) = |J_{\varepsilon,0}|^\alpha C_3$ . We use  $D$  to denote  $(A+BC_2)C_3$  and  $E$  to denote  $CC_3$ , where  $A$ ,  $B$  and  $C$  are the constants in the previous lemma. Now it is easy to show this corollary from the previous lemma. *QED.*

Before we prove more lemmas, we study asymptotic behavior of the maximal invariant set of  $f_\varepsilon$  in  $\mathcal{H}$  when  $f_\varepsilon$  approaches  $\mathcal{BH}$ .

#### **§2.4. Determination of the geometry of Cantor set by the leading gap.**

Suppose  $f_\varepsilon$  is a mapping in  $\mathcal{H}$ . We suppress  $\varepsilon$  when there can be no confusion. Let  $\Lambda$  be the maximal invariant set of  $f_\varepsilon$  and  $\{\eta_n\}_{n=0}^\infty$  be the sequence determined by  $f_\varepsilon$ . For any positive integer  $n$  and any  $I_w$  in  $\eta_n$ , let  $I_{w0}$  and  $I_{w1}$  be the two intervals in  $\eta_{n+1}$  which are contained

in  $I_w$ . We call the complement of  $I_{w_0}$  and  $I_{w_1}$  in  $I_w$  the gap on  $I_w$  and denote it by  $G_w$ . Let  $G$  be the complement of  $I_0$  and  $I_1$  in  $[-1, 1]$ . We call  $G$  the leading gap.

DEFINITION 2.5. *We call the set of ratios,  $\{|G_{\varepsilon,w}|/|I_{\varepsilon,w}|\}$ , for all finite strings  $w$  of zeroes and ones the gap geometry of  $\Lambda_\varepsilon$  or  $f_\varepsilon$ .*

Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a family in  $\mathcal{H} \cup \mathcal{BH}$  and  $\{\Lambda_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is the family of the corresponding maximum invariant sets.

DEFINITION 2.6. *Suppose  $\beta$  is a function defined on  $[0, \varepsilon_0]$ . We say  $\beta$  determines asymptotically the gap geometry of  $\{\Lambda_\varepsilon\}_{0 < \varepsilon \leq 1}$  if there is a positive constant  $C$  such that for all  $0 \leq \varepsilon \leq \varepsilon_0$ , all finite strings  $w$  of zeroes and ones and  $i = 0$  or  $1$ ,*

1.  $C^{-1}\beta(\varepsilon) \leq |G_{\varepsilon,w}|/|I_{\varepsilon,w}| \leq C\beta(\varepsilon)$ ,
2.  $|I_{\varepsilon,wi}|/|I_{\varepsilon,w}| \geq C^{-1}$ .

*The constant  $C$  is called a determining constant.*

THEOREM 2.2. *Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family. Then the family  $\{\Lambda_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$  is a family of Cantor sets. Moreover, the function  $\varepsilon^{\frac{1}{\gamma}}$  on  $[0, \varepsilon_0]$  determines asymptotically the gap geometry of  $\{\Lambda_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ .*

*Proof.* For each  $0 \leq \varepsilon \leq \varepsilon_0$ ,  $\eta_1$  contains four interval  $I_{00}$ ,  $I_{01}$ ,  $I_{11}$  and  $I_{10}$ . We call  $I_{00}$  the left interval,  $I_{10}$  the right interval and  $I_{01}$  and  $I_{11}$  the middle intervals. We also call  $\eta_1$  the first level and  $\eta_2$  the second level.

By the conditions (2) in Definition 2.4, there is a positive constant  $C_1$  which does not depend on  $\varepsilon$  such that  $C_1^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G_0| \leq C_1\varepsilon^{\frac{1}{\gamma}}$ .

By condition (1), there is a positive constant  $C_2$  which does not depend on  $\varepsilon$  such that for any triple  $(G, J, I)$  where  $I$  is an interval in the first level or  $\eta_0$ ,  $G$  is the gap on  $I$  and  $J \subset I$  is an interval in the second interval or the first level,  $|J|/|I| \geq C_2$  and  $C_2^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq C_2\varepsilon^{\frac{1}{\gamma}}$ .

For any integer  $n > 1$  and any triple  $(G, J, I)$  where  $I$  is in  $\eta_n$ ,  $G$  is the gap on  $I$  and  $J \subset I$  is in  $\eta_{n+1}$ , let  $G_i$ ,  $J_i$  and  $I_i$  be the images of  $G$ ,  $J$  and  $I$  under the  $i^{\text{th}}$  iterate of  $f_\varepsilon$  for  $0 \leq i \leq n-1$ . Then  $I_{n-1}$  is in the first level,  $G_{n-1}$  is the gap on  $I_{n-1}$  and  $J_{n-1} \subset I_{n-1}$  is in the second level.

