Towards Topological Classification Of Critically Finite Polynomials

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Abstract: In this note we give a construction of branched cover maps of S^2 to itself which are topologically equivalent to critically finite polynomials in the sense of Thurston. The construction are on the invariant laminations. It applies to all degrees. This way we confirm the conjecture given by Goldberg and Milnor about the fixed point portrait of critically finite polynomials.

1 Background and definitions

Definition 1(critically finite rational map): Given a rational map $f : \mathbf{C} \to \mathbf{C}$, let P_f be the set of its critical points. Let $\Omega_f = \bigcup_{n\geq 1} f^{\circ n}(P_f)$ be the post-critical orbits of f. The map f is said to be critically finite if Ω_f is finite. In other words, all the critical points are either periodic or pre-periodic (forward iterate landing on a periodic orbit).

To describe critically finite rational maps is by means of constructing topological branched cover maps of the sphere. Similar definitions to Definition 1 can be given to branched cover maps. This means that we just replace the term, critical point, by the term, branched point. Now we can talk about critically finite branched cover maps f with the same notations, P_f for branched points and Ω_f for positive orbits of the branched points.

Definition 2(Thurston equivalence): Given two branched cover maps f and g from S^2 to itself with both Ω_f and Ω_g finite. The maps f and g are Thurston equivalent if there is an orientation preserving homeomorphism h from S^2 to itself such that $h \circ f = g \circ h$ on Ω_f and $h \circ f$ is isotopic to $g \circ h$ rel Ω_f, that is, isotopic is restricted to the space S^2/Ω_f. An equivalent class of branched cover maps is called a combinatorial equivalent class.

Given a critically finite branched cover map f of the sphere, we can construct an orbifold $\mathcal{O}_f$, or punctured sphere with marked points. All positive

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critical orbits are removed as punctures. For the points in Ω_f, we will associate an integer with each of them. Consider the function $\nu : \Omega_f \to N$, we choose the smallest ν with the property that if $x, y \in \Omega_f, y \in f^{-1}(x)$, then $\nu(y)$ is a multiple of $\nu(x)\deg_y f$.

It was shown by Thurston that each combinatorial class contains at most one rational map up to conformal conjugacy. In this sense the critically finite rational maps are determined by their isotopic classes.

2 Invariant lamination

Now we confine ourselve to consider critically finite polynomials only. There is a natural way to describe the dynamics at least combinatorially, i.e. by laminations.

A polynomial f has an attracting critical fixed point ∞. It is known that if the orbits of critical points are bounded (true for critically finite case) then the Julia set of f is connected. For such a map we can take an open disk U of ∞ on which f is conjugate to $z \to z^d$ where d is the degree of f. Then take $V = \bigcup_{i \geq 0} f^{-i}(U)$, we see that V is still a disk because there is no critical point goes to ∞. It is known that $\partial V =$ Julia set of f. The conjugacy over V can be extended to ∂V as a continuous map h. Usually h is no longer a homeomorphism on the circle, i.e. there can be several preimages corresponding to one point in the Julia set. [The image of $\{re^{2\pi i\theta} | r \geq 0, \theta \text{ fixed }\}$ under the conjugacy is called external ray.] There is a remarkable property observed by Thurston *[8]* that the *convex hulls of preimages of the points in Julia set are disjoint.* If one connects all the preimages of every points in Julia set by line segments (leaves), it forms a lamination, this means that the closed set of leaves. Such a lamination is forward invariant under the map $z \to z^d$ as we can see from the conjugacy. It means the leaves will be mapped to leaves or point under the map f (forward invariant). The preimages of leaves will form leaves too. Given a forward invariant lamination one can saturate it to an invariant lamination if the forward invariant lamination satisfies the following condition:

$\sum($ degree of critical gap $-1)+$number of critical leaves $= d-1$

Here a critical leaf means a leaf mapped to a point by $z \to z^d$, a critical gap means a gap, a connected component of the complement of the leaves, mapped to other gaps with degree greater than one.

