Renormalization on One-dimensional Folding Maps

Yuping Jiang
Department of Mathematics, Queens College of CUNY
65-30 Kissena Blvd, Flushing, NY 11367

Abstract

Some techniques and results in the renormalization theory of real and complex dynamical systems are summarized. The construction of the induced Markov map of [-1,1] from a Feigenbaum-like map is presented. We show that this induced Markov map has bounded geometry. We discuss some property of infinitely renormalisable quadratic polynomials and show that the Julia set of an infinitely renormalisable quadratic polynomial satisfying complex a priori bounds is locally connected at its critical point. In addition, if this polynomial is also unbranched, then the Julia set is locally connected. Some result about nonconformal maps and a generalised version of Sullivan's sector theorem are discussed.

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§1. Family of folding maps and infinitely renormalisable maps.

Consider the quadratic family \( q_t(x) = t - (t+1)x^2 \). One can calculate bifurcation parameters \( t_n \) for \( 0 \leq n < \infty \), i.e., \( q_t \) has a unique attractive periodic orbit of period \( 2^n \) for \( t_n < t < t_{n+1} \). Let \( \delta_n = (t_{n+1} - t_n) / (t_{n+1} - t_n) \). A remarkable discovery in [10, 11] (see also [5]) is that

- The limit \( \delta = \lim_{n \to \infty} \delta_n \) exists and is universal, i.e., for any analytic full family \( f_t(x) = h_t(-x^2) \) in \( F \) (see below), one obtains the same limit by calculating ratios of bifurcation parameters, where a full family means that \( h_0(0) = 0, h_1(0) = 1 \), and \( dh_0(0)/dt > 0 \),

- the limiting map \( q_\infty \), which is called the Feigenbaum polynomial, is unique in the quadratic family, i.e., there is no \( q_t \) for \( t \neq t_\infty \) such that \( q_t \) is topologically conjugate to \( q_\infty \),

- \( f_t = q_\infty^{-t} \circ q_\infty^t \circ q_\infty \) converges exponentially to a map \( g \) where \( q_\infty(x) = -p_n x \) and \( p_n \) is the periodic point of \( q_\infty \) of period \( 2^n \) with the smallest \( |p_n| \); the map \( g \) is also universal.
Suppose $f(z) = h(-z^2)$ is a folding map of $[-1,1]$ where $h$ is an orientation-preserving diffeomorphism from $[-1,0]$ onto $[-1,h(0)]$. Suppose $f$ satisfies the condition $f$ has a unique fixed point $p_1 > 0$ in $(-1,1)$ and $f^{21}(0)$ is in $[-p_1, p_1)$.

Then $\mathcal{R}(f) = \alpha^{-1} \circ f^{41} \circ \alpha$ is a folding map again where $\alpha(z) = -p_1 z$. The operator $\mathcal{R}$ is called the period doubling operator. Feigenbaum [10, 11] used the following conjecture to explain the discovery. Consider the space $\mathcal{F}$ of folding maps $f(z) = h(-z^2)$ where $h$ can be extended holomorphically on a fixed open domain $D \supset [-1,0]$ and continuously on $\overline{D}$. Then $\mathcal{F}$ with topology of uniform convergence is a Banach space. Let $\mathcal{R}\mathcal{F}$ be the subspace of maps in $\mathcal{F}$ satisfying (a).

Conjecture 1. The compact operator $\mathcal{R}$ from $\mathcal{R}\mathcal{F}$ to $\mathcal{F}$ has a unique fixed point $g$. The transfer map $\mathcal{T}_{1}\mathcal{R}$ has a unique eigenvalue $\delta > 1$ and all other eigenvalues are in a disk centered at $0$ with radius $0 < a < 1$.