We divide the possible itineraries of the sequence of triples  $\{(G_i, J_i, I_i)\}_{i=0}^{n-1}$  into two cases. The first case is that no one of  $I_i$  is in the union

of the middle intervals. The second case is that some of  $I_i$  is in one of the middle intervals.

In the first case,  $G_i$ ,  $J_i$  and  $I_i$  are in the union of the left interval and the right interval for all  $0 \leq i < n$ . By the conditions (1) and (3) in Definition 2.4, there is a positive constant  $c$  which does not depend on  $\varepsilon$  such that the minimum value of the restriction of  $f'_\varepsilon$  to the union of the left and right intervals is greater than  $c$ . Suppose  $\lambda$ ,  $C_0$ ,  $\alpha'$  and  $K'$  are the constants in the conditions (3) and (5) in Definition 2.4. By the naive distortion lemma, there is a constant  $C_3$  which equals  $C_2 \exp(K'C_0/(c(1 - \lambda^{\alpha'})))$  such that  $|J|/|I| \geq C_3^{-1}$  and  $C_3^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq C_3\varepsilon^{\frac{1}{\gamma}}$  because  $G$ ,  $J$  and  $I$  are the images of  $G_{n-1}$ ,  $J_{n-1}$  and  $I_{n-1}$  under the  $(n-1)^{th}$  iterate of  $f_\varepsilon$ .

In the second case, let  $m$  be the largest positive integer such that  $I_m$  is in one of the middle intervals. We can divide this case into two subcases according to  $m$ . One is that  $m$  is  $n-1$ . The other is that  $m$  is less than  $n-1$ .

If  $m = n-1$ , then  $I_{n-1}$  is one of the middle intervals. By the condition (1) in Definition 2.4, there is a positive constant  $C_4$  which

does not depend on  $\varepsilon$  such that the lengths of the left interval and the right interval are greater than  $C_4$ . By Corollary 2.1, there is a constant  $C_5$  which equals  $C_2 \exp(D + E/C_4)$  and does not depend on  $\varepsilon$  such that  $|J|/|I| \geq C_5^{-1}$  and  $C_5^{-1} \varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq C_5 \varepsilon^{\frac{1}{\gamma}}$  because the restriction of the  $(n-1)^{th}$  iterate of  $f$  to  $I$  embeds  $I$  to  $I_{n-1}$  and the distance from  $I_{n-1}$  to  $\{-1, 1\}$  is greater than  $C_4$ .

If  $m < n-1$ , then  $I_i$  is in the union of the left and right intervals for  $m < i \leq n-1$ . Because  $I_{n-1}$  is one of the left and the right intervals,  $I_{m+1}$  has 1 as a boundary and  $I_m$  is the one closing 0 in  $\eta_{n-m, \varepsilon}$ . For the sequence  $I_{m+1}, \dots, I_{n-1}$ , no one of them is in the union of the middle intervals. By the same arguments as those in the first case imply that  $|J_{m+1}|/|I_{m+1}| \geq C_3^{-1}$  and  $C_3^{-1} \varepsilon^{\frac{1}{\gamma}} \leq |G_{m+1}|/|I_{m+1}| \leq C_3 \varepsilon^{\frac{1}{\gamma}}$ . For the sequence  $I_0, \dots, I_m$ , the last one  $I_m$  is in one of the middle intervals. Similar arguments to those in the subcase  $m = n-1$  imply that  $|J|/|I| \geq C_6^{-1} |J_m|/|I_m|$  and  $C_6^{-1} |G_m|/|I_m| \leq |G|/|I| \leq C_6 |G_m|/|I_m|$ , where  $C_6 = \exp(D + E/C_4)$ . Because the restriction of  $f_\varepsilon$  to  $I_m$  is comparable with the power law mapping  $|x|^\gamma$  uniformly on  $\varepsilon$  by the condition (2) in Definition 2.4, we may assume  $f_\varepsilon|_{I_m} = 1 + \varepsilon - |x|^\gamma$ . By similar arguments to those in subcase  $m < n-1$  in the proof of Theo-

rem 1.1 and the naive distortion lemma, there is a positive constant  $C_7$  which does not depend on  $\varepsilon$  such that  $|J_m|/|I_m| \geq C_7^{-1}|J_{m+1}|/|I_{m+1}|$  and  $C_7^{-1}|G_{m+1}|/|I_{m+1}| \leq |G_m|/|I_m| \leq C_7|G_{m+1}|/|I_{m+1}|$ . We get that  $|J|/|I| \geq (C_6C_7C_3)^{-1}$  and  $(C_6C_7C_3)^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq (C_6C_7C_3)\varepsilon^{\frac{1}{\gamma}}$ .

Let  $C$  be  $C_7C_6C_3$ . It is a determining constant of the gap geometry of  $\{\Lambda_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ . *QED.*

In [J1], we also studied some universal property of determining constant  $C$  for a special subspace of  $\mathcal{H}$ .