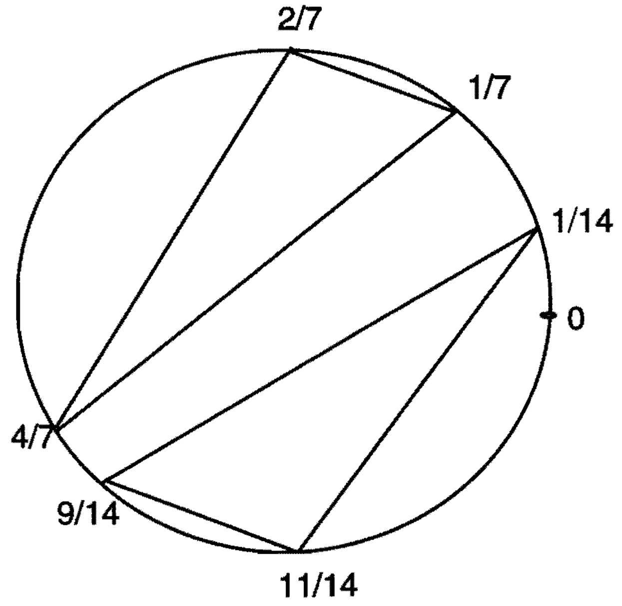
We see here that laminations serve as nice models to describe the dynamics of polynomials. What we will do in this note is to construct branched cover maps out of forward invariant laminations.

3 Two examples in the degree two case

a). The Douady rabbit given by a branched cover map

Consider the following set of numbers $\{\{0\}, \{1/7, 2/7, 4/7\}\}$ from the unit circle (θ is corresponding to $e^{2\pi i\theta}$) it forms a forward invariant lamination of degree two. We will construct a branched cover map from this lamination.

The three points $\{1/7, 2/7, 4/7\}$ form a triangle T_1. Their preimages $\{1/14, 9/14, 11$ form another triangle T_2. Consider the complement of these two triangles T_1 and T_2 in the unit disk. It has five connected components, say U_0, U_1, U_2, U_3 and U_4 (see figure below). The circle is divided into 6 segments, let us denote them as B_{01} and B_{02}, B_1, B_2, B_3 and B_4. The set $\{B_{0j}\}_{j=0}^{1} \cup \{B_i\}_{i=0}^{4}$ gives a Markov partition of the unit circle under the standard map $s(z) = z^2$. We see that B_1 is mapped to B_2, B_2 is mapped to $B_{01} \cup B_{02} \cup B_3 \cup B_4, B_3$ is mapped to B_2, B_4 is mapped to $B_{01} \cup B_{02} \cup B_3 \cup B_4$, B_{01} andB_{02} are mapped to B_1. We see that $\{B_{0j}\}_{j=0}^{1} \cup \{B_i\}_{i=1}^{4}$ is a Markov partition on the unit circle under $s(z) = z^2$.



We put a branched point O in the component U_0 since s on the boundary of U_0 covers the boundary of U_1 twice. A map $f_{0,1}$ from U_0 to U_1 is a branched cover with only branched point O of degree two.

We can define the map over the disk as the continous extension of the map over the boundary. Since $B_{01} \mapsto B_1 \mapsto B_2 \mapsto B_{01}$ and $B_{02} \mapsto B_1 \mapsto B_2 \mapsto B_{01}$ are both admissible sequences, we can put into orbit of branched point according to this sequence. There is one ambiguity in the definition of the map f. We want to extend the map over each component, i.e. the disk with marked points of designated points of branched points. There are infinitely many isotopy classes of the map over disk with marked points if the number of marked points are more than one. However by considering the analytic structure the isotopy class will be limited to one. See section 5, part b).