Lanford [25] and others (see [29]) proved this conjecture by using rigorous computer estimates. Sullivan made a great progress towards a conceptual proof of this conjecture and proved that $q_n$ is unique [40]. The reader may refer to Cvitanović’s book [6], Vul, Khanin and Sinaǐ’s survey article [43] and de Melo and van Strien’s book [28] for background and recent developments. The reader may refer to [7, 9, 24, 34] for some recent study of the universal number $\delta$ and some relation between $\mathcal{T}_{1}\mathcal{R}$ and a transfer operator [2, 36]. The discovery and the conjecture can be generalized as follows:

1. Infinitely renormalizable folding maps. Let us consider a map $g_1(z) = z - (t + 1)z^2$. It is renormalizable if there is a (minimum) integer $n > 1$ and a (maximum) closed interval $I$ containing 0 as its interior point such that $g_n^m I$ from $g_n$ into itself is a folding map. (This is a generalization of (a)). For a renormalizable $g_i$, let $\alpha_i(t)$ be the linear map from $[-1,1]$ onto $I$ such that

$\mathcal{R}(g_i) = \alpha_i^{-1} \circ g_i^m \circ \alpha_i$

is a folding map of the form $h(-z^2)$. Then $\mathcal{R}$ is the renormalization operator. The map $g_i$ is infinitely renormalizable if $g_i, \mathcal{R}(g_i), \ldots, \mathcal{R}^{n_i}(g_i), \ldots$, are all renormalizable. There are uncountably many infinitely renormalizable maps in the quadratic family (see [29]).

2. Folding maps with arbitrary exponents. Consider the family $f_r(z) = z - (t + 1)|z|^r$ of folding maps with exponent $r > 1$. For this family, one can obtain a similar bifurcation diagram and calculate

$\delta = \lim_{n \to \infty} \frac{f_{n+1} - f_n}{f_{n+1} - f_n}$

Then one has similar observations.

i. $\delta = \delta(r)$ is a universal function of $r$, i.e., for analytic full family $h_n(\cdot)$ with fixed $r > 1$, one obtains the same limit by calculating ratios of bifurcation parameters,

ii. the limiting map $f_\infty$ is unique in the family $f_r$.

Consider the space $\mathcal{F}$ of folding maps $f(z) = h(\cdot)$ of $[-1,1]$ where $h$ are orientation-preserving diffeomorphisms from $[-1,0]$ onto $[-1,h(0)]$ and can be extended holomorphically on a fixed open domain $D \supset [-1,0]$ and continuously on $\overline{D}$. Then $\mathcal{F}$ with topology of uniform convergence is a Banach space. Let $\mathcal{R}\mathcal{F}$ be the subspace of renormalizable maps in $\mathcal{F}$.
Conjecture 2. The renormalization operator \( R \) from \( \mathcal{RF} \) to \( \mathcal{F} \) is hyperbolic \([15, 39]\) and its maximal invariant set 
\[ A_r = \cap_{n=0}^\infty R^{-n}(F_r) \]
is a Cantor set.

For an even integer \( r = 2n \), Sullivan \([40]\) made progress towards a conceptual understanding of this conjecture by connecting real and holomorphic dynamical systems. For \( r \neq 2n \), one is still looking for a program towards a conceptual understanding of this conjecture. We will discuss some techniques and results in this direction.

\section{Markov partitions and infinitely renormalizable folding maps.}

There is an important technique, the Markov partition, in the study of hyperbolic dynamical systems introduced by Sinai \([38]\) for Anosov systems and generalized by Bowen \([4]\) for Axiom A systems. The basic philosophy of this technique in one-dimensional dynamics is that for a (piecewise) expanding map \( f \) from a one-dimensional manifold \( M \) into itself, to cut the phase space \( M \) into pieces so that \( f \) maps each piece homeomorphically onto a union of some other pieces \([2, 14]\). An infinitely renormalizable folding map is not expanding at all because its critical orbit is recurrent. But in \([19]\), an induced Markov map \( F \) is constructed from an infinitely renormalizable folding map \( f(z) = h(-|z|^r) \) and is used in the study of \( f \).