Let  $HD(\varepsilon)$  be the Hausdorff dimension of  $\Lambda_\varepsilon$ . An immediate consequence of Theorem 2.2 is the following corollary.

**COROLLARY 2.2.** *There is a positive constant  $C$  which does not depend on  $\varepsilon$  such that*

$$0 < HD(\varepsilon) \leq 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all  $0 \leq \varepsilon \leq \varepsilon_0$ .

**COROLLARY 2.3.** *There is a positive constant  $C$  which does not depend on  $\varepsilon$  such that*

$$C^{-1}\varepsilon^{\frac{1}{\gamma}} \leq s_\varepsilon((a^*0.)) + s_\varepsilon((a^*1.)) \leq C\varepsilon^{\frac{1}{\gamma}}$$

for all  $a^*$  in  $\mathcal{C}^*$  and  $0 \leq \varepsilon \leq \varepsilon_0$ .

*Proof.* For any  $0 < \varepsilon \leq \varepsilon_0$  and any  $a^* \in \mathcal{C}^*$ ,  $s(w1) + s(w0) = |G_w|/|I_w|$  where  $a^* = (\cdots w.)$  and  $G_w$  is the gap on  $I_w$ . Now this corollary is a consequence of Theorem 2.2 and Lemma 2.2. *QED.*

## §2.5. Scaling functions of mappings on $\mathcal{BH}$ .

Let us go on to prove Theorem 2.1. Suppose  $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$  is a good family. For any  $a^*$  in  $\mathcal{C}^*$ , we always use  $w_n i$  to denote the first  $(n+1)$  coordinates of  $a^*$ , that is,  $a^* = (\cdots w_n i.)$ , and use  $s(\varepsilon, w_n i)$  to denote the ratio of lengths,  $|I_{\varepsilon, w_n i}|/|I_{\varepsilon, w_n}|$ . Suppose  $s_\varepsilon$  is the scaling function of  $f_\varepsilon$  for  $0 < \varepsilon \leq \varepsilon_0$ . We suppress  $\varepsilon$  when there can be no confusion.

LEMMA 2.8. *For each  $0 < \varepsilon_1 \leq \varepsilon_0$ ,  $s_\varepsilon$  converges to  $s_{\varepsilon_1}$  uniformly on  $\mathcal{C}^*$  as  $\varepsilon$  tends to  $\varepsilon_1$ .*

*Proof.* Suppose  $C_\varepsilon$  is the constant obtained in Lemma 2.2. By the condition (1) in Definition 2.4,  $C_\varepsilon$  is continuous on  $0 < \varepsilon \leq \varepsilon_0$ . We can find a positive number  $\delta$  such that  $C_\varepsilon \leq 2C_{\varepsilon_1}$  for all  $\varepsilon$  in  $(\varepsilon_1 - \delta, \varepsilon_1 + \delta)$ .

From the proof of Lemma 2.2, we have that for any  $a^*$  in  $\mathcal{C}^*$ ,  $|s_\varepsilon(a^*) - s(\varepsilon, w_n i)| \leq C_\varepsilon |I_{\varepsilon, w_n}|^{\alpha'}$  and  $|s_{\varepsilon_1}(a^*) - s(\varepsilon_1, w_n i)| \leq C_{\varepsilon_1} |I_{\varepsilon_1, w_n}|^{\alpha'}$ .

Because we may write  $|s_\varepsilon(a^*) - s_{\varepsilon_1}(a^*)|$  in  $|s_\varepsilon(a^*) - s(\varepsilon, w_n i) + s(\varepsilon, w_n i) - s(\varepsilon_1, w_n i) + s(\varepsilon_1, w_n i) - s_{\varepsilon_1}(a^*)|$ , this implies that

$$|s_\varepsilon(a^*) - s_{\varepsilon_1}(a^*)| \leq 2C_{\varepsilon_1} |I_{\varepsilon, w_n}|^{\alpha'} + C_{\varepsilon_1} |I_{\varepsilon_1, w_n}|^{\alpha'} + |s(\varepsilon, w_n i) - s(\varepsilon_1, w_n i)|$$

for all  $n > 0$  and  $\varepsilon$  in  $(\varepsilon_1 - \delta, \varepsilon_1 + \delta)$ . Now the last inequality and the conditions (1) and (5) in Definition 2.4 imply this lemma. *QED.*

**LEMMA 2.9.** *For every  $a^*$  in  $C^*$ , the limit of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  exists as  $\varepsilon$  decreases to zero.*

*Proof.* For every  $0 \leq \varepsilon \leq \varepsilon_0$ ,  $\eta_{1, \varepsilon}$  contains four intervals  $I_{\varepsilon, 00}$ ,  $I_{\varepsilon, 01}$ ,  $I_{\varepsilon, 11}$  and  $I_{\varepsilon, 10}$ . We call  $I_{\varepsilon, 01}$  and  $I_{\varepsilon, 11}$  the middle intervals,  $I_{\varepsilon, 00}$  the left interval and  $I_{\varepsilon, 10}$  the right interval.