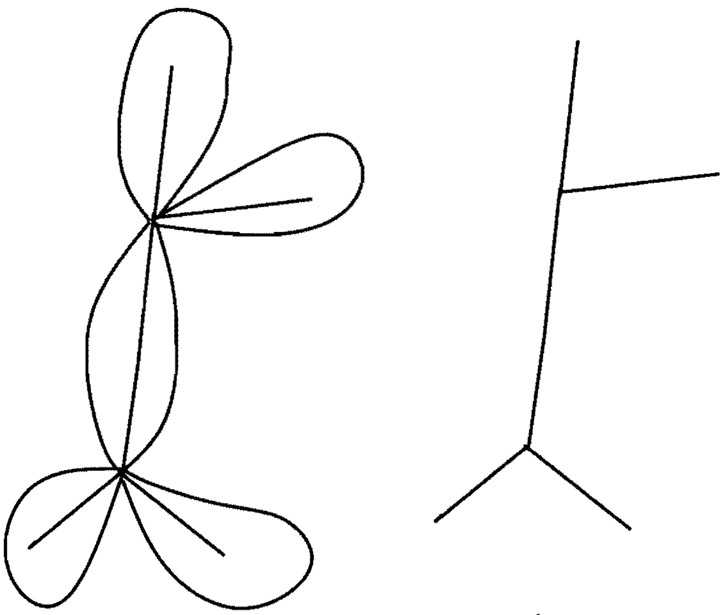
Let us denote the copy of the unit disk we use to define the map f_0 as D_1. Take another copy D_2 of the unit disk and consider the map $s(z) = z^2$ of D_2.

Since the two maps, f_0 from D_1 to itself and s from D_2 to itself, are the same when they restrict on the boundaries of D_1 and D_2, we can glue the boundaries of D_1 and D_2 by the identity and collapse T_1 and T_2 into two points. The resulted space is a topological sphere S^2.

Since the branched points of f are periodic, by a theorem in Levy's thesis *[5]*, there is no Thurston obstruction. So the map f is equivalent to a polynomial in the sense of Thurston. It is easy to check that it is equivalent to the Douady rabbit, $z \to z^2 + \omega(2-\omega)/4, \omega = e^{2\pi i/3}$.

Remark: What we will show in this paper is that the idea for the example above applies for all critically finite hyperbolic polynomials.

Remark: There is a close relationship between the invariant lamination and the Hubbard tree. The Hubbard tree is the dual graph of the pinched disk D^1/L. See for example the Douady rabbit.

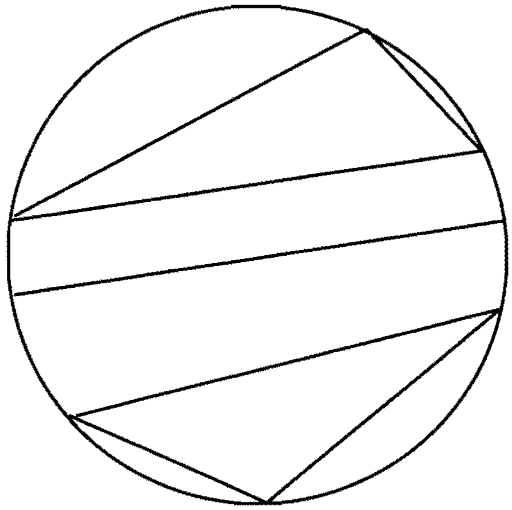


b). The preperiodic case

In this case the lamination is similar except we add a critical leaf L_0 in the middle of leaves. The critical leaf split U_0 into two components U_{01}, U_{02}. We then define a map $k_{01,1}$ from U_{01} to U_1 and a map $k_{02,1}$ from U_{02} to U_1 are homeomorphisms which agree with $s(z) = z^2$ on the boundary $\partial U_0 = B_{01} \cup B_{02}$. Also we define a map j_0 from L_0 to a point. Again we can choose all the maps here as those we defined in a) such that they form a continuous map f_0 on the unit disk. We collapse the critical leaf L_0 to a point, which will be the branch point. One may denote this point as c and its image under j_0 as v. Since $\{B_{0j}\}_{j=0}^{1} \cup \{B_i\}_{i=1}^{4}$ is a Markov partition on the unit circle under $s(z) = z^2$ and

