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On Rigidity of One-Dimensional Maps

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Abstract. The regularity of the conjugacy between two one-dimensional maps with singular points is considered. We prove that the conjugacy between two nice and mixing quasi-hyperbolic one-dimensional maps is a diffeomorphism if it is an absolutely continuous homeomorphism and the exponents and the asymmetries of these two maps at all corresponding singular points are the same. We also discuss the application to geometrically finite one-dimensional maps, to Ulam-von Neumann transformations, and to circle expanding maps.

1. Introduction

A well-recognised program in the study of dynamical systems is to "fill in" the dictionary between the theory of one-dimensional dynamical systems and the theory of Kleinian groups (see the papers of, among others, Bowen [**BO**], Shub and Sullivan [**SS**], Thurston [**TH**], McMullen [**MC1,MC2**], Gardiner and Sullivan [**GS1,GS2**], and Sullivan [**SU1**]). In Kleinian groups, a remarkable result is Mostow's rigidity theorem [**MO**]. The theorem says that if two closed hyperbolic n-manifolds for $n \geq 3$ are homeomorphic, then they must be isometric. A closed hyperbolic n-manifold M^n can be treated as the quotient space $M^n = \mathbb{H}^n/\Gamma$ of a Kleinian group Γ which preserves the upper-half space $\mathbb{H}^n$. For $n = 2$, Mostow's rigidity theorem can

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be stated as follows. Let $\mathbb{D}$ be the open unit disk. A Kleinian group Γ acting on $\mathbb{R}^2$ is called Fuchsian if it preserves $\mathbb{D}$. A Fuchsian group Γ is called co-compact if the quotient space $\mathcal{S} = \mathbb{D}/\Gamma$ is a closed hyperbolic Riemann surface. Suppose Γ_1 and Γ_2 are two isomorphic co-compact Fuchsian groups. Then there is a homeomorphism H of $\overline{\mathbb{D}}$ such that

$$H\Gamma_1 = \Gamma_2 H.$$

Let $\mathbb{S}^1$ be the unit circle which is the boundary of $\mathbb{D}$. Mostow's rigidity theorem in this case says that if $H|\mathbb{S}^1$ is an absolutely continuous homeomorphism, then H must be a Möbius transformation. A corresponding result in the theory of one-dimensional dynamics was proved by Shub and Sullivan [**SS**]. Their result can be described as follows: let f be a C^2 orientation-preserving endomorphism of the circle $\mathbb{S}^1$. It is said to be expanding if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^{\circ n})'| \geq C\lambda^n$ for all x in $\mathbb{S}^1$ and all $n \geq 1$. Shub [**SH**] proved that any C^2 orientation-preserving expanding endomorphism f of $\mathbb{S}^1$ is topologically conjugate to $z \mapsto z^d$ on $\mathbb{S}^1$ for a unique integer $d \geq 2$, where d is called the degree of f. Therefore, any two C^2 orientation-preserving circle endomorphisms f and g with the same degree is topologically conjugate. Let h be the conjugacy from f to g, i.e.,

$$h \circ f = g \circ h.$$

Shub and Sullivan [**SS**] proved that if both f and g are C^k for some $k \geq 2$ and if h is an absolutely continuous homeomorphism, then h must be also a C^k diffeomorphism. We would like to note that there is a proof of Mostow's rigidity theorem from dynamical system point of view (see [**BO**]) by considering the induced expanding Markov maps r_1 and r_2 of $\mathbb{S}^1$ such that $h \circ r_1 = r_2 \circ h$ for $h = H|\mathbb{S}^1$.

Mostow's rigidity theorem is generalised for a bigger class of Fuchsian groups Γ such that the quotient space $\mathcal{S} = \mathbb{D}/\Gamma$ has finite hyperbolic volume (see, for example, [**MO,TU,TH,SU2**]). The corresponding bigger class in one-dimensional dynamics should be the class of one-dimensional dynamical systems having critical points and having certain mixing property. The first corresponding result in the bigger class is proved in [**JI1,JI4**] for generalised Ulam-von Neumann transformations which are certain interval maps with one power law type critical point (where we used the Gibbs theory in the study of this problem). Other corresponding results in the bigger class are developed in [**JI3,JI5**] later for geometrically finite one-dimensional maps. (In [**JI5**], the scaling function for a geometrically finite one-dimensional map is defined, studied, and used as a complete smooth invariant along with exponents and asymmetries at all singular points). However, the maps considered in [**SS,JI4,JI5**] are all geometrically finite from the combinatorics point of view. In this paper, we further develop our method in [**JI1,JI2,JI3,JI4,JI5**] to a space including some geometrically infinite one-dimensional dynamical systems.

The paper is organised as follows. In §2, we gave the definition of a quasi-hyperbolic one-dimensional map (Definition 2), which is certain one-dimensional map with non-recurrent critical points. One of the main technique tools [**JI2**] in our study is discussed in §3. This technique tool (Lemma 2) estimates the nonlinearity of the dynamical system generated by a quasi-hyperbolic one-dimensional map. We also define the nice and mixing conditions for a quasi-hyperbolic one-dimensional map in the beginning of §4. Let SP be the set of all singular points of a quasi-hyperbolic one-dimensional map f, let $PSO = \cup_{i=1}^{\infty} f^{\circ i}(SP)$ be the set of post-singular orbits of f, let $\overline{PSO}$ be the closure of PSO. A point p in SP is fold if $f'(x)/f'(y) < 0$ for points $x < p < y$ close to p. Let FSP be the set of all fold singular points of f and let $NFSP = SP \setminus FSP$ be the set of all non-fold singular points of f. Let $PNFSO = \cup_{n=1}^{\infty} f^{\circ n}(NFSP)$ be the set of post non-fold singular orbits of f. Let $\Gamma = \overline{PSO} \setminus PNFSO$ and $M_0 = M \setminus \Gamma$. Let η be the set of closures of intervals of M_0. In §4, we prove that

Main Theorem. *Suppose f and g are conjugate nice and mixing quasi-hyperbolic one-dimensional maps. Suppose the conjugacy h from f to g, i.e., $h \circ f = g \circ h$, is orientation-preserving homeomorphism of M. The map h restricted to every interval J in η is a $C^{1+\beta}$ for some fixed $0 < \beta \leq 1$ diffeomorphism if*

- **a** *h is an absolutely continuous homeomorphism and*
- **b** *the exponents and the asymmetries of f and g at all corresponding singular points are the same.*

In §5, §6, and §7, we discuss some application of the main result to geometrically finite one-dimensional maps (Corollary 1), to generalised Ulam-von Neumann transformations (Corollaries 2 and 3), and to expanding circle endomorphisms (Corollary 4).