Suppose \( f(z) = h(-|z|^r) \) is a folding map of \([-1, 1]\) satisfying the condition that \( r > 1 \) and \( h \) is a \( C^3 \)-diffeomorphism with nonpositive Schwarzian derivative, i.e.,
\[ S(h)(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2} \frac{h''(x)}{h'(x)} \leq 0 \]
for all \(-1 \leq x \leq 0\). Suppose \( f \) is a Feigenbaum-like map, i.e., it is also topologically conjugate to the Feigenbaum polynomial \( q_{\infty} \) (refer to \([29]\)). Suppose \( R^{\infty}(f) = a_{\infty}^{-1} \circ f^{n_{\infty}} \circ a_{\infty} \) is its \( n_{\infty} \)-renormalization where \( a_{\infty}(z) \) is a linear map from \([-1, 1]\) onto \( I_n \) where \( I_n \) is the interval bounded by the periodic point \( p_n \) of period \( 2^{n-1} \) and \(-p_n \). Using the sequence of nested intervals \( (I_n)_{n=1}^{\infty} \), we construct a partition of \([-1, 1]\). Let \( P_{-1} \) and \( P_1 \) be the closures of the left and right connected components of \([-1, 1] \setminus I_1 \). Inductively, let \( P_{-n} \) and \( P_n \) be the closures of the left and right connected components of \( I_n \setminus I_{n+1} \). Finally set \( P_0 = \{0\} \). The collection \( \beta_0 = \{ P_{-0}, P_0, P_{-1}, P_1, \ldots, P_{-m}, P_m, \ldots P_{-m} \} \) forms a partition of \([-1, 1]\), i.e., \( P_i \) and \( P_j \) have disjoint interiors for \( i \neq j \) and \([-1, 1]\) = \( P_0 \cup \cup_{n=1}^{\infty} (P_{-n} \cup P_n) \). Let \( F \) be the function defined as \( F(0) = 0 \) and 
\[
F(x) = \begin{cases} 
 f(x), & x \in P_{-0} \cup P_0; \\
 f^{n_{\infty}}(x), & x \in P_{-1} \cup P_1; \\
 \vdots \\
 f^{n_{\infty}}(x), & x \in P_{-m} \cup P_m; \\
 \vdots 
\end{cases}
\]
Then \( F \) is continuous on \([-1, 1]\).

\textbf{Lemma 1.} For every even integer \( n = 2m \geq 0 \), \( F(P_{-m}) = \cup_{n=0}^{m} P_{-n} \cup \cup_{n=1}^{m} P_{n} \) and for every odd integer \( n = 2m + 1 > 0 \), \( F(P_{-m}) = \cup_{n=0}^{m} P_{-n} \cup \cup_{n=1}^{m} P_{n} \).

From Lemma 1, the map \( F \) and the partition \( \beta_0 \) satisfy the Markov property in the sense that the image of every interval in \( \beta_0 \) is the union of some intervals in \( \beta_0 \). (But \( \beta_0 \) consists of an infinite number of intervals). Thus we call \( (F, \beta_0) \) an induced Markov map from \( f \). The following definition \([19]\) is similar to the one for a geometrically finite one-dimensional map in \([17]\).
Definition 1. The induced Markov map \( (F, \mu) \) from \( f \) is said to have bounded geometry if there is a constant \( C = C(f) > 0 \) such that

\[
\begin{align*}
(a) & \quad C^{-1} \leq \frac{|F^n(x)|}{|F^n(y)|} \leq C \quad \text{and} \quad C^{-1} \leq \frac{|F^n(x)|}{|F^n(y)|} \leq C \quad \text{for all} \quad n \geq 0, \quad \text{and} \\
(b) & \quad C^{-1} \leq \frac{|F^n(x)|}{|F^n(y)|} \leq C \quad \text{for each pair of} \quad x \quad \text{and} \quad y \quad \text{such that} \quad F^m(x) \quad \text{and} \quad F^m(y) \quad \text{are in the same interval in} \quad \Delta \quad \text{for every} \quad 0 \leq i < n.
\end{align*}
\]

The following theorem is proved in [19] by applying the real a priori bounds obtained by Guckenheimer [13] and Sullivan [40].