$B_{01} \mapsto B_1 \mapsto B_2 \mapsto B_3 \mapsto B_2$ is an admissible sequence. we can find two point $c_1 \in B_{01}$ and $c_2 \in B_{02}$ such that their images are like $c_1 \to v \to v' \to v'' \to v'$ and $c_2 \to v \to v' \to v'' \to v'$ under s. The critical leaf now is just the line in U_0 connects c_1 and c_2.

Combining the arguments in a), we can get a branched cover f from S^2 to S^2.



4 What happens for higher degree, fixed point portrait

Given a critically finite polynomial of degree d, it has several repelling fixed points all lying on the Julia set. Each fixed point has several external rays. The corresponding points in the unit circle of the end points of the external rays forms a forward invariant set under the map $z \to z^d$. Call such a set T, a rotation subset of S^1. Consider all the repelling fixed points, their corresponding rotatioal subsets $\{T_1, T_2, ..., T_k\}$ are called a fixed point portrait by L. Goldberg and J. Milnor *[3]*. They proved that the fixed point portrait has the following properties.

Theorem 4.1: *([3]) Let $\{T_1, T_2, ...T_k\}$ be the rotatioal subsets of a critically finite polynomial f of degree d, then it satisfies the following conditions:*

- *C_1: Each T_i is a rotation subset of S^1 in the sense that it is invariant under the map $z \to z^d$ and has a well defined rotation number p_i/q_i.*
- *C_2: The convex hulls of different T_i's are disjoint.*
- *C_3: The union of the rotatioal subsets of rotation number zero are $\{0, 1/(d-1), 2/(d-2), ..., (d-2)/(d-1)\}$.*
- *C_4: rotatioal subsets with non-zero rotation number are separated by rotatioal subset with zero rotation number.*

Remark: From condition 2 we see that $\{T_1, T_2, ..., T_k\}$ forms a forward invariant lamination. The data given above, i.e. $\{T_1, T_2, ..., T_k\}$ satisfying the above conditions, is purely combinatorial. It was conjectured *[3]* that all such fixed point portraits can be realized by polynomials.

5 Realization of a fixed point portrait

In light of the above construction of the branched cover map, we give a construction of a branched cover map in all degree to realize Goldberg and Milnor's fixed point portrait. To show the map is equivalent to a polynomial we will use Thurston's topological characterization of critically finite rational maps.

a). Fixed point portrait and Markov partition of $s : z \to z^d$ of S^1.

Given a forward invariant lamination L, consider the preimage of L under $s : z \to z^d$, we get another forward invariant lamination $L' = s^{-1}(L)$. To make sure that such a construction is possible we have to verify that the number of critical gaps and critical leaves satisfy the equation given in Section 2.

Proposition: *Given a forward invariant lamination L arising from a fixed point portrait we have*

$$\Sigma(\textit{degree of critical gaps} - 1) + \textit{number of critical leaves} = d - 1.$$

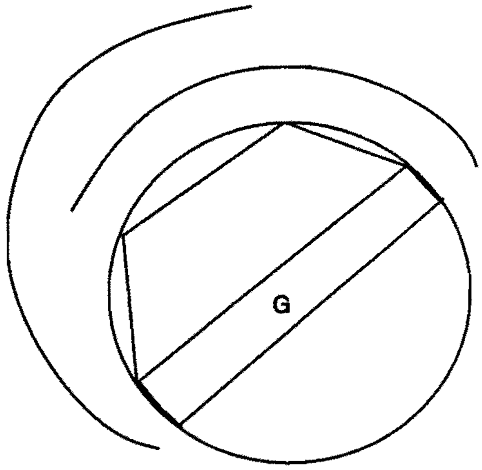
Proof:

We will construct a bijection of critical gaps to the set $\{0, 1/(d-1), 2/(d-2), ..., (d-2)/(d-1)\}$.