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2. Sub-hyperbolic one-dimensional maps

Let M be the interval $[-1, 1]$ or the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Let $f : M \to M$ be a piecewise C^1 map. A point $c \in M$ is said to be *singular* if either $f'(c)$ does not exist or $f'(c)$ exists but $f'(c) = 0$. The later one is called a *critical* point. A singular point c of f is said to be *power law* if there is a real number $\gamma \geq 1$ such that the limits

$$\lim_{x\to c-} \frac{f'(x)}{|x-c|^{\gamma-1}} = B_- \qquad \text{and} \qquad \lim_{x\to c+} \frac{f'(x)}{|x-c|^{\gamma-1}} = B_+$$

exist and are non-zero. The numbers γ and $A = B_-/B_+$ are called the *exponent* and *asymmetry* of f at c. Let

$$r_c(x) = \frac{f'(x)}{|x-c|^{\gamma-1}}$$

for any power law singular point c.

REMARK 1. The exponent and the asymmetry of f at a power law singular point are (orientation) C^1-invariant, this means that if $h \circ f = g \circ h$ for an orientation-preserving C^1 diffeomorphism h, then the exponents and the asymmetries of f and g are the same at corresponding power law singular points.

Henceforth, we assume that $f : M \to M$ has only power law singular points and the number of power law singular points is finite (could be zero). Let SP denote the set of all singular points of f and let CP denote the set of all critical points of f. Let $PSO = \cup_{i=1}^{\infty} f^{\circ i}(SP)$ be the set of post-singular orbits. For a critical point $c \in CP$ and a real number $\tau > 0$, let $U_c(\tau) = [c - \tau, c + \tau]$ and let $U(\tau) = \cup_{c\in CP} U_c(\tau)$. Let $V(\tau) = \overline{M \setminus U(\tau/2)}$. A sequence of intervals $\{I_i\}_{i=0}^n$ is said to be a *chain* if $I_i \subset M \setminus SP$ and if $f : I_i \to I_{i+1}$ is bijective for all $i = 0, 1, \ldots, n-1$.

DEFINITION 1. We say a map $f : M \to M$ is C^{1+} if there is a real number $0 < \alpha \leq 1$ such that

(1) f' is α-Hölder continuous when restricted to every interval in $M \setminus SP$ and
(2) there is a real number $\tau > 0$ such that for every $c \in CP$, r_c is α-Hölder when restricted to $U_c(\tau) \cap \{x < c\}$ and to $U_c(\tau) \cap \{x > c\}$.

DEFINITION 2. A C^{1+} map $f : M \to M$ is said to be *quasi-hyperbolic* if there is a constant $\tau > 0$ such that $\overline{PSO} \cap U(\tau) = \emptyset$ and such that there exist constants $C > 0$ and $0 < \lambda < 1$ satisfying that for any chain $\{I_i\}_{i=0}^{n}$, if either

(i) $I_i \subseteq V(\tau)$ for all $0 \leq i \leq n-1$ or

(ii) $I_n \subseteq U(\tau)$, then

$$|I_0| \leq C\lambda^n |I_n|.$$

The space of quasi-hyperbolic one-dimensional maps contains many interesting maps. To show this let us give some examples. A point q in M is called periodic of period k of f if $f^{\circ i}(q) \neq q$ for all $0 < i < k$ but $f^{\circ k}(q) = q$. For a periodic point q of period k of f, let $O = \{f^{\circ i}(p)\}_{i=0}^{k-1}$ be the periodic orbit and let $E_{q,f} = (f^{\circ k})'(q)$ be the eigenvalue of f at O (or at q). The periodic orbit O is called attractive if $|E_{q,f}| < 1$; parabolic if $|E_{q,f}| = 1$; expanding if $|E_{q,f}| > 1$. The first example comes from a theorem (see [**MV**, Theorem 6.3, pp. 261- 262]). A critical point c of a C^2 map is called non-degenerate if $f'(c) = 0$ and if $f''(c) \neq 0$.

EXAMPLE 1. A C^2 map $f : M \to M$ with only non-degenerate critical points such that $\overline{PSO} \cap SP = \emptyset$ and such that all periodic points are expanding.

The Schwarzian derivative of a C^3 map $f : M \to M$ is, by definition,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2.$$

A C^3 map $f : M \to M$ has negative Schwarzian derivative if $S(f)(x) < 0$ for all x in M. Singer (see [**MV**]) proved that if f has negative Schwarzian derivative, then the immediate basin of every attractive or parabolic periodic orbit contains at least one critical orbit. Therefore, if f has negative Schwarzian derivative, if $\overline{PSO} \cap SP = \emptyset$, and if $\overline{PSO}$ contains neither attractive nor parabolic periodic points, then all periodic points of f are expanding. We then have that

EXAMPLE 2. A C^3 map $f : M \to M$ having negative Schwarzian derivative and satisfying that $\overline{PSO} \cap SP = \emptyset$ and that $\overline{PSO}$ contains neither attractive periodic points nor parabolic periodic points.

Suppose $f : M \to M$ is a C^3 map. If f satisfies the condition that for every critical point c, $p = f^{\circ m}(c)$ is an expanding periodic point of f for some integer $m \geq 1$, then $\overline{PSO} = PSO$ contains neither attractive nor parabolic periodic points. The condition here is called the preperiodic condition. In particular, Example 2 says that a C^3 map $f : M \to M$ having negative Schwarzian derivative and satisfying the preperiodic condition is quasi-hyperbolic.

There are two more examples. One is a $C^{1+\alpha}$ circle expanding map where $0 < \alpha \leq 1$ (see §7). The other is a geometrically finite one-dimensional map (see §5 and [**JI3,JI5**]). In the space of geometrically finite one-dimensional maps, a more interesting example is a generalised Ulam-von Neumann transformation (see §6 and [**JI1,JI4**]). The definitions of these maps are given in §5, §6, and §7, respectively. We also discuss some applications of the Main Theorem to these maps in these sections.

3. Distortion of a quasi-hyperbolic map

Suppose $f : M \to M$ is a quasi-hyperbolic map. Let $CP = \{c_1, \cdots, c_d\}$ be the set of all critical points of f (it may be empty) and let $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents. Let $\tau > 0$ be a fixed number satisfying Definitions 1 and 2. Let $U_i(\tau) = [c_i - \tau, c_i + \tau]$ and let

$$U = U(\tau) = \cup_{i=1}^{d} U_i(\tau).$$

Let $U_i(\tau/2) = [c_i - \tau/2, c_i + \tau/2]$, let $\tilde{U} = \cup_{i=1}^{k} U_i(\frac{\tau}{2})$, and let

$$V = \overline{M \setminus \tilde{U}}.$$

Note that U and V are closed sets. They are fixed in the rest of the paper.

Dividing the space of M into U and V is one of the key points in the paper: the set V is away from all critical points CP and the set U is away from post-singular orbit $PSO = \cup_{n=1}^{\infty} f^{\circ n}(SP)$. In the set V we can use the naive distortion property to control the distortion of the iterates of f (Lemma 1), and in the set U we can prove a Koebe type distortion property to control the distortion of the iterates of f (Lemma 2).