**Theorem 1** [19]. Suppose \( f(x) = h(\alpha|z|) \) is a Feigenbaum-like map. Then the induced Markov map \( (F, \mu) \) from \( f \) has bounded geometry.

A homeomorphism \( H \) of \([-1, 1]\) is said to be quasisymmetric [1] if there is a constant \( C > 0 \) so that for any \( x \) and \( y \) in \( I \),

\[
C^{-1} \leq \frac{|h(x) - h(y)|}{|h(z) - h(y)|} \leq C
\]

where \( z = (x + y)/2 \). (The quasisymmetric property of a conjugacy between two quadratic-like maps preserving the real line (see §3) is a bridge to connect real and holomorphic dynamical systems (see [40])). We proved in [19] the following theorem by using Theorem 1 and the result in [17] (see [18] for some remark).

**Theorem 2** [19, 32, 40]. Suppose \( f(x) = h_1(\alpha|z|) \) and \( g(x) = h_2(\alpha|z|) \) are two Feigenbaum-like maps and \( H \) is the topological conjugacy between them, i.e., \( H \circ f = g \circ H \). Then \( H \) is quasisymmetric.

In §3, we will show that a similar idea in the construction of the Markov partition here can be used as a tool to study some property of infinitely renormalizable quadratic polynomials.

§3. Infinitely renormalizable quadratic-like maps.

Let \( P(z) = z^2 + c \) be a quadratic polynomial where \( c \) is a complex number. We write \( K_c \), for its filled-in Julia set, i.e., the set of complex numbers such that \( \{P^m(z)\}_{m=0}^{\infty} \) is bounded. The Mandelbrot set \( M \) is the set of \( c \) so that \( K_c \) is connected [30]. Consider the Riemann mapping \( h \) from \( \overline{C} \setminus D_1 \) to \( \overline{C} \setminus K_c \) so that \( h(\infty) = 1 \) and \( h \circ P_\theta = P \circ h \) where \( \overline{C} \) is the extended complex plane, \( D_1 \) is the closed unit disk and \( P_\theta(z) = z^2 \). The image \( S_\infty \) of a circle centered at \( 0 \) with radius \( t > 1 \) under \( h \) is called an equipotential curve. The image \( R_\theta \) of a ray \( r_\theta = \{re^{i\theta} : |\theta| < \infty \} \) under \( h \) is called an external ray. Then \( P(S_\theta) = S_\theta \) for \( t > 1 \) and \( P(R_\theta) = R_{\theta(\mod \theta)} \) for \( 0 \leq \theta < 1 \). It is known that every repelling periodic point of \( P \) is a landing point of infinitely many periodic external rays (see [16, 28]). From Carathéodory theorem [28, 30], external rays land continuously at \( K_c \) for \( c \in M \) if and only if \( K_c \) is locally connected.

**Question.** For which \( c \) in \( M \), is \( K_c \) locally connected?

Douady and Hubbard [16, 31], Shishikura [39], Petersen [33], and Hu and Jiang [20] gave some (positive or negative) answers to this question. Yoccoz made a great progress recently. He proved that the Julia set of a nonrenormalizable (or finitely renormalizable) quadratic polynomial is locally connected (see [16, 31]). In [20, 21], we proved some results about an infinitely renormalizable quadratic polynomial by combining the Yoccoz' idea and the renormalization technique.

Let \( E_c \) be the open domain bounded by \( S_\theta \), then \( P \) from \( E_c \) to \( E_c \) is a model of a quadratic-like map defined by Douady and Hubbard [8].
Definition 2. A quadratic-like map $f : U \to V$ is a holomorphic, proper, branched cover of degree two where $U \subset V$ and $U$ and $V$ are simply connected open domains.