From the conditions C3 and C4 of Goldberg-Milnor's theorem we see that there are two kinds of components:

i). Gaps with only one leaf coming from non-zero rotation subset. Those gaps are not critical.

ii). Gaps with leaves coming from both non-zero rotation subset and zero rotation subset. We first consider the case with only two leaves. It looks like the following:



We look at the map of $G \cap S^1$ under s. Since one of the leaves is in the rotational subset and other leaf is invariant under s we get where is the other picture.

It is clearly a critical gap of degree two. If the other gap from the other side of the leaf of the zero rotational subset happen has only one non-zero leaf we got another critical gap of degree two. For this case we have the correspondence of the critical gaps to the zero rotational subset.

If the gap contains more than one leaves or point of zero rotaional subset we can subdivide it into several gaps by cutting along the leaves or points of zero rotational subset. Each piece will be a critical gap of degree two. Put them together we still have the correspondence of critical gaps to the leaves of zero rotational subset. ∎

Now we interpret the forward invariant lamination as a Markov partition of $s: z \to z^d$ of S^1.

Definition: Markov partition of $s: z \to z^d$ of S^1 is a partition $\{U_i\}$ of S^1, such that

1. Each U_i is a set of closed intervals contained in S^1;
2. $int(U_i) \cap int(U_j) = \phi$ if $i \neq j$;
3. $s(U_i)$ is a union of several intervals from $\{U_j\}$.

Given a forward invariant lamination L, consider its preimage $L' = s^{-1}(L)$. Delete all the intersections of L' with S^1, we get a collection of open intervals $\{J_i\} \subset S^1$. Its closure is a partition of S^1. Two intervals J_i and J_j are said to be connected if the convex hull of J_i and J_j in D don't contain any leaf in L'. It's an equivalence relation. Put all connected $J_i's$ together and let U_i be its closure. It is clear that $\{U_i\}$ forms a partition of S^1. Each U_i is called a component of the partition from S^1/L'.

Proposition: *The partition $\{U_i\}$ constructed above is a Markov partition of $s: z \to z^d$ of S^1.*

Proof: It suffices to verify (3) in the definition.

By collapsing all convex hulls of L (or L') into points we got the quotient space S^1/L (or S^1/L') which are pinched disks, say V_i's. The boundary of those disks, ∂V_i, is a components of S^1/L. We call the disks the components of D^1/L.

We see that s maps each component of S^1/L' to S^1/L. Each component of S^1/L is union of several components of S^1/L'. So we see that the partition given by S^1/L' is a Markov partition of s. ∎

b). Construction of branched cover maps of S^2 to itself

To construct a branched cover map of S^2 to itself we need to put a branched point into components of D^1/L. From above we see that the map s over each gap V_i has well-defined degree. The gap V_i iscalled critical if degree of s over V_i is greater than one. It is clear that we should put branched point into V_i. However what we need is to locate the orbits of branched points. To do so we introduce another definition.

Definition: (Admissible sequence of the partition) A sequence $V_{i_1} \to V_{i_2} \to \ldots \to V_{i_k}$ is called an admissible sequence of the partition if we have the following:

1. Each V_{i_j} is a component of D^1/L',
2. $s(\partial V_{i_j}) \supset \partial V_{i_{j+1}}, j = 1, 2, ..., k$ and
3. $s(\partial V_{i_k}) \supset \partial V_{i_1}$.