Let $0 < \alpha \leq 1$ be a number in Definition 1 satisfied by f. A chain $\mathcal{I} = \{I_i\}_{i=0}^{n}$ of closed intervals of M is said to be *admissible* if $f : I_i \to I_{i+1}$ is a $C^{1+\alpha}$-diffeomorphism and if either $I_i \subseteq V$ or $I_i \subseteq U$ for all $0 \leq i \leq n$. For any x in M, let $x_n = f^{\circ n}(x)$.

LEMMA 1. *There is a constant $C > 0$ such that for any admissible chain $\mathcal{I} = \{I_i\}_{i=0}^{n}$ if $I_i \subseteq V$ for all $1 \leq i \leq n-1$, then for any x and* *y in* I_0

$$\Big|\log\Big(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\Big)\Big| \leq C|x_n - y_n|^{\alpha}.$$

PROOF. Let ξ be the set of the closures of intervals in $V\setminus SP$. Let

$$a_0 = \inf_{x\in V\setminus SP} |f'(x)| > 0$$

and let

$$b_0 = \sup_{x\neq y\in I, I\in\xi} \frac{|f'(x)-f'(y)|}{|x-y|^{\alpha}} < \infty.$$

For any x and y in I_0, let $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$ for $0\leq i\leq n$. Then

$$\mathcal{A} = \frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|} = \prod_{i=0}^{n-1}\frac{|f'(x_i)|}{|f'(y_i)|}.$$

This implies that

$$\Big|\log\mathcal{A}\Big| \leq \frac{b_0}{a_0}\sum_{i=0}^{n-1}|x_i-y_i|^{\alpha}.$$

From Definition 2, there are constants $C_0>0$ and $0<\lambda_0<1$ such that

$$|x_i - y_i| \leq C_0\lambda_0^{n-i}|x_n-y_n|$$

for $0\leq i\leq n-1$. Therefore,

$$\Big|\log\mathcal{A}\Big| \leq \frac{b_0C_0}{a_0(1-\lambda_0^{\alpha})}|x_n-y_n|^{\alpha} = C|x_n-y_n|^{\alpha}$$

where $C = (b_0C_0)/(a_0(1-\lambda_0^{\alpha}))$. □

The following lemma is one of the key lemmas (refer to [**JI2**]) and is a property similar to Koebe's distortion theorem in one complex variable.

LEMMA 2. *There is a constant* $C>0$ *such that for any admissible chain* $\mathcal{I}=\{I_i\}_{i=0}^{n}$ *if* $I_n\subset U$, *then for any* x *and* y *in* I_0,

$$\Big|\log\Big(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\Big)\Big| \leq C|x_n - y_n|^{\alpha}.$$

Proof. The ratio $|(f^{\circ n})'(x)|/|(f^{\circ n})'(y)|$ equals the product

$$\mathcal{A} = \prod_{i=0}^{n-1} \frac{|f'(x_i)|}{|f'(y_i)|}$$

where $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$ for $0 \leq i \leq n$. Let (x_i, y_i) mean the open interval in I_i bounded by x_i and y_i. We divide the set of intervals $\mathcal{J} = \{(x_i, y_i)\}_{i=0}^{n-1}$ into two subsets

$$\mathcal{J}_1 = \{(x_i, y_i) \mid (x_i, y_i) \subseteq V\} \text{ and } \mathcal{J}_2 = \{(x_i, y_i) \mid (x_i, y_i) \subseteq U \setminus V\}.$$

Consider

$$\prod_{x_i, y_i \in \mathcal{J}_1} \frac{|f'(x_i)|}{|f'(y_i)|} \qquad \text{and} \qquad \prod_{x_i, y_i \in \mathcal{J}_2} \frac{|f'(x_i)|}{|f'(y_i)|}.$$

Following the proof of Lemma 1, there are constants $C_1, C_2 > 0$ such that

$$\Big|\log\Big(\prod_{x_i, y_i \in \mathcal{J}_1} \frac{|f'(x_i)|}{|f'(y_i)|}\Big)\Big| \leq C_1 \sum_{x_i, y_i \in \mathcal{J}_1} |x_i - y_i|^{\alpha} \leq C_2 \sum_{x_i, y_i \in \mathcal{J}_2} |x_i - y_i|^{\alpha}.$$

To estimate the product $\prod_{x_i, y_i \in \mathcal{J}_2} |f'(x_i)|/|f'(y_i)|$, we write it as the product of three factors:

$$\mathcal{B} = \prod_{x_i, y_i \in \mathcal{J}_2} \Big(\frac{|x_i - c_{k_i}|^{\gamma_{k_i}}}{|f(x_i) - f(c_{k_i})|} \frac{|f(y_i) - f(c_{k_i})|}{|y_i - c_{k_i}|^{\gamma_{k_i}}}\Big)^{t_{k_i}},$$

$$\mathcal{C} = \prod_{x_i, y_i \in \mathcal{J}_2} \frac{|y_i - c_{k_i}|^{\gamma_{k_i} - 1}}{|f'(y_i)|} \frac{|f'(x_i)|}{|x_i - c_{k_i}|^{\gamma_{k_i} - 1}},$$

and

$$\mathcal{D} = \prod_{x_i, y_i \in \mathcal{J}_2} \Big(\frac{|f(x_i) - f(c_{k_i})|}{|f(y_i) - f(c_{k_i})|}\Big)^{t_{k_i}},$$

where x_i and y_i are in $U_{k_i}(\tau)$ and $t_{k_i} = (\gamma_{k_i} - 1)/\gamma_{k_i}$. From Definition 1 and following the proof of Lemma 1, there is a constant $C_3 > 0$ such that

$$\Big|\log \mathcal{B}\Big|, \Big|\log \mathcal{C}\Big| \leq C_3 \sum_{x_i, y_i \in \mathcal{J}_2} |x_i - y_i|^{\alpha}.$$

Now we concentrate on the estimate of $\mathcal{D}$. Let

$$\frac{f(x_i) - f(c_{k_i})}{f(y_i) - f(c_{k_i})} = 1 + \frac{f(x_i) - f(y_i)}{f(y_i) - f(c_{k_i})}.$$