The filled-in Julia set of $f$ is $K_f = \cap_{n=0}^{\infty} f^{-n}(U)$. Henceforth, we will assume $K_f$ is connected and 0 is the branch point of $f$. A quadratic-like map $f$ with connected filled-in Julia set $K_f$ is once renormalizable if there is an integer $n \geq 2$ and an open domain $U_1 \subset U$ so that $f_1 = f^n : U_1 \to V_1 \subset V$ is a quadratic-like map with connected filled-in Julia set $K_{f_1} = K(n, U_1, V_1)$. Similarly, one can define $k$-times renormalizable and infinitely renormalizable quadratic-like maps $f$ (refer to §1 or [21]).

Suppose $P(z) = z^2 + c$ is an infinitely renormalizable polynomial, i.e., $P : E_0 \to E_0$ is infinitely renormalizable for some $t > 1$. (We will not distinguish $P$ and $P : E_0 \to E_0$ anymore.) Suppose

$$\{k_i = P^{n_i} : U_i \to V_i\}_{i=1}^\infty$$

is a sequence of renormalizations where $(n_i)_{i=1}^\infty$ is a strictly increasing sequence of integers. Each $f_i$ has a unique non-separating fixed point $\beta_i$ and a unique separating fixed point $\alpha_i$, i.e., $K_{f_i} \setminus \beta_i$ is connected, and $K_{f_i} \setminus \alpha_i$ is not, where $K_{f_i}$ is the filled-in Julia set of $f_i$. Let $Y_i$ be the (minimal) simply connected domain containing 0, bounded by $S_i$ and some external rays landing at $\alpha_i$ and $\beta_i$, where $\beta_i$ is another preimage of $0$, under $f_i$. Let

$$K_i = \cap_{j=1}^{\infty} P^{-j}(Y_i).$$

Suppose $c_i = P^n(0)$ and $CO = \{c_i\}_{i=1}^\infty$ and $CO_i$ is the critical orbit of $f_i$. Then $CO_i = \{c_{i+m}\}_{m=0}^\infty$. Let us assume that $(\ast\ast) K_i \cap CO = CO_i$. Under this assumption (it is satisfied automatically by all real infinitely renormalizable quadratic polynomials), we prove that (see also [28, 41])

Proposition 1 [21]. Suppose $G = P^{\infty} : U \to V$ is any renormalization with renormalization time $m_i$. Then the filled-in Julia set $K_G$ of $G$ is $K_i$.

From this proposition, the filled-in Julia set of any renormalization for a fixed renormalization time is canonical (it can be also obtained from a sequence of Yoccoz' puzzles (refer to [16])). It is called a renormalized filled-in Julia set of $K_i$. Let $m(A)$ be the modulus of an annulus $A$ (see [1]).

Definition 3. We say an infinitely renormalizable polynomial $P(z) = z^2 + c$ satisfies complex a priori bounds if there are a sequence of renormalizations

$$\{k_i = P^{n_i} : U_i \to V_i\}_{i=1}^\infty$$

and a constant $\lambda > 0$ such that $m(V_i \setminus U_i) > \lambda$ and $f_i$ satisfies $(\ast\ast)$ for every $i \geq 1$.

Let $GCO = \cup_{i=1}^\infty \cup_{j=0}^\infty P^{-j}(P^n(0))$.

Theorem 3 [21]. Suppose $P(z) = z^2 + c$ is an infinitely renormalizable polynomial. If $P$ satisfies complex a priori bounds, then its filled-in Julia set $K_i$ is locally connected at every point in $GCO$.

The following unbranched condition is adapted from [28].

Definition 4. We say an infinitely renormalizable polynomial $P(z) = z^2 + c$ is unbranched if there are a sequence $(K_i)_{i=1}^\infty$ of renormalized filled-in Julia sets of $K_i$, a constant $\lambda > 0$ and domains $W_i \supset K_i$ such that $m(W_i \setminus K_i) > \lambda$ and $W_i \cap CO = W_i \cap CO_i$ for all $i \geq 1$.