Since s is expanding over S^1 we see that for each critical gap V, there exists an admissible sequence of the partition

$$V \to V' \to V" \to \ldots \to V^{(k)}.$$

Put sequences of all critical gaps together we get an admissible sequence of gaps. With such a sequence we can construct a branched cover map of S^2 to itself. Actually what we need to do is to extend the map over S^1/L' to D^1/L'. Notice that L' is a s invariant lamination. If we take another copy of D^1 and map s we can glue s with the map above to get a map of S^2. The reason we can glue them together is they are agree over the boundary.

For each admissible sequence we put points corresponding to symbolic sequences. The points will be the points of orbits of branched points, denoted by $\mathcal{B}$. Now we only consider them as a finite set which address concides with the admissible sequences. We want to extend the map over S^1/L' to D^1/L' and sending the points above according to the address of the admissible sequences.

We now describe the extension over each component of D^1/L'. In doing so we keep in mind that what matters are only the isotopy class of the branched cover map. We can divide them into two cases.

1). If the component don't intersect $\mathcal{B}$, there will be a unique extension of the map over the boundary up to isotopy(Alexsanda's lemma).

2). If the component intersect with $\mathcal{B}$, we want to extend the map over the boundary and points in $\mathcal{B}$ according to their address. Since the set $\mathcal{B}$ comes from admissable sequence, we are able to find the image of $\mathcal{B} \cap G'$ in G. We can always extend the map over $\partial G' \cup (\mathcal{B} \cap G')$ to G'. Usually there are infinitely many isotopy classes for such extension. We only choose the trivial one from the consideration below.

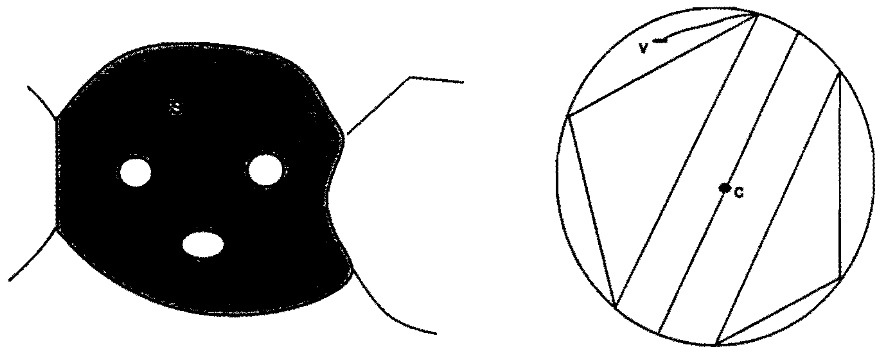
Proposition: *Let M be a Riemann surface of punctured sphere with boundaries. Let $f : M \to M$ be an analytic homeomorphism. Then the isotopy classes of f is finite.*

Outline of the Proof: From the conformal structure of the Riemann surface there corresponds to a hyperbolic structure. The analytic automorphism is correspondent to isometry to the hyperbolic structure. The isotopy classes of isometry is finite.

From the proposition above we see that we should take the extension so that the isotopy class be trivial by the following consideration.

For each gap we construct a Riemann surface, or a subset of **C** with finitely many boundaries as the following.

i). First we consider the case that the gap is not critical. Draw small circles around each point in $\mathcal{B}$. Delete these small disks. Close the two external rays outside S^1. We get a Riemann surface S. See the picture below.



If the map is a polynomial map we see that the map is an analytic homeomorphism of this surface to another one of same topological type. From the proposition above we see the extension must be trivial.

ii). Now we consider how to extend the map over a critical gap. Actually we can subdivide the gap into several pieces by adding more leaves to decompose the map into several copies each of them is analytic homeomorphism. Then applying the argument above. For example, for degree two case, we draw one line in G to connect the branched value to ∂G, two lines in G' to connect the branched points to $\partial G'$. See the picture.

c). The maps are equivalent to polynomials.

To check that the branched cover maps defined above are equivalent to polynomials we use Thurston's topological characterization of rational maps. The theorem was stated and proved in [DH]. We will not state the theorem, instead we will use the consequence derived from the theorem.