Then

$$\mathcal{D} = \exp\Big(\sum_{s=1}^{r-1} \frac{1}{t_{k_{i_s}}} \log\Big|1 + \frac{f(x_{i_s}) - f(y_{i_s})}{f(y_{i_s}) - f(c_{k_{i_s}})}\Big|\Big)$$

where $i_1 < i_2 < \cdots < i_{r-1} < n$. Let $i_r = n$. For each i_s, $1 \leq s < r$, consider the interval L_s bounded by y_{i_s} and $c_{k_{i_s}}$ and the map $h_s = f^{\circ(i_{s+1}-i_s)}$. Let $R_s \subseteq L_s$ be the maximal interval containing y_{i_s} such that h_s on R_s is C^1 and injective. One of the endpoints of R_s is y_{i_s} and the other is a preimage e of a singular point q in SP under $f^{\circ k_s}$ for some $0 \leq k_s < i_{s+1} - i_s$. Let $l_s = i_{s+1} - i_s - k_s$. Then h_s on the minimal interval J_s containing x_{i_s} and R_s is injective and maps J_s onto an interval containing the points $y_{i_{s+1}}$, $x_{i_{s+1}}$ and $f^{\circ l_s}(q)$. We enlarge every interval J of V into a closed interval $J' \supset J$ such that $J' \cap CP = \emptyset$ and such that the length of $J' \cap U$ is greater than a constant $a > 0$. Let $V' = \cup_{J\in V} J'$ be the union of all these enlarged intervals and let $U' = M \setminus V'$. If $f^{\circ i}(J_s) \subseteq V'$ for all $1 \leq i < i_{s+1} - i_s$, by following the proof of Lemma 1, there is a constant $C_4 > 0$ such that

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_4 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{|y_{i_{s+1}} - f^{\circ l_s}(q)|}.$$

Since $y_{i_{s+1}}$ is in U and $f^{\circ l_s}(q)$ is in PSO,

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_4 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{C_0}$$

where $C_0 > 0$ is the distance between U and $\overline{PSO}$. Otherwise, let $0 < k < i_{s+1} - i_s$ be the smallest integer such that $f^{\circ k}(J_s) \cap U' \neq \emptyset$. Since $f^{\circ i}(J_s) \subseteq V'$ for all $1 \leq i < k$, following the proof of Lemma 1, there is a constant $C_5 > 0$ such that

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_5 \frac{|x_{i_s+k} - y_{i_s+k}|}{|y_{i_s+k} - f^{\circ k}(e)|}.$$

Since y_{i_s+k} is in V and $f^{\circ k}(e)$ is in U',

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \leq C_5 \frac{|x_{i_s+k} - y_{i_s+k}|}{C_0'}$$

where $C_0' > 0$ is the distance between V and U'. From Definition 2 again, there is a constant $C_5' > 0$ such that

$$|x_{i_s+k} - y_{i_s+k}| \leq C_5' |x_{i_{s+1}} - y_{i_{s+1}}|.$$

Hence, there is a constant $C_6 > 0$ such that

$$\Big|\log \mathcal{D}\Big| \leq C_6 \sum_{x_i, y_i \in \mathcal{J}_2} |x_i - y_i|^{\alpha}.$$

Combining all the estimates, we have a constant $C_7 > 0$ such that

$$\Big|\log\Big(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\Big)\Big| \leq C_7 \sum_{x_i, y_i \in \mathcal{J}_2} |x_i - y_i|^{\alpha}.$$

Applying (ii) of Definition 2, there are two constants $C_8 > 0$ and $0 < \lambda < 1$ such that

$$|x_i - y_i| \leq C_8 \lambda^{n-i} |x_n - y_n|.$$

Therefore, we have a constant $C > 0$ such that

$$\Big|\log\Big(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\Big)\Big| \leq C |x_n - y_n|^{\alpha}.$$

□

4. Conjugacies between quasi-hyperbolic one-dimensional maps

Suppose $f : M \to M$ is a quasi-hyperbolic one-dimensional map. Remember that SP is the set of all singular points of f, that $PSO = \cup_{i=1}^{\infty} f^{\circ i}(SP)$ is the set of post-singular orbits of f, and that $\overline{PSO}$ is the closure of PSO. If $\overline{PSO}$ occupies too much space in M, it will be difficult to have a nice result. To avoid this, we will only consider a nice quasi-hyperbolic one-dimensional map. Let m be the Lebesgue measure on M. A quasi-hyperbolic one-dimensional map is said to be *nice* if

(I) $m(\overline{PSO}) = 0$ and

(II) there is an open neighborhood W about $\overline{PSO}$ (i.e., $\overline{PSO} \subset W$) such that $M \setminus W$ is non-empty and such that for any point p in M either $\{f^{\circ n}(p)\}_{n=N}^{\infty} \subseteq \overline{PSO}$ for some $N > 0$ or there is a subsequence $\{f^{\circ n_i}(p)\}_{i=1}^{\infty} \subseteq M \setminus W$.

We also need the mixing condition on f so that $\{f^{\circ n}\}_{n=0}^{\infty}$ can not be decomposed into several dynamical systems. We say that f is *mixing* if for any intervals I and J of M, there is an integer $n \geq 0$ such that $f^{\circ n}(J) \supseteq I$. The mixing condition is topologically invariant.

A point p in SP is fold if the asymmetry $A_{p,f}$ of f at p is negative. (It is equivalent to say that p is a local maximum or minimum point or to say that $f'(x)/f'(y) < 0$ for points $x < p < y$ close to p.) Let FSP be the set of all fold singular points of f and let $NFSP = SP \setminus FSP$ be the set of all non-fold singular points of f. Let $PNFSO = \cup_{n=1}^{\infty} f^{\circ n}(NFSP)$ be the set of post non-fold singular orbits of f. Let $\Gamma = \overline{PSO} \setminus PNFSO$ and $M_0 = M \setminus \Gamma$. Let η be the set of closures of intervals of M_0. A bijective map $h : M \to M$ is called an absolutely continuous homeomorphism if h and h^{-1} are absolutely continuous.

Main Theorem. *Suppose f and g are conjugate nice and mixing quasi-hyperbolic one-dimensional maps. Suppose the conjugacy h from f to g, i.e., $h \circ f = g \circ h$, is orientation-preserving homeomorphism of M. The map h restricted to every interval I in η is a $C^{1+\beta}$ for some fixed $0 < \beta \le 1$ diffeomorphism if*

a) *h is an absolutely continuous homeomorphism and*

b) *the exponents and the asymmetries of f and g at all corresponding singular points are the same.*

Let f and g be maps and h be the conjugacy in the theorem. Let $0 < \alpha \le 1$ be a number in Definition 1 satisfied by both f and g. Let $CP = \{c_1, \cdots, c_d\}$ be the set of critical points of f. Let $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents of f at critical points. Let $\gamma = \max\{\gamma_i \mid 1 \le i \le d\}$.

Remark 2. Then in the Main Theorem, $\beta = \alpha/\gamma$ if h satisfies **a)** and **b)**.