For  $a^*$  in  $C^*$ , we may arrange it into two cases according to its coordinates. The first case is that the coordinates of  $a^*$  are eventually all zeroes. The second case is that there are infinite many ones in the coordinates of  $a^*$ .

In the first case, we can find a positive integer  $N$  such that  $I_{\varepsilon, w_n}$  is in the left interval for every  $n \geq N$ . By the condition (1) in Definition 2.4, there is a positive constant  $c$  which does not depend on  $\varepsilon$  such

that the minimum value of  $|f'_\varepsilon|$  on the left interval is greater than  $c$ . By the naive distortion lemma and the conditions (3) and (5) in Definition 2.4 and similar arguments to the proof of Lemma 2.8, there is a constant  $C_1$ , which equals  $\exp(K'C_0/(c(1 - \lambda^{\alpha'})))$ , such that

$$|s_\varepsilon(a^*) - s_{\varepsilon'}(a^*)| \leq C_1(|I_{\varepsilon, w_n}|^{\alpha'} + |I_{\varepsilon', w_n}|^{\alpha'}) + |s(\varepsilon, w_n i) - s(\varepsilon', w_n i)|$$

for all  $\varepsilon$  and  $\varepsilon'$  in  $(0, \varepsilon_0]$  and  $n \geq N$ . Now we can show that the limit of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  as  $\varepsilon$  decreases to zero exists.

In the second case, by the condition (1) in Definition 2.4, there is a positive constant  $C_2$  which does not depend on  $\varepsilon$  such that the lengths of the left interval and the right interval are greater than  $C_2$  for any  $0 \leq \varepsilon \leq \varepsilon_0$ . By Corollary 2.1, there is a constant  $C_3 = D + E/C_2$  which does not depend on  $\varepsilon$  such that if  $I_{\varepsilon, w_n}$  is in one of the middle intervals, then  $|s(\varepsilon, w_m i) - s(\varepsilon, w_n i)| \leq C_3 |I_{\varepsilon, w_n}|^\alpha$  for all  $\varepsilon$  in  $(0, \varepsilon_0]$  and  $m > n > 0$  because  $s(\varepsilon, w_m i) = (|(f^{\circ(m-n)})'(x)|/|(f^{\circ(m-n)})'(y)|)s(\varepsilon, w_n i)$  for some  $x$  and  $y$  in  $I_{\varepsilon, w_m}$ . Moreover, let  $m$  increases to infinity, then  $|s_\varepsilon(a^*) - s(\varepsilon, w_n i)| \leq C_3 |I_{\varepsilon, w_n}|^\alpha$  if  $I_{\varepsilon, w_n}$  is in one of the middle intervals.

If the  $n^{\text{th}}$  coordinate of  $a^*$  is one, then for any  $\varepsilon$  and  $\varepsilon'$  in  $(0, \varepsilon_0]$ ,  $I_{\varepsilon, w_n}$  is in one of the middle intervals for  $\varepsilon$  and  $I_{\varepsilon', w_n}$  is in one of the

middle intervals for  $\varepsilon'$ . Then

$$|s_\varepsilon(a^*) - s_{\varepsilon'}(a^*)| \leq |s(\varepsilon, w_n i) - s(\varepsilon', w_n i)| + C_3(|I_{\varepsilon, w_n}|^\alpha + |I_{\varepsilon', w_n}|^\alpha).$$

Because there are infinite many ones in the coordinates of  $a^*$ , the last inequality implies that the limit of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  as  $\varepsilon$  decreases to zero exists. *QED*.

Let  $s_0(a^*)$  be the limit of  $\{s_\varepsilon(a^*)\}_{0 < \varepsilon \leq \varepsilon_0}$  as  $\varepsilon$  decreases to zero. Then  $s_0$  defines a function on  $\mathcal{C}^*$ .

LEMMA 2.10. The limiting function  $s_0$  is the scaling function of  $f_0$ .

*Proof.* The proof is similar to the proof of Lemma 2.9. Let us outline the proof. There are four intervals in the first partition  $\eta_1$  determined by  $f_0$ . We call the one adjacent to  $-1$  the left interval, the two adjacent to  $0$  the middle intervals and the one adjacent to  $1$  the right interval. For any  $a^*$  in  $\mathcal{C}^*$ , its coordinates either are eventually all zeroes or contains infinite many ones.

If the coordinates of  $a^*$  are eventually all zeroes, then all  $I_{w_n}$  are eventually in the left interval. There is a positive constant  $C_1$  such that for any  $m > n > 0$ ,  $|s(w_m i) - s(w_n i)| \leq C_1 |I_{w_n}|^{\alpha'}$  by the naive

distortion lemma.