Moreover, we extend the result in [20].
Theorem 4 [21]. Suppose $P(c) = z^2 + c$ is an infinitely renormalizable polynomial. If $P$ is unbranched and satisfies complex a priori bounds, then its filled-in Julia set $K_c$ is locally connected.

Examples of polynomials $P(z) = z^2 + c$ satisfying complex a priori bounds and being unbranched are infinitely renormalizable real quadratic polynomials of bounded type (e.g., the Feigenbaum polynomial $a_x$). This follows the real and complex a priori bounds which was proved by Sullivan [40]. Lyubich had some result about complex a priori bounds for some infinitely renormalizable quadratic polynomials [27].

Proof of Theorem 3. Remember that $\{U_i\}_{i=1}^\infty$ is a sequence of nested domains about $0$. Consider the annulus $A_i = U_i \setminus U_{i+1}$ for $i \geq 1$. For each $i \geq 1$, there is a domain $U_i'$ such that $U_{i+1} \subset U_i' \subset U_i$, $f_i(U_i') \subset U_i$, and $f_i$ from $U_i' \setminus U_i'$ onto $V_i \setminus f_i(U_i')$ is a branch cover of degree two. Thus

$$m(U_i' \setminus U_i') = \frac{1}{2} m(V_i \setminus f_i(U_i')) \geq \frac{1}{2} m(V_i \setminus U_i) > \frac{\lambda}{2}.$$ 

Therefore $m(A_i) > \lambda/2$ for all $i \geq 1$. This implies that the diameter of $U_i$ tends to zero as $i$ goes to infinity.

From Proposition 1, there is a preimage $B_i$ of $Y_i$ under some iterate of $f_i$ such that $B_i \subset U_i$. The set $B_i \cap K_c$ is connected since $B_i$ is bounded by some external rays and some equipotential curve. So $\{B_i\}_{i=1}^\infty$ is a basis of connected neighborhoods at $c$. Now it is easy to argue that $K_c$ is locally connected at every point in GCO.

The proof of Theorem 4 is divided into two lemmas in [31]. The proof of Lemma 2 is easy. We will outline the proof of Lemma 3. Let us assume that the sequences of renormalizations in Definitions 3 and 4 are the same and $B_i$ for every $i \geq 1$ is the domain in the proof of Theorem 3. A point $p$ in $K_c$ is recurrent to zero if its orbit $O(p)$ intersects with an infinite number of $B_i$, otherwise it is off-critical.

Lemma 2. The set $K_c$ is locally connected at every off-critical point.

Lemma 3. The set $K_c$ is locally connected at every recurrent to zero point.

Proof. Let $\{K_i\}_{i=1}^\infty$ be the sequence of renormalized filled-in Julia sets of $K_c$ in Definition 4. From Yoccoz' puzzle (see [16, 31]), $K_c$ is partitioned into countably many homeomorphic copies of $K_c$ union the set of off-critical points for every $i \geq 1$.

There is a preimage $B_i$ of $Y_i$ under some iterate of $f_i$ such that $m(W_i \setminus B_i') > \lambda$. Without loss of generality, let us assume $B_i = B_i'$.

Suppose $p$ is recurrent to zero in $K_c$. For each $i \geq 1$, there is a homeomorphic copy $K_{i,p}$ of $K_c$ containing $p$. Let $f_{i,p} = P^i|K_{i,p}$ be the homeomorphism from $K_{i,p}$ to $K_i$ where $i > 0$ is an integer. Since $W_i \cap CO = W_i \cap CO_i$, there is no critical value of $P^i$ in $W_i \setminus B_i$. Thus the inverse $g_{i,p}$ of $f_{i,p}$ can be extended as a holomorphic homeomorphism to $W_i$. Let $W_{i,p}$ and $B_{i,p}$ be the images of $W_i$ and $B_i$ under $g_{i,p}$. Then $B_{i,p} \cap K_c$ is connected.