For any point p in M, from the equation $h \circ f = g \circ h$, we have that if p is not a singular point of f, then h is differentiable at p if and only if h is differentiable at $f(p)$. Moreover, if h is differentiable at p and $f(p)$, we have

$$h'(p) = \frac{f'(p)}{g'(h(p))} h'(f(p)).$$

Remark 3. If p is a singular point, let $\gamma_{f,p}$ and $\gamma_{g,h(p)}$ be the exponents of f and g at p and at $h(p)$. Let $A_{f,p} = B_{-,f}/B_{+,f}$ and $A_{g,h(p)} = B_{-,g}/B_{+,g}$ be the asymmetry of f and g at p and at $h(p)$. If $\gamma_{f,p} = \gamma_{g,h(p)}$ and if $h'(f(p)) \neq 0$ exists, then

$$\big(h'(p-)\big)^{\gamma_{f,p}} = \frac{B_{-,f}}{B_{-,g}} h'(f(p))$$

and

$$\big(h'(p+)\big)^{\gamma_{f,p}} = \frac{B_{+,f}}{B_{+,g}} h'(f(p))$$

Furthermore, if $A_{f,p} = A_{g,h(p)}$, then $h'(p-) = h'(p+)$. Therefore, if $h'(f(p))$ exists and if $\gamma_{f,p} = \gamma_{g,h(p)}$ and $A_{f,p} = A_{g,h(p)}$, then $h'(p)$ exists. Similarly, if $h'(p)$ exists, if p is not a fold singular point, and if $\gamma_{f,p} = \gamma_{g,h(p)}$ and $A_{f,p} = A_{g,h(p)}$, then $h'(f(p))$ exists.

Remember that m is the Lebesgue measure on M. We say that a measurable set E in an interval I has full measure if $m(E) = m(I)$. A point p in a measurable set E_0 is said to be a density point if

$$\lim_{m(J)\to 0} \frac{m(E_0 \cap J)}{m(J)} = 1$$

where J runs over all intervals containing p. We prove the Main Theorem through several lemmas (Lemma 3 to Lemma 5).

Lemma 3. *If h is absolutely continuous, then we can find a small interval $I \subset M \setminus \overline{PSO}$ such that $h|I$ is C^1.*

Proof. Since h is absolutely continuous, h' exists a.e. in M and is integrable and $h = \int h'(x)dx$ (refer to [**AB,VE**]). Let $\Lambda = \cup_{n=0}^{\infty} f^{-n}(\overline{PSO})$. Then $m(\Lambda) = 0$ since $\overline{PSO}$ has zero measure. Since $h'(x)$ is measurable and h is a homeomorphism, we can find a point p in $M\setminus\Lambda$ and a subset $p \in E_0$ such that h is differentiable at every point in E_0, such that p is a density point of E_0, such that $h'(p) \neq 0$, and such that the derivative $h'|E_0$ is continuous at p (refer to [**AB,VE**]). Since f is nice, there is a subsequence $\{f^{\circ n_k}(p)\}_{k=1}^{\infty} \subseteq M \setminus W$ such that it converges to a point q in $M \setminus W$, where W is the open set in (II) in the beginning of §4.

Let us first assume that q is an interior point of U. Take a small interval $q \in I = (a, b) \subset U$. Assume $\{f^{\circ n_k}(p)\}_{k=1}^{\infty} \subset I$. Since $I \cap \overline{PSO} = \emptyset$, there is a sequence of interval $\{J_k\}_{k=1}^{\infty}$ such that $p \in J_k$, such that $f^{\circ n_k} : J_k \to I$ is a C^1-diffeomorphism, and such that $\{J_{l,k} = f^{\circ l}(J_k)\}_{l=0}^{n_k}$ is admissible for every $k \geq 1$. From (ii) of Definition 2, $m(J_k)$ goes to zero as k tends to infinity.

From Lemma 2, there is a constant $C > 0$, such that

$$\Big| \log\Big(\frac{|(f^{\circ n_k})'(w)|}{|(f^{\circ n_k})'(z)|}\Big)\Big| \leq C$$

for any w and z in J_k. (The inequality means that $\{f^{\circ n_k}|J_k\}_{k=1}^{\infty}$ have uniformly bounded distortion.) For any positive integer s, there is an integer $N_s > 0$ such that

$$\frac{m(E_0 \cap J_k)}{m(J_k)} \geq 1 - \frac{1}{s}$$

for all $k > N_s$. Let $E_k = f^{\circ n_k}(E_0 \cap J_k)$. Then h is differentiable at every point in E_k and

$$\frac{m(E_k \cap I)}{m(I)} \geq 1 - \frac{C}{s}$$

for all $k > N_s$ because $\{f^{\circ n_k}|J_k\}_{k=1}^{\infty}$ have uniformly bounded distortion where $C > 0$ is a constant. Let $E = \cap_{s=1}^{\infty} \cup_{k>N_s} E_k$. Then E has full measure in I and h is differentiable at every point in E with non-zero derivative.

Next, we are going to prove that $h'|E$ is uniformly continuous. For any x and y in E, let z_k and w_k be the preimages of x and y under the diffeomorphism $f^{\circ n_k} : J_k \to I$. Then z_k and w_k are in E_0. From $h \circ f = g \circ h$, we have that

$$h'(x) = \frac{(g^{\circ n_k})'(h(z_k))}{(f^{\circ n_k})'(z_k)} h'(z_k)$$

and

$$h'(y) = \frac{(g^{\circ n_k})'(h(w_k))}{(f^{\circ n_k})'(w_k)} h'(w_k).$$

So

$$\Big| \log \Big(\frac{h'(x)}{h'(y)} \Big) \Big| \leq \Big| \log \Big| \frac{(g^{\circ n_k})'(h(z_k))}{(g^{\circ n_k})'(h(w_k))} \Big| \Big| + \Big| \log \Big| \frac{(f^{\circ n_k})'(w_k)}{(f^{\circ n_k})'(z_k)} \Big| \Big| + \Big| \log \Big(\frac{h'(z_k)}{h'(w_k)} \Big) \Big|$$

From Lemma 2, there is a constant $C > 0$ such that

$$\Big| \log \Big| \frac{(f^{\circ n_k})'(w_k)}{(f^{\circ n_k})'(z_k)} \Big| \Big| \leq C|x - y|^{\alpha}$$

and

$$\Big| \log \Big| \frac{(g^{\circ n_k})'(h(z_k))}{(g^{\circ n_k})'(h(w_k))} \Big| \Big| \leq C|h(x) - h(y)|^{\alpha}.$$

Therefore,

$$\Big| \log \Big(\frac{h'(x)}{h'(y)} \Big) \Big| \leq C\Big(|x - y|^{\alpha} + |h(x) - h(y)|^{\alpha} \Big) + \Big| \log \Big(\frac{h'(z_k)}{h'(w_k)} \Big) \Big|.$$

Since $h'|E_0$ is continuous at p, the last term in the last inequality tends to zero as k goes to infinity. Hence

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq C\Big(|x-y|^{\alpha} + |h(x)-h(y)|^{\alpha}\Big).$$

This means that $h'|E$ is uniformly continuous. It can be extended to a continuous function ϕ on I. Because $h|I$ is absolutely continuous and E has full measure in I,

$$h(x) = h(a) + \int_a^x h'(x)dx = h(a) + \int_a^x \phi(x)dx$$

on I. This implies that $h|I$ is actually C^1.