If there are infinite many ones in the coordinates of  $a^*$ . Suppose the  $n^{\text{th}}$  coordinate of  $a^*$  is one. Then  $I_{w_n}$  is in one of the middle interval. There is positive constant  $C_2$  such that for any  $m > n > 0$ ,  $|s(w_m i) - s(w_n i)| \leq C_2 |I_{w_n}|^\alpha$  by the Corollary 2.1.

In both of cases, the limit of  $\{s(w_n i)\}_{n=0}^\infty$  as the length of  $w_n i$  increases to infinity exists. The scaling function of  $f_0$  exists. Allowed  $\varepsilon = 0$  in the proof of lemma 2.9, we can show that this scaling function is  $s_0$ . *QED*.

LEMMA 2.11. *The scaling function  $s_0$  has jump discontinuities at all points in  $\mathcal{A}$ .*

*Proof.* Suppose  $a^*$  is in  $\mathcal{A}$  and  $a^* = (0_\infty w i.)$  where  $0_\infty$  is the one-sided infinite string of zeroes extending to the left,  $w$  is a finite string of zeroes and ones and  $i$  is either zero or one. Let  $0_n$  be the finite string of zeroes of length  $n$ . The interval  $I_{0_n w}$  is eventually in the left interval  $I_{00}$ . We use  $b_n$  to denote the length of  $I_{0_n w}$  and  $a_n$  or  $a'_n$  to denote the length of  $I_{0_n w i}$ . Let  $c_n$  be the distance from  $I_{0_n w}$  to  $-1$  (see Figure 12).

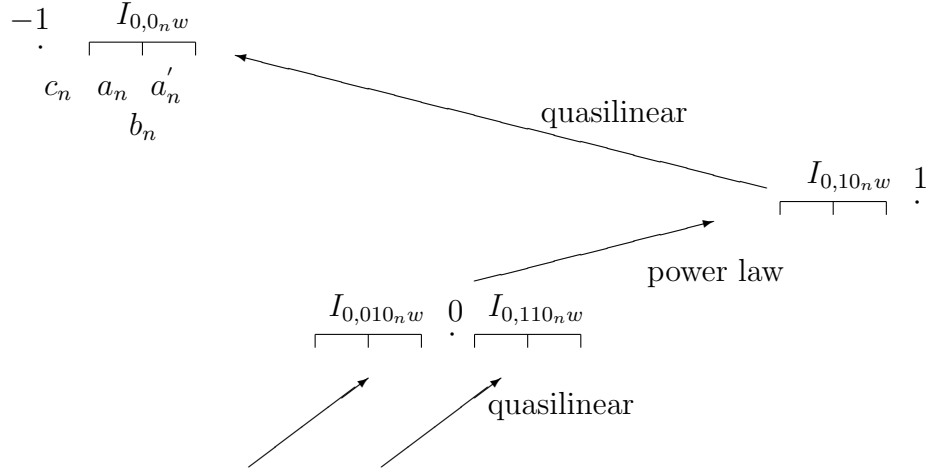


Figure 12

Let  $j$  be either zero or one. Because  $f_0$  has the power law  $|x|^\gamma$  with  $\gamma > 1$  at the critical point and  $I_{j10nw}$  close to the critical point, the limit of  $\{s(j10nw_i)\}_{n=0}^\infty$  equals the limit of

$$s_{n,1} = \frac{(a_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}{(b_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}$$

or the limit of

$$s_{n,2} = \frac{(b_n + c_n)^{\frac{1}{\gamma}} - (a_n + c_n)^{\frac{1}{\gamma}}}{(b_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}$$

as  $n$  increases to infinity if the limits of  $s_{n,1}$  and  $s_{n,2}$  as  $n$  increases to infinity exist.

Because the minimum value of the restriction of  $f_0$  to the left interval  $I_{00}$  is positive, by using the naive distortion lemma, we can show the

limit of  $\{b_n/c_n\}_{n=0}^{\infty}$  and the limit  $\{a_n/c_n\}_{n=0}^{\infty}$  as  $n$  increases to infinity exist. We use  $\tau_1(a^*)$  and  $\tau_2(a^*)$  to denote these limit, respectively. Now we conclude that the limit of  $\{s(j10_nwi)\}_{n=0}^{\infty}$  as  $n$  increases to infinity exists and

$$\lim_{n \rightarrow +\infty} s(j10_nwi) = \frac{(1 + \tau_2(a^*))^{\frac{1}{\gamma}} - 1}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1} \quad \text{or}$$

$$\lim_{n \rightarrow +\infty} s(j10_nwi) = \frac{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - (1 + \tau_2(a^*))^{\frac{1}{\gamma}}}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1}.$$