Since the diameter of $B_i$ tends to zero as $i$ goes to infinity, by modifying $W_i$, we can find a subsequence $\{i_k\}_{k=1}^\infty$ of integers such that

$$B_{i_{k+1}} \subset W_{i_{k+1}} \subset B_{i_k} \subset W_{i_k}.$$ 

To simplify notations, assume $i_k = k$. Consider $f_{i,p}$ from $W_{i,p}$ to $W_i$ and from $B_{i,p}$ to $B_i$. Suppose $q = f_{i,p}(p), B_{i+1,p} = f_{i,p}(B_{i+1,p}),$ and $W_{i+1,p} = f_{i,p}(W_{i+1,p})$. Since $W_{i+1,p} \subset B_{i+1}$ and $B_i$ is bounded by external rays and some equipotential curve, we have $W_{i+1,p} \subset B_i$ (Remember that $W_{i+1,p}$ is a homeomorphic copy of $W_{i+1}$ under some inverse branch of some iterate of $f_i$).
Therefore, \( m(W_k \setminus W_{k+1}) > \lambda \). Because the annulus \( W_{k_0} \setminus W_{k+1} \) under a holomorphic homeomorphism, \( m(W_{k_0} \setminus W_{k+1}) > \lambda \). This implies that the diameter of \( W_{k_0} \) tends to zero as \( k \) goes to infinity. So does the diameter of \( B_{k_0} \) for \( B_{k_0} \subset W_{k_0} \).

Therefore, \( \{B_{k_0}\}_{k_0} \) is a basis of connected neighborhoods of \( K \) at \( p \).

§4: Nonconformal maps and a generalized Sullivan's sector theorem.

In the last section, we discuss some possible methods towards a conceptual understanding of Conjecture 2 and related results.

(a) Nonconformal extension. Let \( f_{r,c}(z) = |z|^r + c \) for \( r > 1 \). For an even integer \( r = 2n \), this map has a natural holomorphic extension \( z^{2n} + c \). For \( r \neq 2n \), it has no holomorphic extension on the whole complex plane \( \mathbb{C} \). Consider a complex extension

\[
F_{r,c}(z) = z^r |c|^{-r} + c.
\]

This extension is a degree two branched cover. There is another interesting family

\[
G_{r,c} = z^r + (r-2) |c| + c.
\]

For \( r = 2 \) and \( c = 0 \), both of them are \( \mathcal{J}_0(z) = z^2 \) and the Julia set is the unit circle \( S^1 \). For \( r = 2 \) and \( |c| > 0 \) small, both of them are \( \mathcal{J}_c(z) = z^2 + c \) and the Julia set \( J_c \) is a quasicircle [42]. Moreover, Reule proved that

Theorem 5 [35]. The Hausdorff dimension \( HD(c) \) of the Julia set \( J_c \) of \( \mathcal{J}_c \) is an analytic function of \( c \) with small \( |c| \) and \( HD(c) > 1 \) for \( |c| > 0 \) small.

The proof of this theorem is to apply Koebe distortion theorem [3] and thermodynamical formalism [4, 35]. For \( r \neq 2n \) and \( |c| > 0 \) small, we can define the "Julia set" \( J_{r,c} \) as the boundary of the basin of infinity for \( F_{r,c} \) (or \( G_{r,c} \)). It is a Jordan circle [22].

Theorem 6 [22]. For \( |c| \) small, the Hausdorff dimension \( HD(c,r) \) of \( J_{r,c} \) is continuous at \( (c,2) \), and moreover, for every \( |c| > 0 \) small, there is a number \( r(c) > 0 \) such that \( HD(c,r) > 1 \) for all \( |r - 2| \leq r(c) \).

Proof: Develop a Koebe-like distortion theorem which we call the geometric distortion theorem [22] and then apply thermodynamical formalism.