If q is in V but not an interior point of U. We can find a small interval $I=(a,b)$ such that $q \in I \subseteq V$ and such that $m(I) = b-a \leq \mathrm{dist}(I, \overline{PSO})$ and $m(h(I)) \leq \mathrm{dist}(h(I), \overline{h(PSO)})$ where dist means the distance. Modifying the above argument either by only applying Lemma 1 or by first applying Lemma 1 and then Lemma 2, we can still prove that $h|I$ is C^1 (refer to the proof of Lemma 5). □

Remember that $U_i(\tau) = [c_i - \tau, c_i + \tau]$ for $c_i \in CP$. Let I and q be the interval and the point found in the previous lemma.

LEMMA 4. *If h is absolutely continuous, then the restriction of h to every $U_i(\tau)$ is $C^{1+\alpha}$.*

PROOF. Denote $J = U_i(\tau) = [d,e]$. Since f is mixing and since $J \cap \overline{PSO} = \emptyset$, there is a preimage $J_k \subset I$ of J under $f^{\circ n_k}$ such that J_k tends to q as $k \to \infty$ and such that $f^{\circ n_k} : J_k \to J$ is a C^1-diffeomorphism where $\{n_k\}_{k=1}^{\infty}$ is a subset of the positive integers. Let F_k be the inverse of $f^{\circ n_k} : J_k \to J$ for all $k > 0$. From the equation $h \circ f = g \circ h$, we have that $h|J = g^{\circ n_k} \circ h \circ F_k$. So $h|J$ is C^1 and

$$h'(x) = \frac{(g^{\circ n_k})'(h(z_k))}{(f^{\circ n_k})'(z_k)} h'(z_k) \neq 0$$

for any x in J where $z_k = F_k(x)$.

Without loss of generality, we assume that $\{J_{i,k} = f^{\circ(n_k - i)}(J_k)\}_{i=0}^{n_k}$ are all admissible for all $k > 0$. For any x and y in J, let z_k and w_k be the preimage of x and y under the diffeomorphism $f^{\circ n_k} : J_k \to J$. Since

$$\frac{h'(x)}{h'(y)} = \frac{(g^{\circ n_k})'(h(z_k))}{(f^{\circ n_k})'(z_k)} \cdot \frac{(f^{\circ n_k})'(w_k)}{(g^{\circ n_k})'(h(w_k))} \cdot \frac{h'(z_k)}{h'(w_k)},$$

from Lemma 2, there is a constant $C > 0$ such that

$$\begin{aligned}\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| &\leq \Big|\log\Big|\frac{(g^{\circ n_k})'(h(z_k))}{(g^{\circ n_k})'(h(w_k))}\Big|\Big| + \Big|\log\Big|\frac{(f^{\circ n_k})'(w_k)}{(f^{\circ n_k})'(z_k)}\Big|\Big| \\ &\quad + \Big|\log\Big(\frac{h'(z_k)}{h'(w_k)}\Big)\Big| \\ &\leq C\Big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\Big) + \Big|\log\Big(\frac{h'(z_k)}{h'(w_k)}\Big)\Big|.\end{aligned}$$

Because $h'(z_k), h'(w_k) \to h'(q)$ as $k \to \infty$, we have that

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq C\Big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\Big).$$

This implies that $h|J$ is actually $C^{1+\alpha}$. □

LEMMA 5. *If h is absolutely continuous and if the exponents of f and g at all singular points are the same, then the restriction of h to the closure of every interval J of $V \setminus \overline{PSO}$ is $C^{1+\frac{\alpha}{\gamma}}$.*

PROOF. We always use C to denote a positive constant (although it may be different in different formulas). Since f is mixing, we can find a subsequence $\{n_k\}_{k=1}^\infty$ of the positive integers and intervals $J_k \subset I$ such that J_k tends to q as $k \to \infty$ and such that $f^{\circ n_k} : J_k \to J$ is a diffeomorphism. Let F_k be the inverse of $f^{\circ n_k} : J_k \to J$. From the equation $h \circ f = g \circ h$, we have that $h|J = g^{\circ n_k} \circ h \circ F_k$. So $h|J$ is C^1 and

$$h'(x) = \frac{(g^{\circ n_k})'(h(z_k))}{(f^{\circ n_k})'(z_k)} h'(z_k) \neq 0$$

for any x in J where $z_k = F_k(x)$.

Let $J_{i,k} = f^{\circ(n_k - i)}(J_k)$ for $0 \leq i \leq n_k$. Without loss of generality, we assume that $\{J_{i,k}\}_{i=0}^{n_k}$ is admissible. For any x and y in J, let z_k and w_k be the preimage of x and y under the diffeomorphism $f^{\circ n_k} : J_k \to J$. Since

$$\frac{h'(x)}{h'(y)} = \frac{(g^{\circ n_k})'(h(z_k))}{(f^{\circ n_k})'(w_k)} \cdot \frac{(f^{\circ n_k})'(w_k)}{(g^{\circ n_k})'(h(w_k))} \cdot \frac{h'(z_k)}{h'(w_k)},$$

$$\begin{aligned}\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| &\leq \Big|\log\Big(\frac{|(g^{\circ n_k})'(h(z_k))|}{|(g^{\circ n_k})'(h(w_k))|}\Big)\Big| \\ &\quad + \Big|\log\Big(\frac{|(f^{\circ n_k})'(w_k)|}{|(f^{\circ n_k})'(z_k)|}\Big)\Big| + \Big|\log\Big(\frac{h'(z_k)}{h'(w_k)}\Big)\Big|.\end{aligned}$$

Let $n = n_k$ and let $m = m(n_k) > 0$ be the smallest integer such that $J_{m,k} \subseteq U_j(\tau) \subseteq U$ for some $1 \leq j \leq d$. Let $x_i = f^{\circ(n_k - i)}(z_k)$ and $y_i = f^{\circ(n_k - i)}(w_k)$ for all $0 \leq i \leq n_k$. Then

$$\begin{aligned}
\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq & \Big|\sum_{i=1}^{m-1}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \\
& + \Big|\sum_{i=1}^{m-1}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \\
& + \Big|\log\Big(\frac{|g'(h(x_m))|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|g'(h(y_m))|}\Big)\Big| \\
& + \Big|\sum_{i=m+1}^{n}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \\
& + \Big|\sum_{i=m+1}^{n}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \\
& + \Big|\log\Big(\frac{h'(z_k)}{h'(w_k)}\Big)\Big|.
\end{aligned}$$

The last term tends to zero as k goes to infinity since $h'(w_k), h'(z_k) \to h'(q)$ as $k \to \infty$. We estimate the first five terms. From Lemma 1, there is a constant $C > 0$ such that

$$\Big|\sum_{i=1}^{m-1}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| \leq C|x-y|^{\alpha}$$

and

$$\Big|\sum_{i=1}^{m-1}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \leq C|h(x)-h(y)|^{\alpha}.$$

From Lemma 2, there is a constant $C > 0$ such that

$$\begin{aligned}
\Big|\sum_{i=m+1}^{n}\Big(\log|f'(y_i)| - \log|f'(x_i)|\Big)\Big| & \leq C|x_m - y_m|^{\alpha} \\
& \leq C|x_{m-1} - y_{m-1}|^{\frac{\alpha}{\gamma_j}} \\
& \leq C|x-y|^{\frac{\alpha}{\gamma_j}}.
\end{aligned}$$