Call a branched cover map of the sphere to itself a topologically polynomial map if the map has a fixed branched point (let's fixed it be the point ∞). There is a simple property for topologically polynomial map, i.e. a simple closed curve can be associated to a well-defined disk which is the disk bounded by the curve not containing ∞. This fact is crucial to have the following reduction shown in [F] that a critically finite topological polynomial map is equivalent to a critically finite polynomial if and only if there is no Levy cycle.

A Levy cycle for a critically finite branched cover map f is a collection of disjoint simple closed curves $\{\gamma_1, \gamma_2, ..., \gamma_s\}$ in S^2/Ω_f such that f maps γ_i homeomorphically to a curve which is isotopic to γ_{i+1}, all other components of $f^{-1}(\gamma_{i+1})$ are bounding some disks containing at most one point in Ω_f.

In our case all branched points of the branched cover maps constructed above are all periodic. It's quite easy to use the theorem above to show the main theorem in this paper.

Theorem: *All branched cover maps constructed above are equivalent to polynomials.*

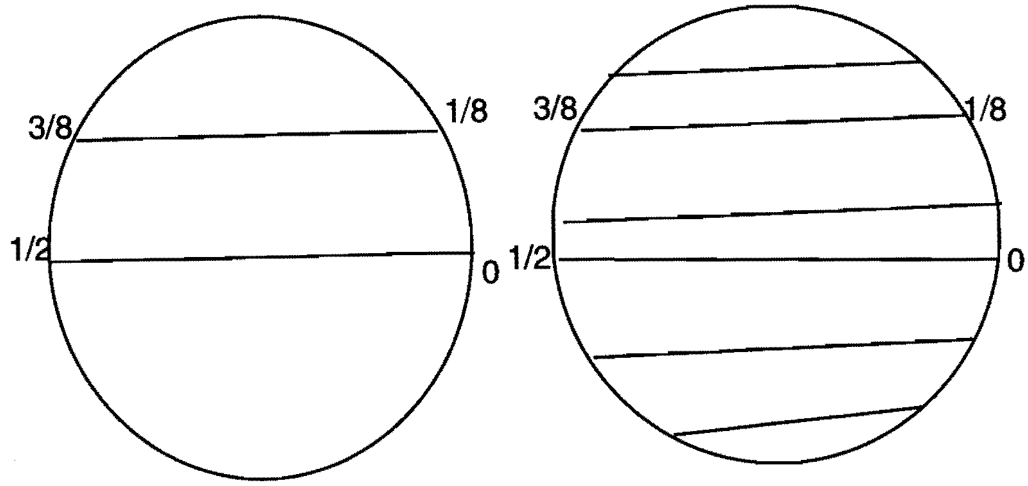
Proof: Suppose on the contrary there exists a Levy cycle for a branched cover map. The map is a topological polynomial. A Levy cycle is associated to a set of disks which are mapped homeomorphically from one to another like the map over the curves. The points in Ω_f contained in those disks have to have a periodic point. By the assumption one of the points has to be branched point. Then the map over the disk cannot be a homeomorphism. Contradiction. ∎

Remark: Another approach to the realization of a fixed point portrait using Hubbard tree is in [P].

Remark: Our method here could be used to classify hyperbolic rational maps. Once we have a partition we can define kneading sequence, i.e. the itenerary of critical orbits of the polynomial. The kneading sequence is clearly a topological invariant. It's nearly complete in the sense that if we count the number of each periodic cycle of critical orbit then it becomes complete. For real multimode map there is another definition of kneading sequence. It's interesting to see if we can use the method in this paper to pick up the kneading sequences realized by polynomials.

d). An example in degree 3

Let's do the following example of $\{\{0, 1/2\}, \{1/8, 3/8\}\}$ in degree 3 to illustrate our construction above. There are three components of S^1/L, denoted by V_1, V_2, V_3. Take the preimages of $\{\{0, 1/2\}, \{1/8, 3/8\}\}$ under the map $z \to z^3$, and connect them to form non-intersecting leaves. We get 6 leaves, i.e. $\{0, 1/2\}$, $\{1/24, 11/24\}$, $\{1/8, 3/8\}$,$\{1/6, 1/3\}$, $\{2/3, 5/6\}$, $\{17/24, 19/24\}$. They divided the unit disk into 7 regions, namely, U_1 to U_7. Those are the components of S^1/L'.