Because  $I_{j10_nwi}$  is in one of the middle intervals,  $I_{01}$  and  $I_{11}$ . from the proof of Lemma 2.10, the error of  $s(j10_nwi)$  to  $s_0(b^*)$  can be estimated by  $|I_{j10_nwi}|^\alpha$ , that is, there is a positive constant  $C_2$  such that

$$|s_0(b^*) - s(j10_nwi)| \leq C_2 |I_{j10_nwi}|^\alpha$$

for any  $b^* = (\dots j10_nwi.)$  in  $\mathcal{C}^*$ . Now we can get that the limit of  $s_0(b^*)$  as  $b^* \neq a^*$  tends to  $a^*$  exists and

$$\lim_{b^* \neq a^*, b^* \rightarrow a^*} s_0(b^*) = \frac{(1 + \tau_2(a^*))^{\frac{1}{\gamma}} - 1}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1} \quad \text{or}$$

$$\lim_{b^* \neq a^*, b^* \rightarrow a^*} s_0(b^*) = \frac{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - (1 + \tau_2(a^*))^{\frac{1}{\gamma}}}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1}.$$

Because  $s_0(a^*) = \tau_2(a^*)/\tau_1(a^*)$ , the limit of  $s_0(b^*)$  as  $b^* \neq a^*$  tends to  $a^*$  does not equal  $s_0(a^*)$ . In other words,  $s_0$  has jump discontinuity at  $a^*$ . *QED.*

LEMMA 2.12. *The scaling function  $s_0$  is continuous at all points in  $\mathcal{B}$ .*

*Proof.* Suppose  $a^*$  is in  $\mathcal{B}$ . Let  $b^*$  be any point in  $\mathcal{C}^*$  with the same first  $(n+1)^{th}$  coordinates  $w_n i$  as that of  $a^*$ . If the  $n^{th}$  coordinate of  $a^*$  is one, then  $I_{w_n}$  is in one of the middle intervals  $I_{01}$  and  $I_{11}$ . The errors from  $s_0(a^*)$  and  $s_0(b^*)$  to  $s(w_n i)$  can be estimated by  $|I_{w_n}|^\alpha$ , that is, there is a positive constant  $C$  such that  $|s_0(a^*) - s(w_n i)|$  and  $|s_0(b^*) - s(w_n i)|$  are less than  $C|I_{0,w_n}|^\alpha$ . Then  $|s_0(a^*) - s_0(b^*)| \leq 2C|I_{w_n}|^\alpha$ . Because there are infinite many ones in the coordinates of  $a^*$ , the limit of  $\{s_0(b^*)\}$  as  $b^*$  tends to  $a^*$  exists and  $\lim_{b^* \rightarrow a^*} s_0(b^*) = s_0(a^*)$ . In other words,  $s_0$  is continuous at  $a^*$ . *QED.*

Suppose  $\tilde{f}_0 = h_\gamma \circ f_0 \circ h_\gamma^{-1}$  is again the representation of  $f_0$  under the singular metric associated to  $f_0$ .

LEMMA 2.13. *There is a Hölder continuous scaling function  $\tilde{s}_0$  of  $\tilde{f}_0$  and the restriction of  $s_0$  to  $\mathcal{B}$  equals the restriction of  $\tilde{s}_0$  to  $\mathcal{B}$ . In particular, the restriction of  $s_0$  to  $\mathcal{B}$  is Hölder continuous on  $\mathcal{B}$ .*

*Proof.* By using similar arguments to the proof of Lemma 2.2, we can show that there is a Hölder continuous scaling function  $\tilde{s}_0$  of  $\tilde{f}_0$ .

The restriction of  $h_\gamma$  to the union of the middle intervals  $I_{10}$  and  $I_{11}$  is a  $C^1$  embedding. For any  $a^*$  in  $\mathcal{B}$ , if the  $n^{\text{th}}$  coordinate of  $a^*$  is one, then  $I_{w_n}$  is in one of the middle intervals. We use  $|h_\gamma(I_{w_n i})|/|h_\gamma(I_{w_n})|$  and  $|I_{w_n i}|/|I_{w_n}|$  to approach  $\tilde{s}_0(a^*)$  and  $s_0(a^*)$ , respectively. Now we can show that  $\tilde{s}_0(a^*) = s_0(a^*)$  because there are infinite many ones in the coordinates of  $a^*$ . *QED.*

Lemma 2.2 and Lemma 2.8 to 2.13 give the proof of Theorem 1.1.

More generally, we have the following theorem.

**THEOREM 2.3.** *Suppose  $f$  is on  $\mathcal{BH}$  and  $\tilde{f}$  is the representation of  $f$  under the singular metric associated to  $f$ . There exist the scaling function  $s_{\tilde{f}}$  of  $\tilde{f}$  and the scaling function  $s_f$  of  $f$  and these scaling functions satisfy that*

- (a)  $s_{\tilde{f}}$  is Hölder continuous on  $\mathcal{C}^*$ ,
- (b)  $s_{\tilde{f}}$  has jump discontinuities at all points in  $\mathcal{A}$  and  $s_f$  is continuous at all points in  $\mathcal{B}$ ,

(c) the restriction of  $s_f$  to  $\mathcal{B}$  equals the restriction of  $s_{\bar{f}}$  to  $\mathcal{B}$ .