Similarly,

$$\Big|\sum_{i=m+1}^{n}\Big(\log|g'(h(x_i))| - \log|g'(h(y_i))|\Big)\Big| \leq C|h(x_m)-h(y_m)|^{\frac{\alpha}{\gamma_j}}$$
$$\leq C|h(x)-h(y)|^{\frac{\alpha}{\gamma_j}}.$$

Now we consider

$$\mathcal{S} = \frac{|g'(h(x_m))|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|g'(h(y_m))|}.$$

Define

$$\mathcal{S} = \mathcal{S}_1\cdot\mathcal{S}_2\cdot\mathcal{S}_3$$

where

$$\mathcal{S}_1 = \frac{|g'(h(x_m))|}{|h(x_m)-h(c_j)|^{\gamma_j-1}}\cdot\frac{|h(y_m)-h(c_j)|^{\gamma_j-1}}{|g'(h(y_m))}, $$
$$\mathcal{S}_2 = \frac{|x_m-c_j|^{\gamma_j-1}}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|y_m-c_j|^{\gamma_j-1}},$$

and

$$\mathcal{S}_3 = \Big(\frac{|h(x_m)-h(c_j)|}{|x_m-c_j|}\Big)^{\gamma_j-1}\cdot\Big(\frac{|y_m-c_j|}{|h(y_m)-h(c_j)|}\Big)^{\gamma_j-1}.$$

Lemma 4 implies that

$$\Big|\log S_3\Big| \leq C|x_m-y_m|^{\alpha} \leq C|x_{m-1}-y_{m-1}|^{\frac{\alpha}{\gamma_j}} \leq C|x-y|^{\frac{\alpha}{\gamma_j}}.$$

From (1) of Definition 2,

$$\Big|\log S_2\Big| \leq C|x_m-y_m|^{\alpha} \leq C|x_{m-1}-y_{m-1}|^{\frac{\alpha}{\gamma_j}} \leq C|x-y|^{\frac{\alpha}{\gamma_j}}$$

and

$$\Big|\log S_1\Big| \leq C|h(x_m)-h(y_m)|^{\alpha} \leq C|h(x_{m-1})-h(y_{m-1})|^{\frac{\alpha}{\gamma_j}}$$
$$\leq C|h(x)-h(y)|^{\frac{\alpha}{\gamma_j}}.$$

Thus,

$$\Big|\log\Big(\frac{h'(x)}{h'(y)}\Big)\Big| \leq C\Big(|x-y|^{\frac{\alpha}{\gamma}} + |h(x)-h(y)|^{\frac{\alpha}{\gamma}}\Big).$$

This implies that $h'|J$ is actually $C^{\frac{\alpha}{\gamma}}$. So is $h'|\overline{J}$. We have that $h|\overline{J}$ is $C^{1+\frac{\alpha}{\gamma}}$. □

Proof of the Main Theorem. Lemmas 4 and 5 and Remark 3 and the calculation before Remark 3 complete the proof. □

5. Geometrically finite one-dimensional maps

Let M be the interval $[-1,1]$ or $\mathbb{S}^1$. Let $f: M \to M$ be a piecewise C^1 map with only power law type singular points. Let SP be the set of singular points of f and $SO = \cup_{n=0}^{\infty} f^{\circ n}(SP)$ be the union of singular orbits of f. If SO is non-empty and finite, let $\xi_1 = \{I_0, \cdots, I_{k-1}\}$ be the set of the closures of intervals in the complement of SO in M, then (f, ξ_1) is a Markov map. By a Markov map, we mean that

- **a.** $I_0, \ldots, I_{k-1}$ have pairwise disjoint interiors,
- **b.** the union $\cup_{i=0}^{k-1} I_i$ of all intervals in ξ_1 is M,
- **c.** the restriction $f|I$ to every interval I in ξ_1 is homeomorphic, and
- **d.** the image $f(I)$ of every interval I in ξ_1 is the union of some intervals in ξ_1.

Suppose (f, ξ_1) is the Markov map in the previous paragraph. Let $g_i = (f|I_i)^{-1}$ be the inverse of $f: I_i \to f(I_i)$ for $0 \leq i < k$. A sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\cdots$, $(k-1)$'s is called admissible if the domain $f(I_{i_l})$ of g_{i_l} contains $I_{i_{l+1}}$ for all $0 \leq l < n-1$. For an admissible sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\cdots$, $(k-1)$'s, we can define $g_{w_n} = g_{i_0} \circ \cdots g_{i_{n-1}}$ and $I_{w_n} = g_{w_n}(f(I_{i_{n-1}}))$. Let ξ_n be the set of the intervals I_{w_n} for all admissible sequences of length n. It is also a Markov partition of M respect to f and called the n^{th}-partition of M induced from (f, ξ_1). Let κ_n be the maximum of the lengths of intervals in ξ_n. A one-dimensional map $f: M \to M$ is said to be geometrically finite if

- **(1)** f is C^{1+},
- **(2)** the set of singular orbits SO is non-empty and finite,
- **(3)** no critical point is periodic, and
- **(4)** there are constants $C > 0$ and $0 < \mu < 1$ such that $\kappa_n \leq C\mu^n$ for all $n > 0$.

Suppose f is a geometrically finite one-dimensional map. It is quasi-hyperbolic from [**JI3,JI5**]. The set of post-singular orbits PSO of f is finite, so it has zero measure, i.e., $m(\overline{PSO}) = 0$. Every singular point u of f lands at an expanding periodic orbit of f, this means that there is an integer $k > 0$ such that $v = f^{\circ k}(u)$ is an expanding periodic point of f (see [**JI3,JI5**]). Therefore f is nice. Let FSP be the set of all fold singular points of f and let $PFSO = \cup_{n=1}^{\infty} f^{\circ n}(FSP)$ be the set of all post fold singular orbits of f (see the beginning of §4). Then $PFSO$ is finite. Let $M_0 = M \setminus PFSO$. Let η be the set of the closures of intervals of $M \setminus PFSO$. It is finite. We have that

Corollary 1. *Suppose f and g are two conjugate mixing geometrically finite one-dimensional maps. Let h be the conjugacy between f and g, i.e., $h \circ f = g \circ h$. Suppose h is orientation-preserving. Then $h|I$ for any I in η is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if (i) h is an absolutely continuous homeomorphism and (ii) the exponents and the asymmetries of f and g at all corresponding critical points are the same.*