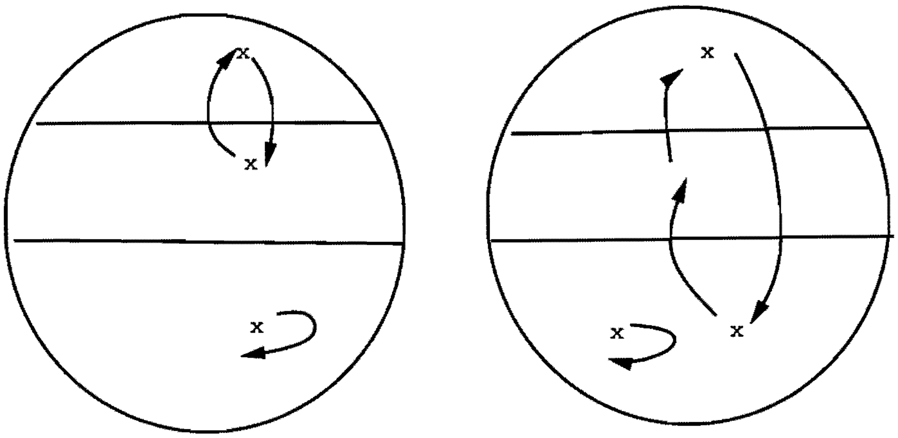
The map from S^1/L' to S^1/L is the following:

$$s(U_1) = V_3, s(U_2) = V_2, s(U_3) = 2V_1, s(U_4) = V_2, s(U_5) = 2V_3, s(U_6) = V_2, s(U_7) = V_1.$$

Here $2V_1$ means the component is covered twice by s. If one consider the maps over V_i's one can get the following:

$$s(V_1) = V_2 \cup V_3, s(V_2) = 2V_1 \cup V_2, s(V_3) = V_1 \cup V_2 \cup V_3$$

One can have several choices of the orbit of a branched point. The figure illustrate two examples.



We can extend the map to the disk as described above and glue the map with another copy of s of D^1 to get a branched map of the sphere. As shown above the map is equivalent to a cubic polynomial.

Remark: There is a nice structure for the space of critically finite hyperbolic polynomials. For the periodic two cycle above we can zoom in a Mandelbrot set into it. By using this construction we can get all critically finite polynomials from simple models plus Mandelbrot set. The classification will be described in subsequent papers. The classification will be useful to compute the mapping class group of the hyperbolic component introduced by Sullivan.

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References

[1] A. Douady and J. H. Hubbard, A Proof of Thurston's Topological Characterization of Rational Functions, Institute Mittag-Leffler Preprint (1984)

[2] Y. Fisher, Thesis, Cornell University, 1989

[3] L. R. Goldberg and J. Milnor, Fixed Points of Polynomial Maps, SUNY IMS Preprint, 1990

[4] S. Hu and Y. Jiang, Toward classification of critically finite polynomials, IHES preprint, 1991.

[5] S. Levy, Critically Finite Rational Maps, Princeton University, Ph.D. Thesis, 1985

[6] A. Poirier, On the Realization of Fixed Point Portrait, SUNY/IMS preprint, 1991.

[7] D. Sullivan, Quasiconformal Homeomorphism and Conformal Dynamics, III, preprint.

[8] W. P. Thurston, On the Combinatorics and Dynamics of Iterated Rational Maps, Preprint, 1985

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