*Proof.* The proof is the same as the proofs of Lemma 2.10 to Lemma 2.13. *QED.*

Suppose  $\mathcal{S} = \{s_f | f \in \mathcal{BH}\}$ . We can use  $s_f$  to compute the eigenvalues of  $f$  at all periodic points (see [J2]) and the power law at the critical point, that is,

$$\gamma = \frac{\log s_f((0_\infty \cdot))}{\log(\lim_{b^* \in \mathcal{B}, b^* \rightarrow 0_\infty} s_f(b^*))}.$$

The absolute value of the asymmetry of  $f$  at the critical point is  $|sv_f| = \lim_{n \rightarrow +\infty} |I_{010_n}| / |I_{110_n}|$ . Suppose  $f$  and  $g$  are two mappings on  $\mathcal{BH}$  and  $h$  is the conjugating mapping between  $f$  and  $g$ . A consequence of Theorem 1.4 is the following corollary.

**COROLLARY 2.4.** *The scaling function and the asymmetry at the critical point form a complete set of  $C^1$ -invariant on the boundary of hyperbolicity.*

**REMARK 2.3.** These invariants are also optimal to form a complete set of  $C^1$ -invariants on  $\mathcal{BH}$  (see Remark 1.5).

An example of a scaling function in  $\mathcal{S}$  is the following proposition.

PROPOSITION 2.2. *Let  $q : x \mapsto 1 - 2x^2$ . Then  $s_q(a^*) = 1/2$  for  $a^* \in \mathcal{B}$  and  $s_q(a^*) \neq 1/2$  for  $a^* \in \mathcal{A}$ .*

*Proof.* Recall that  $\tilde{q}(y) = 1 - 2|y|$ . The proof of this proposition just follows the proof of Lemma 2.11 and Lemma 2.13. *QED.*

### §2.6. The Hausdorff dimension of the maximal invariant set of $q_\varepsilon$ .

From Proposition 2.2, we can observe more on the Hausdorff dimension of the maximal invariant set of  $q_\varepsilon(x) = 1 + \varepsilon - (2 + \varepsilon)x^2$ . Suppose  $\Lambda_\varepsilon$  is the maximal invariant set of  $q_\varepsilon$  and  $HD(\varepsilon)$  is the Hausdorff dimension of  $\Lambda_\varepsilon$ .

PROPOSITION 2.3. *There is a positive constant  $C$  which does not depend on  $\varepsilon$  such that*

$$1 - C^{-1}\sqrt{\varepsilon} \leq HD(\varepsilon) \leq 1 - C\sqrt{\varepsilon}$$

*for all  $0 \leq \varepsilon \leq 1$ .*

*Proof.* Suppose  $dy = dx/\sqrt{(1 + \varepsilon)^2 - x^2}$  is the metric associated to  $q_\varepsilon$  on  $[-1, 1]$ ,  $y = h_{2,\varepsilon}(x)$  is the corresponding change of coordinate and  $\tilde{q}_\varepsilon = h_{2,\varepsilon} \circ q_\varepsilon \circ h_{2,\varepsilon}^{-1}$ . We suppress  $\varepsilon$  when there can be no confusion.

Let  $\varepsilon$  be in  $[0, 1]$ . We call  $I_{0_4}$  and  $I_{10_4}$  the end intervals. Recall that  $I_{0_4}$  is the interval in  $\eta_3$  adjacent to  $-1$  and  $I_{10_4}$  is the interval in  $\eta_4$  adjacent to  $1$ . We call the complement of the interiors of the end intervals the middle interval.

Suppose  $y$  is in  $[-1, 1]$  and  $x$  is the preimage of  $y$  under  $h_2$ . The nonlinearity of  $\tilde{q}$  at  $y$  is

$$n(\tilde{q})(y) = \frac{\varepsilon(1 + \varepsilon)}{(2(1 + \varepsilon) - (2 + \varepsilon)x^2)\sqrt{(1 + \varepsilon)^2 - x^2}}.$$

We can find a positive constant  $C_1$  which does not depend on  $\varepsilon$  such that

$$|n(\tilde{q})(y)| \leq C_1\varepsilon$$

for any  $y$  in the image of the middle interval under  $h_2$ .

Let  $\tilde{\Lambda}$  be the maximal invariant set of  $\tilde{q}$ . It is diffeomorphic to  $\Lambda$ . The sets  $\tilde{\Lambda}$  and  $\Lambda$  have the same Hausdorff dimension. We use  $\tilde{I}$  to denote the image of  $I$  under  $h_2$  and compute the Hausdorff dimension of  $\tilde{\Lambda}$ .