Remark 4. For an orientation-preserving conjugacy class $\mathcal{F}$ of geometrically finite one-dimensional maps, there is the topological model: the symbolic space Σ with the shift map σ. This topological model is called the symbolic dynamical system for $\mathcal{F}$ (see [**JI5**]). The dual symbolic space Σ^* of $\mathcal{F}$ and the scaling function

$$s_f : \Sigma^* \to \mathbb{R}$$

of f in $\mathcal{F}$ were defined in [**JI5**]. The scaling function s_f of f is a smooth invariant in $\mathcal{F}$. We proved in [**JI5**] that the scaling function s_f exists for every geometrically finite one-dimensional map. The property of the scaling function s_f of a geometrically finite one-dimensional map f has been also studied in details in [**JI5**]. Furthermore, the scaling function is used as a complete smooth invariant along with exponents and asymmetries in the smooth classification of geometrically finite one-dimensional maps. We proved in [**JI5**] that if the scaling functions s_f and s_g of f and g in $\mathcal{F}$ are the same, the conjugacy h between f and g is bi-Lipschitz, and further, if the exponents and asymmetries of f and g at corresponding singular points are also the same, then h restricted on every interval in η is a $C^{1+\beta}$-diffeomorphism. The analysis from bi-Lipschitz to $C^{1+\beta}$ in [**JI5**] is not quite complete. However, a bi-Lipschitz homeomorphism is an absolutely continuous homeomorphism, it can be closed by Corollary 1.

6. Generalised Ulam-von Neumann transformations

Let $M = [-1, 1]$ and let $f : M \to M$ be a C^{1+}-folding map with a unique power law type singular point 0. We call $f : M \to M$ a generalised Ulam-von Neumann transformation (see [**JI1,JI4**]) if

[a] $f(-1) = f(1) = -1$ and $f(0) = 1$,

[b] $f|[-1,0]$ and $f|[0,1]$ are orientation-preserving and orientation-reversing homeomorphisms, and
[c] f is a geometrically finite one-dimensional map.

One example of a generalised Ulam-von Neumann transformation is $f(x) = 1 - 2|x|^{\gamma}$ for $\gamma > 1$. Another one is $g(x) = -1 + 2\cos(\pi x/2)$. For a generalised Ulam-von Neumann transformation f, let $I_0 = [-1,0]$ and $I_1 = [0,1]$. We then have that $f(I_0) = f(I_1) = M$. Thus $\xi_1 = \{I_0, I_1\}$ is a Markov partition of M with respect to f. From [**JI1,JI4**] a generalised Ulam-von Neumann transformation f is a nice and mixing quasi-hyperbolic one-dimensional map. The post-singular orbit $PSO = \cup_{i=1}^{\infty} f^{\circ i}(0)$ of f is the boundary $\{-1,1\}$ of M. Any two generalised Ulam-von Neumann transformations f and g are topologically conjugate by an orientation-preserving homeomorphism h (see [**JI4**]). Following the Main Theorem, we have that

Corollary 2. *Let f and g be two generalised Ulam-von Neumann transformations. Let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. Then h is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if h is an absolutely continuous homeomorphism and the exponents and the asymmetries of f and g at 0 are the same.*

If the eigenvalues (refer to §2) at corresponding periodic points and the exponents at 0 of two generalised Ulam-von Neumann transformations of f and g are the same, then h is a bi-Lipschitz homeomorphism (see [**JI4**]). So h is an absolutely continuous homeomorphism in this case. In [**JI4**], we argued from bi-Lipschitz to C^1 by the Gibbs theory. Now it can follow from the Main Theorem.

Corollary 3. *Let f and g be two generalised Ulam-von Neumann transformations. Let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. Then h is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if the eigenvalues of f and g at corresponding periodic points and the exponents and asymmetries of f and g at 0 are the same.*

Remark 5. Let $f : M \to M$ be a generalised Ulam-von Neumann transformation. For any $x \in M$, we can find a unique sequence $\{x_i\}_{i=1}^{\infty} \subset [-1,0]$ such that $f(x_i) = x_{i-1}$ for $1 \leq i < \infty$ where $x_0 = x$. One can check that $x_i \to -1$ as $i \to \infty$. Let $g : M \to M$ be another generalised Ulam-von Neumann transformation and let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. If h is a C^1-diffeomorphism, we have that

$$h'(x) = \Big(\prod_{i=1}^{\infty} \frac{g'(h(x_i))}{f'(x_i)} \Big) \cdot h'(-1).$$

Using this equality, one can discuss the $C^{k+\beta}$ (or real analytic) smoothness of h in Corollaries 2 and 3 if both f and g are $C^{k+\alpha}$ in the sense of Definition 1 (or real analytic) where $k \geq 1$ is an integer and $0 < \beta \leq \alpha \leq 1$ are real numbers.

7. Expanding circle endomorphisms

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a $C^{1+\alpha}$-orientation-preserving endomorphism for some $0 < \alpha \leq 1$. We say f is expanding if there are constants $C > 0$ and $\mu > 1$ such that $|(f^{\circ n})'(x)| \geq C\mu^n$ for all $x \in \mathbb{S}^1$ and all $n \geq 1$. Let $d = \deg(f)$ be the topological degree of f. An example of an expanding circle endomorphism of degree d is $z \mapsto z^d$. Shub [**SH**] proved that every degree d such map is topologically conjugate to $z \mapsto z^d$. Thus any two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms f and g of degree $d > 1$ are topologically conjugate by an orientation-preserving homeomorphism h of $\mathbb{S}^1$. The following result is due to Sullivan [**SU3**] and can now follow from the Main Theorem.

Corollary 4 (Sullivan). *Let f and g be two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms of degree $d > 1$ where $0 < \alpha \leq 1$ is a real number. Let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. Then h is a $C^{1+\alpha}$-diffeomorphism if and only if the eigenvalues of f and g at all periodic points of f are the same.*

Remark 6. A C^1-orientation-preserving expanding circle endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ of degree $d > 1$ has a fixed point q. The preimage $f^{-1}(q)$ contains d points, and $\mathbb{S}^1 \setminus f^{-1}(q)$ has d connected components. Let I_0 be the closure of a fixed interval of $\mathbb{S}^1 \setminus f^{-1}(q)$ having q as an endpoint. For any $x \in \mathbb{S}^1$, we can find a unique sequence $\{x_i\}_{i=1}^{\infty} \subset I_0$ such that $f(x_i) = x_{i-1}$ for $1 \leq i < \infty$ where $x_0 = x$. One can check that $x_i \to q$ as $i \to \infty$. Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be another C^1-orientation-preserving expanding circle endomorphism of degree d and let h be the conjugacy from f to g, i.e., $h \circ f = g \circ h$. If h is a C^1-diffeomorphism, then

$$h'(x) = \Big(\prod_{i=1}^{\infty} \frac{g'(h(x_i))}{f'(x_i)} \Big) \cdot h'(q).$$

Using this equality, one can discuss the $C^{k+\alpha}$ smoothness of h in Corollary 4 if both f and g are $C^{k+\alpha}$ where $k > 1$ is an integer and $0 < \alpha \leq 1$ is a real number. In particular, if f and g in Corollary 4 are both real analytic, then h is real analytic (see [**SS**]).

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