(b) Compositions of inverse branches. Let us consider the inverse branches \( g_+ \) and \( g_- \) of \( f_{r,c}(z) = |z|^r + c \) for \( r > 1 \) and \( \text{real } c \). Both \( g_+ \) and \( g_- \) have holomorphic extensions \( G_+ \) and \( G_- \) on \( \mathbb{C} \setminus \{z \leq c\} \). The study of iterates of \( f_{r,c} \) should be similar to the study of compositions of \( G_+ \) and \( G_- \). (Note that one should notice the branched points of the composition when he considers such a composition.) Sullivan's sector theorem demonstrates an important result on these compositions.

Suppose \( I = (0, 1) \) is the open unit interval. Let \( \mathcal{E}_0 \) be the set of all well-defined functions \( G \) defined on \( \mathcal{E}_0 = (\mathbb{C} \setminus \mathbb{R}^1) \cup I_0 \) where \( I_0 \) is an open interval containing \( T \) such that \( G(T) = 0 \), \( G(1) = 1 \), and \( G \) preserves the upper- and lower-half planes. We assume that \( I_0 \) is the maximum open interval on which \( G \) is a homeomorphism and call it the definition interval of \( G \).

Take \( \mathcal{S}_0(z) = R^2 z^{1/4}, r > 1 \), where \( z = R e^{i\theta} \) for \( R > 0 \) and \( -\pi < \theta < \pi \). For every \( a \leq 0 \), we call \( L_a(z) = ES_a(z - a) + F \) an \( r \)-root at \( a \) where \( E \) and \( F \) are numbers such that \( L_0(a) = 0 \) and \( L_1(1) = 1 \). Then \( L_a \) is in \( \mathcal{E}_0 \) and its definition interval is \( (a, \infty) \). An element \( G \) in \( \mathcal{E}_0 \) is compatible with \( L_a \) if \( [a, 1] \subset G(I_0) \). For a compatible pair \( (L_a, G) \), let \( \sigma' = G^{-1}(a), J = [\sigma', 1], L \cup R = I_0 \setminus J, \), and \( \delta = \min \{|L|, |R|\} \). Let \( \mu = (1 + |\sigma'|)/\delta \).
Suppose $\{(L_i, G_i)\}_{i=0}^n$ is a sequence of compatible pairs in $\mathcal{L}$, where $L_i$ is an $r$-root at $a_i$. Let $\mu_i$ be the number for $(L_i, G_i)$. Let

$$\mathcal{L} = L_n \circ G_n \circ \cdots \circ L_0 \circ G_0 \circ \cdots \circ L_0 \circ G_0.$$ 

Then $\mathcal{L}$ is a schlicht function [3] defined on $C_1 = (C \setminus R^1) \cup f$.

**Definition 5.** We call $\mathcal{L}$ a root-like map if there are constants $C > 0$ and $\lambda > 1$ such that

1. $\alpha_0 = 0$ and $\alpha_1 \geq 1/C$,
2. $|a_i| \geq (\lambda^{i-1}/C)|a_{i-1}|$ for all $1 \leq i < n$, and
3. $\mu_i < C$ for all $0 \leq i \leq n$.

The following is a generalized version of Sullivan's sector theorem proved in [23] and is one of starting points for us to study Conjecture 2.

**Theorem 7** [23]. Suppose $\mathcal{L}$ is a root-like map. Then there is a constant $\theta > 0$ depending only on $\lambda$ and $C$ such that the image of the upper-half plane under $\mathcal{L}$ is contained in the sector

$$\text{Sec}_{\theta} = \{ z \in C | 0 \leq \arg(z) \leq \theta \}.$$ 

The proof of this theorem is to analyse the geometry of one compatible pair $(L, G)$ in details, and apply the sharpest version of Koebe distortion theorem [3] and Sullivan's idea [40] about trapping points by hyperbolic geometry [12, 26].

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E-mail: yunse@qcmunix.acq.acq.edu