By direct computations, there is a positive constant  $C_2$  which does

not depend on  $\varepsilon$  such that for any  $I_w \in \eta_3$ ,

$$1 - C_2^{-1}\varepsilon \leq |\tilde{I}_{w0}|/|\tilde{I}_{w1}| \leq 1 + C_2\varepsilon.$$

Suppose  $w$  is a finite string of zeroes and ones. We call a piece string of consecutive zeros in  $w$  a zero element. We call the maximum length of all the zero elements in  $w$  the zero-length of  $w$ . If the length of  $w$  is greater than 4 and  $I_w$  is in the union of the end intervals, then the zero-length of  $w$  has to be greater then or equal to 4. Using this fact, we can show that for any finite string  $w$  of zeros and ones, if the length of  $w$  is greater than 4 and the zero-length of  $w$  is less than 4, the image under the  $i^{th}$  iterate of  $q$  for any  $0 < i \leq n - 4$  is in the middle interval for any  $0 < i \leq n - 4$ . For any finite string  $w$  of zeroes and ones satisfying that the length of  $w$  is greater then 4 and the zero-length of  $w$  is less than 3, the image of  $I_w$  under the  $(n - 4)^{th}$  iterate of  $q_\varepsilon$  is in  $\eta_3$  and  $|\tilde{q}_\varepsilon^{\circ(n-4)}(\xi)|/|\tilde{q}_\varepsilon^{\circ(n-4)}(\theta)|$  is bounded above by  $\exp(C_1\varepsilon)$  for any  $\xi$  and  $\theta$  in  $\tilde{I}_w$ . We can find a positive constant  $C_3$  which does not depend on  $\varepsilon$  such that

$$1 - C_3\varepsilon \leq \frac{|\tilde{I}_{w0}|}{|\tilde{I}_{w1}|} \leq 1 + C_3\varepsilon$$

for any finite string  $w$  of zeroes and ones satisfying that the length of

$w$  is greater than 4 and the zero-length of  $w$  is less than 3.

The restriction of the nonlinearity,  $n(h_2)(x) = x/((1 + \varepsilon)^2 - x^2)$ , to the middle interval is bounded above by a positive constant  $C_4$  which does not depend on  $\varepsilon$ . For any finite string  $wi$  of zeroes and ones, we use  $\tilde{s}(wi)$  to denote the ratio  $|\tilde{I}_{wi}|/|\tilde{I}_w|$ . A direct consequence of Theorem 2.2 is that there is a positive constant  $C_5$  which does not depend on  $\varepsilon$  such that  $\tilde{s}(w0) + \tilde{s}(w1) \geq 1 - C_5\sqrt{\varepsilon}$  for all finite string  $w$  satisfying that the length of  $w$  is greater than 4 and the zero-length is less than 3. Moreover, there is positive constant  $C_6$  which does not depend on  $\varepsilon$  such that  $\tilde{s}(wi) \geq (1/2)(1 - C_6\sqrt{\varepsilon})$  where  $i = 0$  or  $1$ .

Let  $S_n = \sum |\tilde{I}_w|^\delta$  where sum is over all finite string  $w$  satisfying that the zero-length is less than 3. For  $w = i_0i_1 \cdots i_n$ , let  $w_k = i_0 \cdots i_k$  for  $4 \leq k \leq n$ . We can write  $S_n$  in

$$\sum (\tilde{s}(w_n)\tilde{s}(w_{n-1}) \cdots \tilde{s}(w_4))^\delta |\tilde{I}_{w_4}|^\delta.$$

Suppose  $C_7$  is the minimum length of the intervals in  $\eta_3$ . Then  $S_n$  is greater than  $C_7((1/2)(1 - C_6\sqrt{\varepsilon}))^{\delta(n-4)}N_n$ , where  $N_n$  is the total number of finite strings of zeroes and ones which satisfies that the zero-length are less than 3.

It is easy to check that  $N_2 = 4$  and  $N_n = 2N_{n-1} - 1$  for any  $n > 2$ . We can find a positive constant  $C_8$  such that  $S_n$  is greater than  $C_8((1 - C_6\sqrt{\varepsilon})/2)^{n\delta}2^n$  for  $n \geq 4$ .

Let  $\delta_0 = \log 2 / (\log 2 - \log(1 - C_6\sqrt{\varepsilon}))$ . Then  $2((1 - C_6\sqrt{\varepsilon})/2)^{\delta_0} = 1$  and  $HD(\varepsilon)$  is greater than or equals to  $\delta_0$ . Here  $\delta_0$  is bounded below by  $1 - C\sqrt{\varepsilon}$  for a positive constant  $C$  which does not depend on  $\varepsilon$ .

Another side of the inequality in Proposition 2.3 comes from Corollary 2.2. *QED*.

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